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Sensitivity Analysis of Uncertainties in Dynamic Noncooperative Games Using a Multi-Modeling Method

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Abstract— **In this paper, we formulate a two-person nonzero-sum game with unknown parameters by multiple models and carry out a sensitivity analysis of performance indices with respect to the unknowns. We show that from each player's point of view, the deviation from the indices is bounded from below and above if he/she adopts his/her estimates of the uncertain parameters while assuming the competitors use unknown values of the parameters. Motivated by the model for strategic bidding in a competitive electricity market, performance indices with linear and cross terms in control have been constructed and a simple numerical example is given for illustration purpose.**

I. INTRODUCTION

Game theory is a theory of *rational behavior* in situations where each decision maker (DM) has to make his decision based on what he/she thinks the other DMs' reactions are likely to be. The central theme of game theory is a conflict situation – a collision of interests. Since the 1950s, game theory has been deeply investigated and successfully applied to many areas. However, it is not only the applications in these fields that are important; equally important is the development of suitable concepts to describe and understand conflict situations [1]. As a result, the role of information is very crucial in such problems. As pointed out by Harsanyi [2] that, by suitable modeling, all forms of incomplete information can be reduced to the case in which the DMs have less than full information about each other's objective functionals.

This paper is motivated by the strategic bidding problem in competitive electricity markets in which each power supplier's objective is to maximize benefit by constructing its bidding production/price, given its own production costs and constraints and its anticipation of rival and market behavior. As game theory is the natural framework for investigating the decision-making problem where decision makers have conflicting interests, which is exactly the case of deregulated electricity market, it is not surprising to see a considerable literature addressing the strategic bidding problem using game theory [3], [4], [5], [6], most of them assuming complete information about the competitors' objective functionals. However, in practice the objective functionals of other companies depend on their own power production cost functions which are usually not publicized. Thus it makes little sense to assume that the system parameters are known by all decision makers. Among all situations of games with unknown parameters, there is a case in which the state evolution and the mathematical structure of the performance indices are known by all decision makers, yet the exact values of the indices depend on certain unknown parameters, say fuel price in the power case. Furthermore, it is common in practice that different decision makers perceive different values of the same unknown parameters based on his/her available information and understanding of the game, resulting in his/her state evolution and performance indices, which in turn leads to his/her control strategies and objective functionals calculation. In this sense, the problem should be characterized by multiple decision makers having different models [7].[1]

The paper is organized as follows: in Section II, we study a game with perfect state information structure and formulate the effect of uncertainties using multiple models. Then we show in Section III that the deviation of performance indices from the nominal case is bounded from below and above. A numerical example is given in Section IV and conclusions are drawn in Section V.

II. DYNAMIC NONCOOPERATIVE FINITE GAMES WITH UNCERTAINTIES

A. Control Strategies for a Game with Complete Information

For convenience in analysis, we consider a two-person nonzero-sum game with complete and perfect information structure, which is described by the state equation:

$$x_{k+1} = Ax_k + B^1 u_k^1 + B^2 u_k^2 \tag{1}$$

and the performance index (or objective functional) for the j^{th} DM ($j = 1, 2$):

$$J_k^j = \frac{1}{2} x_N' Q_N^j x_N + \frac{1}{2} \sum_{i=k}^{N-1} (x_i' Q^j x_i + 2x_i' D^{j\prime} u_i^j + u_i^{j\prime} R^j u_i^j + 2G^{j\prime} u_i^j) \tag{2}$$

where the state x_k and its succeeding state x_{k+1} are assumed to belong to some state space $S \subset \Re^n$; the control variables u_k^1 and u_k^2 are chosen by the decision maker 1 (DM1) and decision maker 2 (DM2) in some constraint set U_k^1 and U_k^2, which are in turn subsets of some control space $C \subset \Re^m$; the performance indices J_k^1 and J_k^2 are the total costs incurred starting at the k^{th} stage (cost-to-go starting at stage k) by the DM1 and DM2, respectively. For the j^{th} DM ($j = 1, 2$), $Q_N^j, Q^j, R^j, D^j, G^j$ are matrices of appropriate dimensions, $Q_N^j \geq 0$, $Q^j \geq 0$ and $R^j > 0$ and symmetric.

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Assume both decision makers have full access to the system parameters and state information. Each DM seeks a finite sequence of control (or referred to as a *strategy* or a *policy*) to minimize his/her total cost over N stages.

Lemma 1: For the two-player nonzero-sum game described by the Equations (1) and (2), denote the feedback Nash optimum value of J_k^j by $V_k^j(x_k)$ and denote the optimum feedback Nash strategy for u_k^j by $\gamma_k^j(x_k)$, where $j = 1, 2$ and $k \in \{N-1, \ldots, 0\}$. Let Ψ_{k+1}, K_{k+1}^j be matrices of appropriate dimension, defined by

$$\Psi_{k+1} = \begin{bmatrix} R^1 + B^{1\prime} K_{k+1}^1 B^1 & B^{1\prime} K_{k+1}^1 B^2 \\ B^{2\prime} K_{k+1}^2 B^1 & R^2 + B^{2\prime} K_{k+1}^2 B^2 \end{bmatrix} \quad (3)$$

$$K_k^j = Q^j + 2D^{j\prime} L_k^j + L_k^{j\prime} R^j L_k^j + S_k' K_{k+1}^j S_k \quad (4)$$

where $S_k \triangleq A + B^1 L_k^1 + B^2 L_k^2$.

