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Nonlinear μ Flight Control and Suboptimal Exact Solution on F-16 Application

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***Abstract*—This paper investigates the theory of nonlinear H_∞ analysis to flight vehicles with varying real parameters which arise from the uncertain aerodynamic coefficients. It's so called nonlinear μ flight control. The difficult task involved in applying the nonlinear μ light control is to solve the associated Hamilton-Jacobi inequality for uncertain system. In this paper we derive the suboptimal condition to meet the L_2-gain of the nonlinear uncertain system less than a constant γ. The complete six degree-of-freedom nonlinear equations of motion for F-16 aircraft are considered directly to design the nonlinear μ flight controller by treating the longitudinal and lateral motions as a whole. The associated Hamilton-Jacobi partial differential inequality is solved analytically, resulting in a nonlinear μ controller with simple proportional feedback structure. This paper verify that the derived nonlinear μ control law can ensure global flight envelop and asymptotical stability of the closed loop system with varying aerodynamic characteristics and have strong robustness against wind gusts with varying statistical characteristics.**

I. Introduction

THIS problems of instability and flight performance degradation caused by linearization process can be taken into account by exploiting linear robust control theory, such as H_∞ control [1]-[2], synthesis technique by Packard [3], and quantitative feedback theory (QFT) by Reynolds et al [4], etc. However, a linear single-point design with constant feedback gains is probably inadequately to maintain good control properties over the whole nonlinear flight region, especially near the boundaries of the flight envelope. A popular control design employed for such flight vehicles operating over a large flight envelope is gain scheduling (Nichols, [5]) via on-line updating of linear controllers obtained from linearized models about a set of operating points within the flight envelope.

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An alternative approach to tackle nonlinear flight region is the direct use of nonlinear control theory such as nonlinear H_∞ control. The practical application of nonlinear H_∞ control theory is still very limited due to the difficulties in solving the associated Hamilton-Jacobi partial differential inequality (HJPDI). Algebraic and geometric tools have been used to find a particular solution of the HJPDI for satellite attitude control problem [6]. The nonlinear H_∞ control to general six degree-of-freedom motions with exact solutions of HJPDI is demoed in [7].

In recent years, numerous papers have addressed nonlinear robust stability problems with parameters uncertainty [8] or unstructured uncertainty [9]-[10]. The paper here extended the results in [7] to the complete six degree-of-freedom nonlinear flight vehicle dynamics with varying aerodynamic coefficients, which has not been considered in the literature before. Under this approach, the nonlinear control system is similar to the μ–analysis technique in linear control system. It's called nonlinear μ analysis in this paper. The control surface inverse algorithm (CSIA) [11] is considered in this paper to implement the control surface deflections. CSIA is constructed based on minimizing the global errors between the commands and actual control forces and moments. In our study, control command prefilter is designed such that the nonlinear μ flight control commands can be followed as close as possible by flight vehicles' actuator systems.

II. Introduced Aerodynamic Coefficients into Nonlinear 6-DOF Equations

In general, the changes of aerodynamic and propulsive forces and moments are expressed by means of a Taylor series about the equilibrium point. Only retaining the first-order terms, we have

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0), \quad f'(x_0) \quad fixed \qquad (1)$$

x_0 denotes the states at equilibrium point. On the other hand, based on the theorem of mean value in Calculus, there exists a ξ such that

$$f(x) = f(x_0) + f'(\xi)(x - x_0), \quad x_0 \le \xi \le x \qquad (2)$$

It's noticed that the left term is equal to the right terms in (2) and yet they are approximate each other in (1). The value of ξ is just aware between x_0 and x, so the value of $f'(\xi)$ can only be defined as

$$a \le f'(\xi) \le b$$

where *a, b* are the derivatives about the points x_0 and x, respectively. In applying the small-disturbance theory, we assume that the 6-d.o.f. motions consist of small deviations about a steady flight condition. Obviously, this theory in (1) cannot be applied to problems in which large-amplitude motions are to be expected. However, in (2), the mean-value theory yields sufficient accuracy for large-amplitude motions, even though the aerodynamic derivatives $f'(\xi)$ is indefinite.

