Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

On the Value of Two-person Zero-sum Linear Quadratic Differential Games

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Abstract— **Considered in this paper are the two-person zero-sum linear quadratic differential games. It is shown that the value of the game exists if and only if both the upper value and lower value exist. As an consequence, we prove that another necessary and sufficient condition for the existence of the value of the game is the existence of an open loop-open loop saddle point. An example is also given where lower value exists but upper value does not exist.**

I. Introduction

Since 1980's, many work has been contributed to linear quadratic differential games due to their essential role in modern robust control and H^∞-optimization design, see, e.g., [1], [3], [7]. In this paper, we consider the two-person, zero-sum linear quadratic differential games on a finite horizon. It is well-known that the solvability of the corresponding Riccati differential equations is equivalent to the existence of the H^∞-optimal control ([1]). In [2], the relationship between saddle points of the game and the solvability of various Riccati differential equations is studied. It is shown that

(a) if the Riccati differential equation admits a solution, then, the game admits a closed loop-closed loop saddle point;
(b) if both the Riccati differential equation and the lower Riccati differential equation(for definition, see [2]) admit a solution, then, the game admits a closed loop-open loop saddle point;
(c) if both the Riccati differential equation and the upper Riccati differential equation(for definition, see [2]) admit a solution, then, the game admits an open loop-closed loop saddle point.

Note that nothing is said about the open loop-open loop saddle point. This paper is to address this issue. Obviously, the existence of open loop-open loop saddle points guarantees the existence of the value of the game, we shall show that this is also necessary. This follows easily from our main result which states that a necessary and sufficient condition for the existence of the value of the game is that both the lower value and the upper value of the game exist. Examples show that except open loop-open loop saddle point, existence of any other type of saddle point can not guarantee the existence of the open loop value of the game.

Consider the following dynamic system

$$\dot{x} = Ax + Bu + Gv, \quad x(0) = x_0, \quad x \in [0, T] \tag{1}$$

This work was supported by the Foundation of Guangdong Science and Technology Projects (Project Id:G03B2040770)

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with quadratic index

$$J(u, v) = \int_0^T (|u|^2 - |v|^2 + x'Qx)dt + x(T)'Wx(T) \tag{2}$$

Where $T > 0$ is a given final time, A, B, G, Q, W are matrices of suitable dimension such that Q and W are symmetric, but not necessarily nonnegative. Moreover, without loss of generality, cross terms in (u, v, x) are not present in the quadratic form because they can be eliminated by appropriate transformations. Let

$$v^-(x_0) = \sup_{v \in L^2(0,T;\mathbb{R}^m)} \inf_{u \in L^2(0,T;\mathbb{R}^l)} J(u, v)$$

be the open loop lower value of the game and

$$v^+(x_0) = \inf_{u \in L^2(0,T;\mathbb{R}^l)} \sup_{v \in L^2(0,T;\mathbb{R}^m)} J(u, v)$$

be the open loop upper value of the game. If both $v^-(x_0)$ and $v^+(x_0)$ exist and $v^-(x_0) = v^+(x_0)$, then we say that the open loop value of the game exists and the open loop value is $v(x_0) = v^-(x_0) = v^+(x_0)$. Hereafter, by the value, the lower value and the upper value of the game we mean the open loop value, the open loop lower value and the open loop upper value of the game respectively, unless stated otherwise. Obviously, we always have

$$v^-(x_0) \leq v^+(x_0)$$

A natural question is: do the lower value and the upper value exist? are they equal to each other if both of them exist? how are they related to the existence of saddle points? In section III, we discuss an example where the lower value exists while the upper value does not, we then prove the main theorem of the paper in section IV, using results established on Fredholm integral equations in section II. Some final remarks are given in section V.