If the matrices Ψ_{k+1}, thus recursively defined, are invertible, the game admits a state-feedback Nash equilibrium solution in the form of

$$u_k^{j*} := \arg \min_{u_k^j \in U_k^j} J_k^j = \gamma_k^j(x_k) = L_k^j x_k + \Phi_k^j \quad (5)$$

where

$$L_k = \begin{bmatrix} L_k^1 \\ L_k^2 \end{bmatrix} = -\left(\Psi_{k+1}\right)^{-1} F_{k+1} \quad (6)$$

$$\Phi_k = \begin{bmatrix} \Phi_k^1 \\ \Phi_k^2 \end{bmatrix} = -\left(\Psi_{k+1}\right)^{-1} G \quad (7)$$

in which

$$F_{k+1} = \begin{bmatrix} B^{1\prime} K_{k+1}^1 A + D^1 \\ B^{2\prime} K_{k+1}^2 A + D^2 \end{bmatrix} \quad (8)$$

$$G = \begin{bmatrix} G^1 \\ G^2 \end{bmatrix} \quad (9)$$

Furthermore, the optimum value of the cost function is given by

$$V_k^j(x_k) = \frac{1}{2}\left(x_k' K_k^j x_k + P_k^j x_k + M_k^j\right) \quad (10)$$

where by defining $Z_k \triangleq B^1 \Psi_k^1 + B^2 \Psi_k^2$, P_k^j, M_k^j are recursively generated by

$$\begin{aligned} P_k^j &= 2\Psi_k^{j\prime} D^j + 2G^{j\prime} L_k^j + 2\Psi_k^{j\prime} R^j L_k^j \\ &\quad + 2Z_k' K_{k+1}^j S_k + 2P_{k+1}^j S_k \quad (11) \\ M_k^j &= 2G^{j\prime} \Psi_k^j + \Psi_k^{j\prime} R^j \Psi_k^j + Z_k' K_{k+1}^j Z_k \\ &\quad + P_{k+1}^j Z_k + M_{k+1}^j \quad (12) \end{aligned}$$

with the boundary conditions

$$K_N^j = Q_N^j, \quad P_N^j = 0, \quad M_N^j = 0 \quad (13)$$

Remark 1: The lemma can be obtained by applying a standard dynamic programming algorithm [8]. Here we extend the standard linear quadratic game to a more general one, where the performance indices contain linear and cross terms in control u^j. Special cases of Lemma 1 when the performance indices are quadratic but with no linear and cross terms in control can be found in [1].

As in many applications it is unrealistic to assume that the DMs know the parameters embedded in competitors' cost functions, we will next investigate the case when the cost functions depend on some uncertain parameters.

B. Multi-modeling Formulation with Unknown Parameters in the Performance Indices

In the previous subsection, we formulated a two-person nonzero-sum game with perfect information, which is referred to as the *nominal case* hereafter. However, the assumption in the nominal case that both decision makers have full access to the system parameters may not be valid in many practical cases. For example, in the power electricity auction case, where the objective functionals of the other companies depend on their own power production cost functions, which are usually not publicized, it makes no sense to assume that the system parameters are known by all decision makers. Among all situations of games with unknown parameters, there is a case in which the state evolution and the mathematical structure of the performance indices are known by all decision makers, yet the exact values of the indices depend on certain unknown parameters. We will consider this case in the following subsections, formulating it by multiple models and carrying out a sensitivity analysis of performance indices with respect to the unknowns.

Suppose that there is a parameter $\varepsilon \in \Re$ on which the performance indices depend, such that the game in normal form is described by the cost functional pair $\left[J^1(u^1, u^2, \varepsilon), J^2(u^1, u^2, \varepsilon)\right]$. Consequently, the Nash state feedback control for the j^{th} DM ($j = 1, 2$), is also a function of ε:

$$\gamma_k^j(x_k, \varepsilon) = L_k^j(\varepsilon) x_k + \Phi_k^j(\varepsilon) \quad (14)$$

and the optimum cost function is then in the form of

$$V_k^j(x_k, \varepsilon) = \frac{1}{2}\left(x_k' K_k^j(\varepsilon) x_k + P_k^j(\varepsilon) x_k + M_k^j(\varepsilon)\right) \quad (15)$$

To study the impact of the unknown parameter ε on the control strategies and the values of performance indices, let the performance indices be analytic functions of ε which is assumed to be bounded $|\varepsilon| \le \Delta^0$, where $\Delta^0 > 0$ is given. Let the feedback gains and constants be expanded in Taylor power series about $\varepsilon = \varepsilon_0$, where ε_0 stands for the nominal value of ε; i.e.,

$$L_k^j(\varepsilon) = \sum_{i=0}^{\infty} \frac{(\varepsilon - \varepsilon_0)^i}{i!} \frac{\partial^i L_k^j(\varepsilon)}{\partial \varepsilon^i}\bigg|_{\varepsilon=\varepsilon_0} \quad (16)$$

$$\Phi_k^j(\varepsilon) = \sum_{i=0}^{\infty} \frac{(\varepsilon - \varepsilon_0)^i}{i!} \frac{\partial^i \Phi_k^j(\varepsilon)}{\partial \varepsilon^i}\bigg|_{\varepsilon=\varepsilon_0} \quad (17)$$

In practice, it is common that different decision makers formulate different models for a given game. This type of problem is then called a *multi-modeling* problem [7], and should be analyzed from each decision maker's point of view.