To simplify the notations of 6-Dof equations of motion, the following definitions are used.

$$\Sigma(t) = [U \quad V \quad W]^T = [U_0 \quad V_0 \quad W_0]^T + [u \quad v \quad w]^T = \Sigma_0 + \sigma(t)$$
$$\Omega(t) = [P \quad Q \quad R]^T = [P_0 \quad Q_0 \quad R_0]^T + [p \quad q \quad r]^T = \Omega_0 + \omega(t)$$
$$u_\Sigma(t) = [F_x \quad F_y \quad F_z]^T = [F_{x0} \quad F_{y0} \quad F_{z0}]^T + [f_x \quad f_y \quad f_z]^T = u_{\Sigma_0} + u_\sigma(t)$$
$$u_\Omega(t) = [L \quad M \quad N]^T = [L_0 \quad M_0 \quad N_0]^T + [l \quad m \quad n]^T = u_{\Sigma_0} + u_\omega(t)$$
$$d_\sigma = [d_x \quad d_y \quad d_z]^T,\ d_\omega = [d_l \quad d_m \quad d_n]^T$$

The symbols with subscript zero denote the values at equilibrium point (trim condition), and the lower-case symbols denote the deviation from the equilibrium point. However, it needs to be noted here that we do not make the assumption of small deviation.

In this paper, we are interested in expressing the aerodynamic forces and moments by (2). Note that the aerodynamic derivative $f'(\xi)$ is varying about certain range. For example, the aerodynamic force along the x axis F_x^a can be described by

$$F_x^a(\Sigma_0+\sigma,\Omega_0+\omega) = F_x^a(\Sigma_0,\Omega_0) + \frac{\partial F_x^a(\Sigma_0+\bar{\sigma},\Omega_0)}{\partial \sigma}\sigma + \frac{\partial F_x^a(\Sigma_0,\Omega_0+\bar{\omega})}{\partial \omega}\omega \quad (3)$$

with $0 \le \bar{\sigma} \le \sigma, 0 \le \bar{\omega} \le \omega$. Equation (3) can be simplified to be

$$F_x^a = F_{x_0}^a + \begin{bmatrix} F_{x_u}^a & F_{x_v}^a & F_{x_w}^a & F_{x_p}^a & F_{x_q}^a & F_{x_r}^a \end{bmatrix} \begin{bmatrix} u & v & w & p & q & r \end{bmatrix}^T \quad (4)$$

The partial derivatives, $F_{x_u}^a$, $F_{x_v}^a$, … , etc are varying about the flight envelope range. The other forces and moments can be also represented in the same method. Express all aerodynamic forces and moments in the matrix form:

$$\begin{bmatrix} F_x^a \\ F_y^a \\ F_z^a \\ L^a \\ M^a \\ N^a \end{bmatrix} = \begin{bmatrix} F_{x0}^a \\ F_{y0}^a \\ F_{z0}^a \\ L_0^a \\ M_0^a \\ N_0^a \end{bmatrix} + \begin{bmatrix} F_{xu}^a & F_{xv}^a & F_{xw}^a & F_{xp}^a & F_{xq}^a & F_{xr}^a \\ F_{yu}^a & F_{yv}^a & F_{yw}^a & F_{yp}^a & F_{yq}^a & F_{yr}^a \\ F_{zu}^a & F_{zv}^a & F_{zw}^a & F_{zp}^a & F_{zq}^a & F_{zr}^a \\ L_u^a & L_{vv}^a & L_w^a & L_p^a & L_q^a & L_r^a \\ M_u^a & M_{vv}^a & M_w^a & M_p^a & M_q^a & M_r^a \\ N_u^a & N_{vv}^a & N_w^a & N_p^a & N_q^a & N_r^a \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix} \quad (5)$$
$$= \begin{bmatrix} \Sigma_0^a \\ \Omega_0^\alpha \end{bmatrix} + G \begin{bmatrix} \sigma \\ \omega \end{bmatrix}$$

where G is a matrix, which belongs to a given compact set

$$G \in \left\{ [g_{ij}] \,|\, g_{ij}^- \le g_{ij} \le g_{ij}^+, \quad i,j = 1,\ldots 6 \right\} = S_G \quad (6)$$

with $g_{11} = F_{xu}^a, g_{12} = F_{xv}^a, g_{66} = N_r^a, \ldots,$ etc. g_{ij}^- and g_{ij}^+ are the left and right point of the range which g_{ij} is in. The components of the force and moment acting on the flight vehicle are composed of aerodynamic and control contributions. Hence, the forces and moments are divided into two parts:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} F_x^a \\ F_y^a \\ F_z^a \end{bmatrix} + \begin{bmatrix} F_x^u \\ F_y^u \\ F_z^u \end{bmatrix}, \quad \begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} L^a \\ M^a \\ N^a \end{bmatrix} + \begin{bmatrix} L^u \\ M^u \\ N^u \end{bmatrix} \quad (7)$$

where the symbols with superscript u denote the applied forces created by control surfaces (including gravity and thrust) and

$$u_\Sigma^u = \begin{bmatrix} F_x^u & F_y^u & F_z^u \end{bmatrix}^T, \quad u_\Omega^u = \begin{bmatrix} L^u & M^u & N^u \end{bmatrix} \quad (8)$$