II. Solution set of Fredholm integral equations

It is convenient to introduce some technical notations that will used throughout the paper. Let $M^+ = BB', M^- = GG', M = M^+ - M^-, \mathcal{X} = L^2(0, T; \mathbb{R}^n)$, define the following linear operators:

$$\begin{array}{ll} L : \mathcal{X} \to \mathcal{X}, & (Lx)(t) = \int_0^t e^{A(t-s)} x(s) ds \\ \hat{L} : \mathcal{X} \to \mathbb{R}^n, & (\hat{L}x)(t) = \int_0^T e^{A(T-s)} x(s) ds \\ S : \mathbb{R}^n \to \mathcal{X}, & (Sy)(t) = e^{At} y \\ \hat{S} : \mathbb{R}^n \to \mathbb{R}^n, & \hat{S}y = e^{AT} y \end{array}$$

with the adjoint operators given by

$$\begin{array}{ll} L^*:\mathcal{X}\to\mathcal{X}, & (L^*x)(t)=\int_t^T e^{A'(s-t)}x(s)ds \\ \hat{L}^*:\mathbb{R}^n\to\mathcal{X}, & (\hat{L}^*y)(t)=e^{A'(T-t)}y \\ S^*:\mathcal{X}\to\mathbb{R}^n, & S^*x=\int_0^T e^{A't}x(t)dt \\ \hat{S}^*:\mathbb{R}^n\to\mathbb{R}^n, & \hat{S}^*y=e^{A'T}y \end{array}$$

Moreover, define operators

$$K=L^*QL+\hat{L}^*W\hat{L},\quad \hat{K}=L^*QS+\hat{L}^*W\hat{S}$$

Using these operators, the system (1) can be rewritten as

$$x=Sx_0+LBu+LGv$$

with

$$x(T)=\hat{S}x_0+\hat{L}Bu+\hat{L}Gv$$

and the index (2) can be written as a bilinear form in Hilbert space $\mathcal{U}\times\mathcal{V}=L^2(0,T;\mathbb{R}^l)\times L^2(0,T;\mathbb{R}^m)$:

$$\begin{aligned} J(u,v) &= ((I+B'KB)u,u)-((I-G'KG)v,v) \\ &+2(B'KGv,u) \\ &+2(B'(L^*QS+\hat{L}^*W\hat{S})x_0,u) \\ &+2(G'(L^*QS+\hat{L}^*W\hat{S})x_0,v) \\ &+ <(S^*QS+\hat{S}^*W\hat{S})x_0,x_0> \end{aligned}$$

where $(\cdot,\cdot)$ and $<\cdot,\cdot>$ denote inner product in L^2 and Euclidean space respectively. Therefore, the game problem is converted into the min-max problem of bilinear forms in Hilbert space.

Remark 2.1 Under coercivity assumptions on $I+B'KB$ and $I-G'KG$, the game problems admits a unique open loop-open loop saddle point which can be constructed explicitly.

In this paper, no such coercivity conditions are imposed, and the usual Riccati equation approach can't be applied.

Remark 2.2 The problem setting of this paper can be easily generated to the time-varying systems, where transition matrix $\Phi(t,s)$ replaces $e^{A(t-s)}$ in the operators defined above.

We will first establish some preliminary results.

Lemma 2.3 Let $\mathcal{N}(A)$ denote the kernel of operator A, let $C,D\in\mathcal{L}(X)$ be linear bounded operators on some Hilbert space X, then, $\mathcal{N}(I+CD)=C\mathcal{N}(I+DC)$.

Proof Let $(I+CD)x=0$, then, $x=-CDx$ and $(I+DC)Dx=D(I+CD)x=0$, that is, $Dx\in\mathcal{N}(I+DC)$, hence $x=-CDx\in C\mathcal{N}(I+DC)$. Conversely, let $x=Cp$ for some $p\in\mathcal{N}(I+DC)$, then, $(I+CD)x=(I+CD)Cp=C(I+DC)p=0$, hence $x\in\mathcal{N}(I+CD)$.

Consider the following Fredholm integral equation with parameter v:

$$(I+KM^+)p=Gv+\hat{K}x_0 \tag{3}$$

Let $V(x_0)=\{v\in\mathcal{V}: v$ is such that (3) has solution. $\}$. If $V(x_0)$ is not empty, then we can pick a $v\in V(x_0)$, and let $P(v,x_0)$ be the solution set of (3) corresponding to v. Choosing one $p\in P(v,x_0)$, then, $P(v,x_0)$ can be expressed as

$$P(v,x_0)=p+N$$

where $N=\mathcal{N}(I+KM^+)$. The following theorem characterizes the set $V(x_0)$.