Assume that DM1's estimate of ε is ε^1, $|\varepsilon^1| \le \Delta^0$, such that $\exists \delta^1 \in \Delta^1 \subset \Re$ and $\varepsilon = \varepsilon^1 + \delta^1$, where $\Delta^1 = [-\Delta^0 - \varepsilon^1, \Delta^0 - \varepsilon^1]$. The game perceived by DM1 is then described by $G^A = \left[J^{1A}(u^1, u^2, \varepsilon^1 + \delta^1), J^{2A}(u^1, u^2, \varepsilon^1 + \delta^1)\right]$.

Similarly, assume that ε^2, $|\varepsilon^2| \leq \Delta^0$, is the estimate of DM2 and $\exists \delta^2 \in \Delta^2 \subset \Re$, where $\Delta^2 = [-\Delta^0 - \varepsilon^2, \Delta^0 - \varepsilon^2]$, such that $\varepsilon = \varepsilon^2 + \delta^2$ and the game perceived by DM2 is described by $G^B = \left[J^{1B}(u^1, u^2, \varepsilon^2 + \delta^2), J^{2B}(u^1, u^2, \varepsilon^2 + \delta^2)\right]$. In the sequel, we analyze the problem only from the DM1's perspective as the analysis carried out by DM2 will be similar.

Since each element in the matrices is analytic and the corresponding i^{th} order derivatives are continuous at ε^1, and DM1 expands the feedback gains and constant terms about $\varepsilon = \varepsilon^1$ for both DMs $j = 1, 2$:

$$L_k^{jA}(\varepsilon) = \sum_{i=0}^{\infty} \frac{(\varepsilon - \varepsilon^1)^i}{i!} \frac{\partial^i L_k^{jA}(\varepsilon)}{\partial \varepsilon^i}\bigg|_{\varepsilon=\varepsilon^1} = \bar{L}_k^{jA}(\delta^1) \quad (18)$$

$$\Phi_k^{jA}(\varepsilon) = \sum_{i=0}^{\infty} \frac{(\varepsilon - \varepsilon^1)^i}{i!} \frac{\partial^i \Phi_k^{jA}(\varepsilon)}{\partial \varepsilon^i}\bigg|_{\varepsilon=\varepsilon^1} = \bar{\Phi}_k^{jA}(\delta^1) \quad (19)$$

Similarly, expanding all variables in terms of ε about $\varepsilon = \varepsilon^1$, we obtain:

$$\begin{array}{ll} K_N^{jA}(\varepsilon) = \bar{Q}_N^{jA}(\delta^1) & K_k^{jA}(\varepsilon) = \bar{K}_k^{jA}(\delta^1) \\ P_k^{jA}(\varepsilon) = \bar{P}_k^{jA}(\delta^1) & M_k^{jA}(\varepsilon) = \bar{M}_k^{jA}(\delta^1) \\ S_k^{jA}(\varepsilon) = \bar{S}_k^{jA}(\delta^1) & Z_k^{jA}(\varepsilon) = \bar{Z}_k^{jA}(\delta^1) \end{array}$$

such that for $s \leq i, t \leq i - s$, we have following general expressions for the i^{th} derivatives:

$$\begin{aligned} \bar{K}_k^{jA(i)}(\delta^1) &= \bar{Q}^{jA(i)} + 2\sum_{s=0}^{i} \binom{i}{s} \left(\bar{D}^{jA(s)}\right)' \bar{L}_k^{jA(i-s)} \\ &+ \sum_{s=0}^{i}\sum_{t=0}^{i-s} \binom{i}{s}\binom{i-s}{t} \left(\bar{L}_k^{jA(s)}\right)' \bar{R}^{jA(t)} \bar{L}_k^{jA(i-s-t)} \\ &+ \sum_{s=0}^{i}\sum_{t=0}^{i-s} \binom{i}{s}\binom{i-s}{t} \left(\bar{S}_k^{A(s)}\right)' \bar{K}_{k+1}^{jA(t)} \bar{S}_k^{A(i-s-t)} \quad (20) \end{aligned}$$

$$\begin{aligned} \bar{P}_k^{jA(i)}(\delta^1) &= 2\sum_{s=0}^{i} \binom{i}{s} \left(\bar{P}_{k+1}^{jA(s)}\right)' \bar{S}_k^{jA(i-s)} \\ &+ \sum_{s=0}^{i}\sum_{t=0}^{i-s} \binom{i}{s}\binom{i-s}{t} \left(\bar{Z}_k^{A(s)}\right)' \bar{K}_{k+1}^{jA(t)} \bar{S}_k^{A(i-s-t)} \\ &+ \sum_{s=0}^{i}\sum_{t=0}^{i-s} \binom{i}{s}\binom{i-s}{t} \left(\bar{\Phi}_k^{jA(s)}\right)' \bar{R}^{jA(t)} \bar{L}_k^{jA(i-s-t)} \\ &+ 2\sum_{s=0}^{i} \binom{i}{s} \left(\bar{\Phi}_k^{jA(s)}\right)' \bar{D}^{jA(i-s)} \\ &+ 2\sum_{s=0}^{i} \binom{i}{s} \left(\bar{G}^{jA(s)}\right)' \bar{L}_k^{jA(i-s)} \quad (21) \end{aligned}$$