The moments of inertia matrix I_M and cross- product matrix $S(\omega)$ induced by $\omega=[p\ q\ r]^T$ are defined as

$$I_M = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}, \quad S(\omega) = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \quad (9)$$

The trim force u_{Σ_0} and moment u_{Ω_0} can be solved from

$$\sigma = \dot{\sigma} = \omega = \dot{\omega} = d_\sigma = d_\omega = 0 \quad (10)$$

The nonlinear equations of motion with respect to the equilibrium point are (see [7])

$$\dot{\sigma} = -S(\omega)\sigma - S(\Omega_0)\sigma - S(\omega)\Sigma_0 + \frac{1}{m_s}u_\sigma^u + \frac{1}{m_s}u_\sigma^a + \frac{1}{m_s}d_\sigma \quad (11)$$

$$\begin{aligned} \dot{\omega} &= -I_M^{-1}S(\omega)I_M\omega - I_M^{-1}S(\omega)I_M\Omega_0 \\ &\quad -I_M^{-1}S(\Omega_0)I_M\omega + I_M^{-1}u_\omega^u + I_M^{-1}u_\omega^a + I_M^{-1}d_\omega \end{aligned} \quad (12)$$

Most of the flight control techniques are based on the assumption that the perturbed motion from the equilibrium point is small. Under this assumption, such terms as $S(\omega)\sigma$ and $I_M^{-1}S(\omega)I_M\omega$ can be negligible in comparison with ω and σ , and (11)~(12) is reduced to a linear form. According to the values of Σ_0 and Ω_0 , different control modes can be defined from (11) and (12). Considering $\Sigma_0 \neq 0$ and $\Omega_0 \neq 0$, the nonlinear velocity and body-rate control mode can be expressed in matrix form:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \sigma \\ \omega \end{bmatrix} &= \begin{bmatrix} -S(\Omega_0) & S(\Sigma_0) \\ 0 & I_M^{-1}S(I_M\Omega_0)I_M - I_M^{-1}S(\Omega_0)I_M \end{bmatrix}\begin{bmatrix} \sigma \\ \omega \end{bmatrix} \\ &+ \begin{bmatrix} m_s^{-1}G_{11} & m_s^{-1}G_{12} \\ I_M^{-1}G_{21} & I_M^{-1}G_{22} \end{bmatrix}\begin{bmatrix} \sigma \\ \omega \end{bmatrix} - \begin{bmatrix} S(\omega) & 0 \\ 0 & I_M^{-1}S(\omega)I_M \end{bmatrix}\begin{bmatrix} \sigma \\ \omega \end{bmatrix} \\ &+ \begin{bmatrix} m_s^{-1}I_3 & 0 \\ 0 & I_M^{-1} \end{bmatrix}\begin{bmatrix} d_\sigma \\ d_\omega \end{bmatrix} + \begin{bmatrix} m_s^{-1}I_3 & 0 \\ 0 & I_M^{-1} \end{bmatrix}\begin{bmatrix} u_\sigma^u \\ u_\omega^u \end{bmatrix} \end{aligned} \quad (13)$$

Equation (13) can be recast into the standard state-space form:

$$G: \begin{aligned} \dot{x} &= f(x) + g_1(x)d + g_2(x)u^c \\ z &= \begin{bmatrix} h_1(x) \\ \rho_u u^c \end{bmatrix} \end{aligned} \quad (14)$$

The problem is to find the control u^c such that the L_2-gain of the closed-loop system (14) is less than γ. The following lemma characterizes the desired control u^c (see [12]).

(1) The closed-loop system (14) has L_2-gain less than γ, if there exists a positive C^1 function E satisfying

$$\frac{1}{2}\left(\frac{\partial E}{\partial x}\right)^T\left(\frac{1}{\gamma^2}g_1g_1^T-\frac{1}{\rho_u^2}g_2g_2^T\right)\left(\frac{\partial E}{\partial x}\right)+\left(\frac{\partial E}{\partial x}\right)^T f(x)+\frac{1}{2}h_1^T(x)h_1(x)<0 \quad (15)$$