Theorem 2.4 $V(x_0)$ is given by

$$V(x_0)=\{v\in\mathcal{V}:(v,G'q)+<x_0,q(0)>=0,\forall q\in N\}$$

Proof Let $q\in N$, then, $q=-KM^+q$, and $q(0)=-\hat{K}^*M^+q$ by simple verification. Therefore, by virtue of Lemma 2.3 and the fact that operator K is compact, hence $I+KM^+$ has closed range.

$$\begin{aligned} & (v,G'q)+<x_0,q(0)>=0,\forall q\in N \\ \Leftrightarrow\ & (v,G'KM^+q)+(\hat{K}x_0,M^+q)=0,\forall q\in N \\ \Leftrightarrow\ & (KGv+\hat{K}x_0,M^+q)=0,\forall q\in N \\ \Leftrightarrow\ & KGv+\hat{K}x_0\perp\mathcal{N}(I+M^+K) \\ \Leftrightarrow\ & KGv+\hat{K}x_0\in\mathcal{R}(I+KM^+) \end{aligned}$$

Corollary 2.5 The following statements hold true:

1) $V(0)=(G'N)^\perp$

2) if $V(x_0)$ is not empty, then $V(x_0)=v+V(0)$ for arbitrary $v\in V(x_0)$.

Theorem 2.6 Suppose $V(x_0)$ is not empty, then, for every $v\in V(x_0)$, $G'P(v,x_0)\cap V(x_0)$ contains exactly one element.

Proof Fix $v\in V(x_0)$. Then $G'P(v,x_0)=G'p+G'N$ for some $p\in N$. Let $\mathbf{P}_{G'N}$ and $\mathbf{P}_{(G'N)^\perp}$ be projections onto $G'N$ and $(G'N)^\perp$ respectively. Decompose $v=v_1+v_2$ with $v_1=\mathbf{P}_{G'N}v, v_2=\mathbf{P}_{(G'N)^\perp}v$, and $G'p=q_1+q_2$ with $q_1=\mathbf{P}_{G'N}G'p, q_2=\mathbf{P}_{(G'N)^\perp}G'p$. Then, since $q_2\in(G'N)^\perp$, $v_1+q_2\in V(x_0)$ by Corollary 2.5. Moreover, since $q_2=G'p-q_1\in G'p+G'N$ and $v_1\in G'N$, $v_1+q_2\in G'p+G'N=G'P(v,x_0)$. Thus, we have verified that

$$\mathbf{P}_{G'N}v+\mathbf{P}_{(G'N)^\perp}G'p=v_1+q_2\in G'P(v,x_0)\cap V(x_0)$$

To prove the uniqueness, let $u_1,u_2\in G'P(v,x_0)\cap V(x_0)$, then, $u_1-u_2\in G'N$, on the other hand, $u_1-u_2\in V(0)=(G'N)^\perp$, hence $u_1=u_2$.

Now, denote the only element in $G'P(v,x_0)\cap V(x_0)$ by $G'p_*$, although p_* might not be unique in $P(v,x_0)$, $G'p_*$ is uniquely determined. Therefore, we are able to define operator

$$D^{x_0}:V(x_0)\to V(x_0),\qquad D^{x_0}v=G'p_*$$

especially, we can define

$$D:V(0)\to V(0),\qquad Dv=G'p_*^0$$

To character these operators, let $(I+KM^+)_{N^\perp}$ be the restriction of $I+KM^+$ on $N^\perp$. Then, $(I+KM^+)_{N^\perp}$ is a bijection from $N^\perp$ to $\mathcal{R}(I+KM^+)$, and hence has bounded inverse. Now choose

$$p_*=((I+KM^+)_{N^\perp})^{-1}(KGv+\hat{K}x_0)$$

Then,

$$D^{x_0}v=G'((I+KM^+)_{N^\perp})^{-1}(KGv+\hat{K}x_0) \tag{4}$$

and

$$Dv=G'((I+KM^+)_{N^\perp})^{-1}KGv \tag{5}$$

Theorem 2.7 D is a compact, self-adjoint operator on $V(0)$.

Proof Since K is compact, so is D. To show D is self-adjoint, let $v_i \in V(0), p_i \in P(v_i, 0), i = 1, 2$, then

$$\begin{aligned} & (Dv_2, v_1) - (Dv_1, v_2) \\ = \; & (G'p_2, v_1) - (G'p_1, v_2) \\ = \; & (Gv_1, p_2) - (Gv_2, p_1) \\ = \; & (Gv_1, KGv_2 - KM^+p_2) \\ & -(Gv_2, KGv_1 - KM^+p_1) \\ = \; & -(KGv_1, M^+p_2) + (KGv_2, M^+p_1) \\ = \; & -((I + KM^+)p_1, M^+p_2) \\ & +((I + KM^+)p_2, M^+p_1) \\ = \; & 0 \end{aligned}$$

This completes the proof.