$$\begin{aligned} \bar{M}_k^{jA(i)}(\delta^1) &= \bar{M}_{k+1}^{jA(i)} + \sum_{s=0}^{i} \binom{i}{s} \left(\bar{P}_{k+1}^{jA(s)}\right)' \bar{Z}_k^{jA(i-s)} \\ &+ \sum_{s=0}^{i}\sum_{t=0}^{i-s} \binom{i}{s}\binom{i-s}{t} \left(\bar{\Phi}_k^{jA(s)}\right)' \bar{R}^{jA(t)} \bar{\Phi}_k^{jA(i-s-t)} \\ &+ \sum_{s=0}^{i}\sum_{t=0}^{i-s} \binom{i}{s}\binom{i-s}{t} \left(\bar{Z}_k^{A(s)}\right)' \bar{K}_{k+1}^{jA(t)} \bar{Z}_k^{A(i-s-t)} \\ &+ 2\sum_{s=0}^{i} \binom{i}{s} \left(\bar{G}^{jA(s)}\right)' \bar{\Phi}_k^{jA(i-s)} \quad (22) \end{aligned}$$

Also denote the control gain matrices and constants by:

$$\begin{aligned} \bar{L}_k^A(\delta^1) &= -\left(\bar{\Psi}_{k+1}^A\right)^{-1} \bar{F}_{k+1}^A \\ \bar{\Phi}_k^A(\delta^1) &= -\left(\bar{\Psi}_{k+1}^A\right)^{-1} \bar{G}^A \end{aligned}$$

we obtain the $i^{th}, i \geq 1$ derivatives as follows:

$$\begin{aligned} \bar{L}_k^{A(i)}(\delta^1) &= -\left(\bar{\Psi}_{k+1}^A\right)^{-1} \left(\bar{\Psi}_{k+1}^{A(i)} \bar{L}_k^A + \bar{F}_{k+1}^{A(i)}\right) \\ &\quad - \left(\bar{\Psi}_{k+1}^A\right)^{-1} \sum_{s=1}^{i-1} \binom{i}{s} \bar{\Psi}_{k+1}^{Aj} \bar{L}_k^{A(i-s)} \quad (23) \end{aligned}$$

$$\begin{aligned} \bar{\Phi}_k^A(\delta^1) &= -\left(\bar{\Psi}_{k+1}^A\right)^{-1} \left(\bar{\Psi}_{k+1}^{A(i)} \bar{L}_k^A + \bar{G}^{A(i)}\right) \\ &\quad - \left(\bar{\Psi}_{k+1}^A\right)^{-1} \sum_{s=1}^{i-1} \binom{i}{s} \bar{\Psi}_{k+1}^{Aj} \bar{\Phi}_k^{A(i-s)} \quad (24) \end{aligned}$$

Lemma 2: Consider a two-person nonzero-sum game described by the state equation Eq. (1) and the cost functions:

$$\begin{aligned} J_k^{jA}(\varepsilon) &= \frac{1}{2} x_N' Q_N^j(\varepsilon) x_N + \frac{1}{2} \sum_{i=k}^{N-1} [x_i' Q^j(\varepsilon) x_i \\ &\quad + 2x_i' D^{j\prime}(\varepsilon) u_i^j + u_i^{j\prime} R^j(\varepsilon) u_i^j + 2G^{j\prime}(\varepsilon) u_i^j] \quad (25) \end{aligned}$$

where $j = 1, 2$ and ε is an unknown parameter to both DMs. Let $\delta^1 = \varepsilon - \varepsilon^1$, where ε^1 is the estimate of ε by DM1. Let $\bar{\Psi}_{k+1}^A(\delta^1)$ be a matrix of appropriate dimension, defined by

$$\bar{\Psi}_{k+1}^A(\delta^1) = \begin{bmatrix} \bar{R}^1 + B^{1\prime} \bar{K}_{k+1}^1 B^1 & B^{1\prime} \bar{K}_{k+1}^1 B^2 \\ B^{2\prime} \bar{K}_{k+1}^2 B^1 & \bar{R}^2 + B^{2\prime} \bar{K}_{k+1}^2 B^2 \end{bmatrix} \quad (26)$$

If the matrices $\bar{\Psi}_{k+1}^A$, thus recursively defined, are invertible, the game $G^A = [J^{1A}(u^1, u^2, \varepsilon^1 + \delta^1), J^{2A}(u^1, u^2, \varepsilon^1 + \delta^1)]$ admits a state feedback Nash equilibrium solution, γ_k^{jA}, $(j = 1, 2)$, given by

$$\begin{aligned} \gamma_k^{jA}(x_k, \delta^1) &= \arg \min_{u_k^{jA} \in U_k^j} J_k^{2A}(u_k^{1A}, u_k^{2A}, \varepsilon^1 + \delta^1) \\ &= \bar{L}_k^{jA}(\delta^1) x_k + \bar{\Phi}_k^{jA}(\delta^1) \quad (27) \end{aligned}$$

and the optimum Nash cost function for the j^{th} DM is:

$$\bar{V}_k^{jA}(x_k, \delta^1) = \frac{1}{2} \left(x_k' \bar{K}_k^{jA}(\delta^1) x_k + \bar{P}_k^{jA}(\delta^1) x_k + \bar{M}_k^{jA}(\delta^1)\right)$$

where the matrices involved are defined in Eq. (18) through Eq. (24).

Proof: The proof of the lemma is straightforward by replacing the recursively defined matrices in Eq. (4), Eq. (11) and Eq. (12) by their counterparts in terms of δ^1. ■

III. SENSITIVITY ANALYSIS OF UNCERTAIN PARAMETER

In practice where ε is not available, each DM may estimate the value of ε through his/her knowledge and understanding of the situation and replace ε with its estimate as if it were the true value of ε. For instance, if ε^1 were the estimate of ε by DM1, then DM1 will implement ε^1 through his/her analysis of the game, i.e., DM1 evaluates $\varepsilon = \varepsilon^1$. Meanwhile, DM1 realizes that his/her estimate may be different from DM2's estimate ε^2, which is unknown to DM1. In this case, DM1 assumes that DM2 adopts a different value from ε^1, which is denoted by ε, such that the feedback controls evaluated by DM1 are $\left(\gamma_k^{1A}(x_k, \varepsilon^1 - \varepsilon^1), \gamma_k^{2A}(x_k, \varepsilon - \varepsilon^1)\right) = \left(\gamma_k^{1A}(x_k, 0), \gamma_k^{2A}(x_k, \delta^1)\right)$, where DM1 leaves δ^1 as a variable for DM2. Due to the fact that ε is unavailable, δ^1 is also unavailable to DM1. We are interested in knowing the deviation of $\bar{J}_k^{1A}(x_k, \delta^1)$, the cost-to-go of DM1 should Eq. (25) be evaluated at ε^1 and the control strategies $\left(\gamma_k^{1A}(x_k, 0), \gamma_k^{2A}(x_k, \delta^1)\right)$ be implemented by DM1, from the optimum Nash cost-to-go $\bar{V}_k^{1A}(x_k, \delta^1)$, where DM1 assumes both DMs use ε^1. Similar analysis is carried out from DM2's point of view.

Through this sensitivity analysis of the uncertain parameter in the performance indices, we obtain both upper and lower bounds of the difference between the optimum value and actual value of performance indices, which are summarized in the following theorem.

Theorem 1: Consider a two-person nonzero-sum game described by the state equation Eq. (1) and the cost functions:

$$J_k^j(\varepsilon) = \frac{1}{2} x'_N Q_N^j(\varepsilon) x_N + \frac{1}{2} \sum_{i=k}^{N-1} [x'_i Q^j(\varepsilon) x_i + 2x'_i D^{j\prime}(\varepsilon) u_i^j + u_i^{j\prime} R^j(\varepsilon) u_i^j + 2G^{j\prime}(\varepsilon) u_i^j] \tag{28}$$

where the performance indices depend on an unknown parameter ε, $|\varepsilon| \leq \Delta^0$, $\Delta^0 > 0$. If DMj $(j = 1, 2)$, applies a control strategy using his/her own estimation ε^j and $\delta^j = \varepsilon - \varepsilon^j \in \Delta^j \subset \Re$, where Δ^j is compact, but assumes the other DM adopts a strategy using a different value of ε, then from DMj's point of view, there exist a lower bound $LB_k^j \in \Re$ and a upper bound $UB_k^j \in \Re$, such that the discrepancy between the optimum Nash cost-to-go $\bar{V}_k^j(x_k, \delta^j)$ resulting from single-modeling and the cost-to-go $\bar{J}_k^j(x_k, \delta^j)$ resulting from the multi-modeling is bounded by LB_k^j and UB_k^j at every time step k:

$$LB_k^j \leq \bar{V}_k^j(x_k, \delta^j) - \bar{J}_k^j(x_k, \delta^j) \leq UB_k^j \tag{29}$$

Proof: We will prove the theorem from the perspective of DM1 as the proof from the stand point of DM2 is similar. Since DM1 estimates and implements $\varepsilon = \varepsilon^1$ while conjectures that DM2 adopts an unknown ε, the pair of control strategies perceived by DM1 is $\left(\gamma_k^{1A}(x_k, 0), \gamma_k^{2A}(x_k, \delta^1)\right)$. Applying the controls to the cost functionals, DM1 obtains the following non-optimum cost as two DMs are using different values for ε:

$$\bar{J}_k^{1A}(x_k, \delta^1) = \frac{1}{2}\left(x'_k W_k^{1A}(\delta^1) x_k + T_k^{1A}(\delta^1) x_k + H_k^{1A}(\delta^1)\right) \tag{30}$$

where W_k^{1A}, T_k^{1A} and H_k^{1A} are matrices of appropriate dimensions and are recursively defined by Riccati-like equations. It follows that the discrepancy between the optimum cost defined by Eq. (2) and the non-optimum cost given by Eq. (30) becomes

$$\begin{aligned} &\bar{V}_k^{1A}(x_k, \delta^1) - \bar{J}_k^{1A}(x_k, \delta^1) \\ &= \frac{1}{2}\left(\bar{M}_k^{1A}(\delta^1) - H_k^{1A}(\delta^1)\right) + \frac{1}{2}\left(\bar{P}_k^{1A}(\delta^1) - T_k^{1A}(\delta^1)\right) x_k \\ &+ \frac{1}{2} x'_k \left(\bar{K}_k^{1A}(\delta^1) - W_k^{1A}(\delta^1)\right) x_k \end{aligned} \tag{31}$$