III. A GAME WHERE VALUE DOES NOT EXIST

Consider the following dynamics:

$$\dot{x} = x + u + v, \quad x(0) = 0$$

with index

$$J(u, v) = \int_0^1 (u^2 - v^2 + 2x^2) dt$$

We claim that the lower value of the game is 0 whereas the upper value does not exist. To prove $v^-(0) = 0$ is easy, for each control action v, $J(-v, v) = 0$, thus, $\inf J(u, v) \leq 0$, hence $v^-(0) \leq 0$. On the other hand, $v^-(0) \geq \inf J(u, 0) = 0$. Now let's deal with the hard part, we need to show $v^+(0) = \infty$. Fix control action u, we write the index as

$$\begin{aligned} J(u, v) \; &= -((I - 2L^*L)v, v) \\ & +4(L^*Lu, v) + ((I + 2L^*L)u, u) \end{aligned}$$

where L is the integral operator defined in section 2. Since the underlying Riccati differential equation blows up in interval [0, 1], by virtue of results in [4], [6], $I - 2L^*L$ is not positive definite. i.e., there exists a non zero $\bar{v}$ such that $((I - 2L^*L)\bar{v}, \bar{v}) \leq 0$. In fact, strict inequality holds. To see this, take $\bar{v}(t) = 1$ on [0, 1], we calculate $((I - 2L^*L)\bar{v}, \bar{v}) \approx 1 - 2 * 0.757762 < 0$, thus, $\sup\{J(u, v); v \in L^2(0, 1)\} = \infty$ for every u given, which shows that upper value of the game does not exist.

Remark 3.1 It is easy to construct similar examples where upper value exist but lower value does not exist and examples where neither lower value nor upper value exists.

Remark 3.2 The example game admits open loop - closed loop saddle point as well as closed loop - closed loop saddle point because both Riccati differential equation and lower Riccati differential equation admits solution on [0, 1] ([2]).

IV. MAIN RESULT AND ITS PROOF

The main results in this paper are contained in

Theorem 4.1 Consider the game problem (1)-(2). The following statement are equivalent.

1) There exists an open loop-open loop saddle point $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$ such that

$$J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*), \quad \forall u \in \mathcal{U}, v \in \mathcal{V}$$

2) The value of the game exists.

3) Both lower value and upper value exist.

To prove the theorem, some lemmas are in need. Using notations of section 2, we rewrite index as

$$J(u, v) = ((I + E)u, u) + 2(f(v, x_0), u) + J(0, v)$$

where

$$E = B'L^*QLB + B'\hat{L}^*W\hat{L}B = B'KB$$

and

$$f(v, x_0) = B'KGv + B'\hat{K}x_0$$

Obviously, lower value exists if and only if both the following conditions hold:

(**A**) There exists $v \in \mathcal{V}$ such that

$$\hat{J}(v) = \inf_u J(u, v) > -\infty \tag{6}$$

(**B**) Let $\hat{V}(x_0) = \{v \in \mathcal{V} : v \text{ is such that (6) holds true }\}$, then

$$\sup\{\hat{J}(v) : v \in \hat{V}(x_0)\} < \infty \tag{7}$$

By standard extremal theory (see, e.g., [5]), (6) holds true if and only if $I + E \geq 0$ and

$$f(v, x_0) \in \mathcal{N}(I + E)^\perp = \mathcal{R}(I + E) \tag{8}$$

Moreover, u is an optimizer if and only if

$$f(v, x_0) + (I + E)u = 0 \tag{9}$$

Our first observation is

Lemma 4.2 Suppose lower value exists, then $\hat{V}(x_0)$ coincides with $V(x_0)$.

Proof Let $v \in \hat{V}(x_0)$, then, by definition and (8)

$$(f(v, x_0), u) = (B'KGv + B'\hat{K}x_0, u) = 0, \forall u \in \mathcal{N}(I + E) \tag{10}$$

Notice $B\mathcal{N}(I + E) = \mathcal{N}(I + M^+K)$ by Lemma 2.3, (10) is equivalent to

$$KGv + \hat{K}x_0 \in (\mathcal{N}(I + M^+K))^\perp = \mathcal{R}(I + KM^+) \tag{11}$$

hence $v \in V(x_0)$. Since the above procedure is invertible, the opposite inclusion also holds true.