Represent the matrix variables by Taylor polynomial of degree 0 and notice that

$$\bar{M}_k^{1A}(0) = H_k^{1A}(0), \quad \bar{P}_k^{1A}(0) = T_k^{1A}(0), \quad \bar{K}_k^{1A}(0) = W_k^{1A}(0)$$

we obtain

$$\begin{aligned} &\bar{V}_k^{1A}(x_k, \delta^1) - \bar{J}_k^{1A}(x_k, \delta^1) \\ &= \frac{1}{2}[\left(\bar{M}_k^{1A(1)}(c_6) - H_k^{1A(1)}(c_5)\right) \\ &+ \left(\bar{P}_k^{1A(1)}(c_4) - T_k^{1A(1)}(c_3)\right) x_k \\ &+ x'_k \left(\bar{K}_k^{1A(1)}(c_2) - W_k^{1A(1)}(c_1)\right) x_k]\delta^1 \end{aligned} \tag{32}$$

where c_i $(i = 1, \ldots, 6)$ is some point between 0 and δ^1. As we can see from the above equation that the coefficients of δ^1 consist of three components in terms of x_k: a quadratic term, a linear term and a constant. In the sequel, we will find both upper and lower bound for the cost discrepancy through analyzing each component.

We first consider the quadratic term. For any given vector $x_k \neq 0$, normalize it by $||x_k||$ such that

$$\begin{aligned} &x'_k \left(\bar{K}_k^{1A(1)}(c_2) - W_k^{1A(1)}(c_1)\right) x_k \\ &= \left(\frac{x'_k \bar{K}_k^{1A(1)}(c_2) x_k}{x'_k x_k} - \frac{x'_k W_k^{1A(1)}(c_1) x_k}{x'_k x_k}\right) ||x_k||^2 \end{aligned}$$

As $\bar{K}_k^{1A(1)}(c_2)$ and $W_k^{1A(1)}(c_1)$ are both symmetric, we observe the Rayleigh quotients of x_k with following property

$$\lambda_{\min}^{\bar{K}} \leq \frac{x'_k \bar{K}_k^{1A(1)}(c_2) x_k}{x'_k x_k} \leq \lambda_{\max}^{\bar{K}} \tag{33}$$

$$\lambda_{\min}^{W} \leq \frac{x'_k W_k^{1A(1)}(c_1) x_k}{x'_k x_k} \leq \lambda_{\max}^{W} \tag{34}$$

where $\lambda_{\min}^{\bar{K}}, \lambda_{\max}^{\bar{K}}$ are the smallest and largest eigenvalues of $\bar{K}_k^{1A(1)}(c_2)$, respectively, and $\lambda_{\min}^{W}, \lambda_{\max}^{W}$ are the smallest and largest eigenvalues of $W_k^{1A(1)}(c_1)$. It follows that the quadratic term is bounded at every time step k by

$$\begin{aligned} &\left(\lambda_{\min}^{\bar{K}} - \lambda_{\max}^{W}\right) ||x_k||^2 \\ &\leq x'_k \left(\bar{K}_k^{1A(1)}(c_2) - W_k^{1A(1)}(c_1)\right) x_k \\ &\leq \left(\lambda_{\max}^{\bar{K}} - \lambda_{\min}^{W}\right) ||x_k||^2 \end{aligned} \tag{35}$$

However, the eigenvalues still depend on δ^1 as c_1, c_2 depend on δ^1. As we are considering a problem with finite dimension, $\bar{K}_k^{1A(1)}(c_2)$ and $W_k^{1A(1)}(c_1)$ have a finite number of eigenvalues which are continuous with respect to δ^1. By denoting

$$\lambda_{\min}^{\bar{K}*} = \min_{\delta^1} \lambda_{\min}^{\bar{K}} \qquad \lambda_{\max}^{\bar{K}*} = \max_{\delta^1} \lambda_{\max}^{\bar{K}}$$
$$\lambda_{\min}^{W*} = \min_{\delta^1} \lambda_{\min}^{W} \qquad \lambda_{\max}^{W*} = \max_{\delta^1} \lambda_{\max}^{W}$$

and defining

$$LB_{k,1}^1 \triangleq \left(\lambda_{\min}^{\bar{K}*} - \lambda_{\max}^{W*}\right) ||x_k||^2$$
$$UB_{k,1}^1 \triangleq \left(\lambda_{\max}^{\bar{K}*} - \lambda_{\min}^{W*}\right) ||x_k||^2$$

the quadratic term of x_k is then bounded by

$$LB_{k,1}^1 \leq x_k' \left(\bar{K}_k^{1A(1)}(c_2) - W_k^{1A(1)}(c_1)\right) x_k \leq UB_{k,1}^1 \quad (36)$$

Similarly, for any given vector $x_k \neq 0$, normalize it by $||x_k||$ such that the linear term of x_k becomes

$$\begin{aligned}&\left(\bar{P}_k^{1A(1)}(c_4) - T_k^{1A(1)}(c_3)\right) x_k \\ &= \left(\bar{P}_k^{1A(1)}(c_4)\frac{x_k}{||x_k||} - T_k^{1A(1)}(c_3)\frac{x_k}{||x_k||}\right) ||x_k||\end{aligned}$$

where vectors $\bar{P}_k^{1A(1)}$ and $T_k^{1A(1)}$ are both of dimension $1 \times n$ and x_k is of dimension $n \times 1$, and

$$\bar{P}_k^{1A(1)}(c_4)\frac{x_k}{||x_k||} = \sum_{i=1}^{n} \bar{P}_{k,i}^{1A(1)}(c_4)\frac{x_{k,i}}{||x_k||} \quad (37)$$