Lemma 4.3 Suppose lower value exists, then for $v \in V(x_0)$,

$$\hat{J}(v) = -(v, v) + (Gv, p) + \langle x_0, p(0) \rangle \tag{12}$$

where $p \in P(v, x_0)$ and $\hat{J}(v)$ is independent of the choice of p.

Proof By (9), the optimizer u can be expressed as

$$u = -B'(KBu + KGv + \hat{K}x_0) \tag{13}$$

Let $p = KBu + KGv + \hat{K}x_0$, then $u = -B'p$ and $p = KB(-B'p) + KGv + \hat{K}x_0$, thus, $p \in P(v, x_0)$, conversely, let $p \in P(v, x_0)$, $u = -B'p$ must be the optimizer for $J(u, v)$. Therefore,

$$\hat{J}(v) = ((I+E)u,u) + 2(f(v,x_0),u) + J(0,v)$$
$$= (f(v,x_0),u) + J(0,v)$$
$$= (B'KGv + B'\hat{K}x_0, -B'p) + J(0,v)$$
$$= -(Q(LGv+Sx_0), LM^+p)$$
$$- < W(\hat{L}Gv+\hat{S}x_0), \hat{L}M^+p >$$
$$-\|v\|^2 + (Q(LGv+Sx_0), LGv+Sx_0)$$
$$+ < W(\hat{L}Gv+\hat{S}x_0), \hat{L}Gv+\hat{S}x_0 >$$
$$= -\|v\|^2 + (Gv, L^*Qx + \hat{L}^*Wx(T))$$
$$+ < x(0), S^*Qx + \hat{S}^*Wx(T) >$$
$$= -\|v\|^2 + (Gv,p) + < x_0, p(0) >$$

and $\hat{J}(v)$ does not depend on the choices of p.

Lemma 4.4 Suppose lower value exists, then, it equals $< x_0, p(0) >$ *and is independent of choice of the solution* p *to the following Fredholm equation*:

$$(I+KM)p = \hat{K}x_0 \tag{14}$$

Proof We want to calculate $\sup\{\hat{J}(v); v \in V(x_0)\}$. Fix $\bar{v} \in V(x_0)$ and $\bar{p} \in P(\bar{v}, x_0)$, then $v \in V(x_0)$ can be written as $v = \bar{v} + v^0$ and $p \in P(v,x_0)$ can be written as $p = \bar{p} + p^0$ for some $v^0 \in V(0)$ and $p^0 \in P(v^0, 0)$. Similar to Theorem 2.5, it can be shown that

$$-(G\bar{v}, p^0) + (Gv^0, \bar{p}) = < x_0, p^0(0) >$$

Then,

$$\hat{J}(v) = -\|\bar{v}+v^0\|^2 + (G(\bar{v}+v^0), \bar{p}+p^0)$$
$$+ < x_0, \bar{p}(0) + p^0(0) >$$
$$= -\|\bar{v}\|^2 + (G\bar{v},\bar{p}) + < x_0, \bar{p}(0) >$$
$$-\|v^0\|^2 + (G\bar{v}, p^0) + < x_0, p^0(0) >$$
$$+(Gv^0,\bar{p}) + (Gv^0, p^0) - 2(\bar{v}, v^0)$$
$$= \hat{J}(\bar{v}) - \|v^0\|^2 + (v^0, G'p^0) - 2(v^0, \bar{v} - G'\bar{p})$$

Hence,

$$\sup_{v\in V(x_0)} \hat{J}(v)$$
$$= \hat{J}(\bar{v}) + \sup_{v^0\in V(0)} (-((I-D)v^0, v^0) - 2(v^0, \bar{v} - D^{x_0}\bar{v}))$$

Again, by standard extremal theory, $\sup_{v\in V(x_0)} \hat{J}(v)$ is finite if and only if

$$I - D \geq 0$$

and

$$\bar{v} - D^{x_0}\bar{v} \in \mathcal{N}(I-D)^\perp = \mathcal{R}(I-D)$$

Moreover, if $v^0 \in V(0)$ is such that

$$\bar{v} - D^{x_0}\bar{v} + (I-D)v^0 = 0$$

Then, $v^* = \bar{v} + v^0$ is the optimizer for $\hat{J}(v)$. Recalling the definitions (4)-(5), we have

$$v^* = D^{x_0}\bar{v} + Dv^0 = D^{x_0}(\bar{v}+v^0) = D^{x_0}v^*$$

Now let $v = v^*$, p be any $p^* \in P(v^*, x_0)$ in (12), then

$$\sup_{v\in V(x_0)} \hat{J}(v) = \hat{J}(v^*)$$
$$= -\|v^*\|^2 + (v^*, G'p^*) + < x_0, p^*(0) > = < x_0, p^*(0) >$$

and is independent of the choice of p^*.

Finally, we verify that p^* satisfies the Fredholm equation (14). In fact, since $p^* \in P(v^*, x_0)$, $(I + KM^+)p^* = KGv^* + \hat{K}x_0$, moreover, $v^* = D^{x_0}v^* = G'p^*$, hence, $(I+KM)p^* = \hat{K}x_0$.

Lemma 4.5 Suppose the upper value exists, then, it equals $< x_0, p(0) >$ *and is independent of choice of the solution* p *to the Following Fredholm equation (14).*

Proof Notice that

$$\inf_u \sup_v J(u,v) = -\sup_v \inf_u (-J(u,v))$$

Hence, if the upper value exists, then, by Lemma 4.4,

$$\sup_v \inf_u (-J(u,v)) = < x_0, q(0) > \tag{15}$$

where q satisfies

$$(I+KM)q = -\hat{K}x_0$$

and (15) is independent of choice of q, or equivalently, the upper value equals $< x_0, p(0) >$ and is independent of choice of the solution p to the Following Fredholm equation (14).

Proof of Theorem 4.1 1) $\Rightarrow$ 2) and 2) $\Rightarrow$ are obvious. We need only prove 3) $\Rightarrow$ 1).

By virtue of Lemma 4.4, if lower value exists, then, there exists $v^* = G'p^* \in V(x_0)$ where p^* satisfies (14). Let u^* be the optimizer of $J(u, v^*)$, then, $u^* = -B'p^*$, and

$$J(u^*, v^*) \leq J(u, v^*)$$

Similarly, if upper value exists, the same (u^*, v^*) verifies

$$-J(u^*, v^*) \leq -J(u^*, v)$$

Thus, (u^*, v^*) constitutes the desired open loop-open loop saddle point.

V. CONCLUDING REMARKS

This paper studies the two-person, zero-sum linear quadratic differential games on finite horizon. Some necessary and sufficient conditions for the existence of the value of the game are derived. Although we obtain the open loop-open loop saddle points whenever the value of the game exists, nothing is said about their synthesis as state feedback. In subsequent papers, we shall further investigate relationship among open loop saddle points, closed loop saddle points, value of the game and the Riccati differential equations. Another future research work would be extensions to infinite horizontal differential game problems.

REFERENCES

[1] T. Basar & P. Bernhard, H^∞-optimal control and related minmax design problems: A dynamical game approach, Birkhaüser, Boston, 1995.

[2] P. Bernhard, Linear-quadratic, two-person, zero-sum differential game: necessary and sufficient conditions, Journal of Optimization Theory and Appliations, Vol. 27 (1979), 51-69.

[3] K. Glover, D. J. N. Limebeer, J. C. Doyle, E. M. Kasenally & M. G. Safonov, A characterization of all solutions to the four block general distance problems, SIAM J. Control Optim. Vol. 29 (1991), 283-324.

[4] D. Hinrichsen and A. J. Prichard, The Riccati equation, Lecture Notes (1991).

[5] A. D. Ioffe and V. M. Tikhomirov, Theory of Extremal Problems, Nauka, Moscow 1974, (in Russian), English transl. North Holland 1979.
[6] B. Jacob, Linear quadratic optimal control of time-varying systems with indefinite cost on Hilbert spaces:the finite horizon problem, Journal of Mathematical Systems, Estimation and Control, Vol. 8 (1995), no. 3, 261-288.
[7] K. Zhou & P. Khargonekar, An algebraic Riccati equation approach to H^∞ optimization, System & Control Letter, Vol.11 (1988), 85-91.