Denote $\left|\bar{P}_k^{1A(1)}(c_4)\right| = \sum_{i=1}^n \left|\bar{P}_{k,i}^{1A(1)}(c_4)\right|$ and note the fact that $\frac{|x_{k,i}|}{||x_k||} \leq 1$ for $i = 1, \ldots, n$, we reach the following relationship

$$-\left|\bar{P}_k^{1A(1)}(c_4)\right| \leq \bar{P}_k^{1A(1)}(c_4)\frac{x_k}{||x_k||} \leq \left|\bar{P}_k^{1A(1)}(c_4)\right| \quad (38)$$

Similarly

$$-\left|T_k^{1A(1)}(c_4)\right| \leq T_k^{1A(1)}(c_4)\frac{x_k}{||x_k||} \leq \left|T_k^{1A(1)}(c_4)\right| \quad (39)$$

Consider the entire range of δ^1 and denote

$$\left|\bar{P}_k^{1A(1)}\right|^* = \max_{\delta^1} \left|\bar{P}_k^{1A(1)}(c_4)\right|$$
$$\left|T_k^{1A(1)}\right|^* = \max_{\delta^1} \left|T_k^{1A(1)}(c_3)\right|$$

we obtain the bounds for the linear term as

$$LB_{k,2}^1 \leq \left(\bar{P}_k^{1A(1)}(c_4) - T_k^{1A(1)}(c_3)\right) x_k \leq UB_{k,2}^1 \quad (40)$$

where

$$LB_{k,2}^1 \triangleq -\left(\left|\bar{P}_k^{1A(1)}\right|^* + \left|T_k^{1A(1)}\right|^*\right) ||x_k||$$
$$UB_{k,2}^1 \triangleq \left(\left|\bar{P}_k^{1A(1)}\right|^* + \left|T_k^{1A(1)}\right|^*\right) ||x_k||$$

Finally we consider the constant term of x_k. By denoting

$$\bar{M}_{k,\max}^{1A(1)} = \max_{\delta^1} \bar{M}_k^{1A(1)}(c_6) \qquad \bar{M}_{k,\min}^{1A(1)} = \min_{\delta^1} \bar{M}_k^{1A(1)}(c_6)$$
$$H_{k,\max}^{1A(1)} = \max_{\delta^1} H_k^{1A(1)}(c_5) \qquad H_{k,\min}^{1A(1)} = \min_{\delta^1} H_k^{1A(1)}(c_5)$$

we have

$$LB_{k,3}^1 \leq \bar{M}_k^{1A(1)}(c_6) - H_k^{1A(1)}(c_5) \leq UB_{k,3}^1 \quad (41)$$

where

$$LB_{k,3}^1 \triangleq \bar{M}_{k,\min}^{1A(1)} - H_{k,\max}^{1A(1)}$$
$$UB_{k,3}^1 \triangleq \bar{M}_{k,\max}^{1A(1)} - H_{k,\min}^{1A(1)}$$

Summarizing the results given in (36), (40) and (41), and denoting

$$\Sigma_k^L = \left(LB_{k,1}^1 + LB_{k,2}^1 + LB_{k,3}^1\right) \quad (42)$$
$$\Sigma_k^U = \left(UB_{k,1}^1 + UB_{k,2}^1 + UB_{k,3}^1\right) \quad (43)$$

the square bracket term in Eq. (32) is thus bounded by

$$\Sigma_k^L \leq [\cdot] \leq \Sigma_k^U \quad (44)$$

As $-2\Delta^0 \leq \delta^1 \leq 2\Delta^0$, we define

$$LB_k^1 \triangleq \min\{\Sigma_k^L \Delta^0, -\Sigma_k^U \Delta^0\} \quad (45)$$
$$UB_k^1 \triangleq \Sigma_k^U \Delta^0 \quad (46)$$

and we conclude that the discrepancy between the optimum and the non-optimum cost functionals is bounded by

$$LB_k^1 \leq \bar{V}_k^{1A}(x_k, \delta^1) - \bar{J}_k^{1A}(x_k, \delta^1) \leq UB_k^1 \quad (47)$$

■

IV. NUMERICAL EXAMPLE

A multi-model situation is illustrated by a strategic bidding example in the competitive electricity market.

Suppose there are P power suppliers, where P is a positive integer, which will be called *players* or *decision makers* (DMs) hereafter, and the power production cost of the j^{th} player, $j \in \{1, \ldots, P\}$, at time step $k \in \{0, \ldots, N\}$ is modeled by a quadratic function [9]

$$C_k^j = C^j(q_k^j) = \frac{1}{2}a^i\left(q_k^j\right)^2 + b^j q_k^j + c^j \quad (48)$$

where a^j, b^j and c^j are production coefficients of the j^{th} player, which assumed to be fixed during the course of one auction, q_k^j is the production quantity of the j^{th} player at time step k, and N is a positive integer. The market clearing price (MCP) at each time step k, λ_k, is assumed to follow a linear dynamic model governed by:

$$\lambda_{k+1} = A_k\lambda_k + \sum_{i=1}^{p} B_k^j q_k^j \quad (49)$$

such that the profit π_k^j of the j^{th} supplier at the k^{th} stage is given by:

$$\pi_k^j = \lambda_k q_k^j - C_k^j = -\frac{1}{2}a^j\left(q_k^j\right)^2 + \left(\lambda_k - b^j\right) q_k^j - c^j \quad (50)$$

Practically, each supplier would bid to maximize its total profit over a certain planning horizon, say N stages, rather than in a single step.

If we think of the market clearing price λ_k as the state variable x_k, and the quantities of production q_k^j as the control variables u_k^j, Eq. (49) and Eq. (50) show that each supplier's profit depends not only on his/her own decision variable, but also on his rivals' decision variables in the dynamic electricity market, which can be formulated by a N-person nonzero-sum dynamic game. In addition, in practice each player knows its own production coefficients, a^j, b^j and c^j, but may not know those of its competitors. The situation then falls under the framework of games with uncertainties.

For convenience we consider a market consisting of two power suppliers. Consider the system defined in Eq. (1) and Eq. (28) with following coefficients:

$$\begin{aligned} A &= 0.97, & B^1 &= -0.5, & B^2 &= -0.7, \\ Q^1 &= 1.25\varepsilon^2 + 4\varepsilon + 1, & Q^2 &= 0.75\varepsilon^2 + 8\varepsilon + 2, \\ R^1 &= 10\varepsilon, & R^2 &= 5\varepsilon, & D^1 &= 1.4\varepsilon, \\ D^2 &= 1.6\varepsilon, & G^1 &= 2.6\varepsilon, & G^2 &= 1.2\varepsilon \end{aligned}$$

where ε is unknown to both DMs. Suppose for $j = 1, 2$, the boundary conditions are

$$K_N^j = Q_N^j = Q^j, \quad P_N^j = 0, \quad M_N^j = 0$$

A. Model formulated by DM1

If DM1 uses the estimate of $\varepsilon^1 = 0.2$ to calculate his/her own control strategy but assumes DM2 adopts a different value of ε, the control strategies applied by DM1 will be $\left(\gamma_k^{1A}(x_k, 0), \gamma_k^{2A}(x_k, \delta^1)\right)$ and the simulation carried out by DM1 is given in Fig. 1.

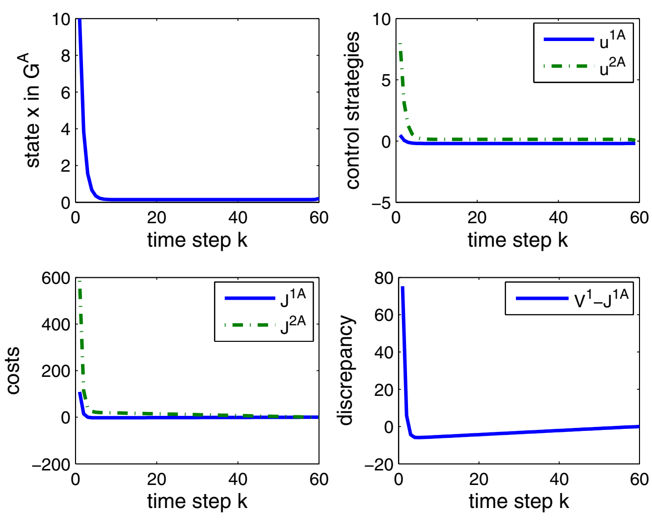

Fig. 1. Game A: DM1 uses $\varepsilon^1 = 0.2$ but DM2 uses $\varepsilon = 0.5$

B. Model formulated by DM2

Similarly, if DM2 implements $\varepsilon = \varepsilon^2 = 0.75$ but assumes DM1 adopts a different value of ε, the control strategies applied by DM2 will be $\left(\gamma_k^{1B}(x_k, \delta^2), \gamma_k^{2B}(x_k, 0)\right)$ and the simulation is given in Fig. 2. In both cases, we observe that the discrepancies in both cases are bounded.

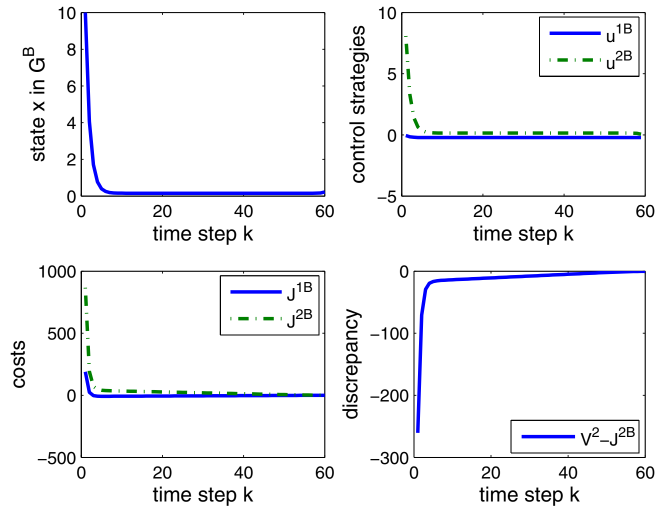

Fig. 2. Game B: DM1 uses $\varepsilon^1 = 0.5$ but DM2 uses $\varepsilon = 0.75$

V. CONCLUSIONS

In this paper, we present a generic two-person nonzero-sum game as a model to be used in the design of optimal bidding strategies in a competitive electricity market. Dynamic noncooperative games with unknown parameters are formulated by multiple models and the sensitivities of performance indices with respect to the unknown are analyzed. We show both theoretically and experimentally that the deviations of the indices are bounded from below and above from each DM's point of view.

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