(2) If such an E function exists, the desired feedback control u^c can be constructed from E as

$$u(x)=-\frac{1}{\rho_u^2}g_2^T(x)\frac{\partial E}{\partial x} \quad (16)$$

Comparing (13) with (14), we have $x^T=[\sigma^T \quad \omega^T]$ and

$$f(x)=\begin{bmatrix}-S(\Omega)+m_s^{-1}G_{11} & S(\Sigma_0)+m_s^{-1}G_{12}\\ I_M^{-1}G_{21} & I_M^{-1}S(I_M\Omega_0)I_M-I_M^{-1}S(\Omega)I_M+I_M^{-1}G_{22}\end{bmatrix} \quad (17)$$

$$g_1(x)=g_2(x)=\begin{bmatrix}\frac{1}{m_s}I_3 & 0\\ 0 & I_M^{-1}\end{bmatrix},\ u^c=\begin{bmatrix}u_\sigma^u\\ u_\omega^u\end{bmatrix},\ d=\begin{bmatrix}d_\sigma\\ d_\omega\end{bmatrix} \quad (18)$$

To reflect these requirements, we choose the output signal z in the form of (14) and

$$h_1=\left(\frac{1}{2}\rho_\sigma m_s\sigma^T\sigma+\frac{1}{2}\rho_\omega\omega^T I_M\omega\right)^{\frac{1}{2}} \quad (19)$$

Substitute (17)~(19) into (15), the HJPDI can be expressed a

$$\begin{aligned}&\frac{1}{2}\left(\frac{1}{\gamma^2}-\frac{1}{\rho_u^2}\right)\left[\frac{1}{m_s^2}\left(\frac{\partial E}{\partial\sigma}\right)^T\left(\frac{\partial E}{\partial\sigma}\right)+\left(\frac{\partial E}{\partial\omega}\right)^T I_M^{-2}\left(\frac{\partial E}{\partial\omega}\right)\right]\\&+\left(\frac{\partial E}{\partial\sigma}\right)^T\left(-S(\omega)\sigma-S(\Omega_0)\sigma+S(\Sigma_0)\omega+\frac{1}{m_s}G_{11}\sigma+\frac{1}{m_s}G_{12}\omega\right)\\&+\left(\frac{\partial E}{\partial\omega}\right)^T\left(I_M^{-1}S(I_M\Omega_0)\omega-I_M^{-1}S(\omega)I_M\omega-I_M^{-1}S(\Omega_0)I_M\omega\right.\\&\left.+I_M^{-1}G_{21}\sigma+I_M^{-1}G_{22}\omega\right)+\frac{1}{4}\rho_\omega\omega^T I_M\omega+\frac{1}{4}m_s\rho_\sigma\sigma^T\sigma<0\end{aligned} \quad (20)$$

Motivated from the linear result, we search for a possible quadratic solution in the form

$$E(x)=\frac{1}{2}[\sigma \quad \omega]^T\begin{bmatrix}C_\sigma m_s I_3 & 0\\ 0 & C_\omega I_M\end{bmatrix}\begin{bmatrix}\sigma\\ \omega\end{bmatrix}>0 \quad (21)$$

C_σ, C_ω, are scalar constants to be determined. To guarantee the positiveness of $E(x)$, these scalar constants must be positive. By substituting (21) into (20), the following inequality can be derived

$$x\left(\begin{bmatrix}\frac{1}{2}\left(\frac{1}{\gamma^2}-\frac{1}{\rho_u^2}\right)C_\sigma^2 I_3+\frac{m_s\rho_\sigma}{4}I_3 & C_\sigma m_s S(\Sigma_0)\\ 0 & \frac{1}{2}\left(\frac{1}{\gamma^2}-\frac{1}{\rho_u^2}\right)C_\omega^2 I_3-C_\omega S(\Omega_0)I_M+\frac{1}{4}\rho_\omega I_M\end{bmatrix}+\begin{bmatrix}C_\sigma I_3 & 0\\ 0 & C_\omega I_3\end{bmatrix}\begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}\right)x<0 \quad (22)$$

To guarantee the positiveness of (22), it can be rewritten as

$$\begin{bmatrix}\sigma\\ \omega\end{bmatrix}^T\left(\begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}+\begin{bmatrix}C_\sigma I_3 & 0\\ 0 & C_\omega I_M\end{bmatrix}\overline{G}\right)\begin{bmatrix}\sigma\\ \omega\end{bmatrix}<0 \quad (23)$$

where

$$G=\begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix},\ \overline{G}=G+G^T$$

M is defined as follows:

$$x\begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}x<0$$

$$M_{11}=\frac{1}{2}\left(\frac{1}{\gamma^2}-\frac{1}{\rho_u^2}\right)C_\sigma^2 I_3+\frac{m_s\rho_\sigma}{4}I_3,\quad M_{12}=\frac{1}{2}C_\sigma m_s S(\Sigma_0)$$

$$M_{22}=\frac{1}{2}\left(\frac{1}{\gamma^2}-\frac{1}{\rho_u^2}\right)C_\omega^2 I_3-\frac{1}{2}C_\omega I_M S^T(\Omega_0)-\frac{1}{2}C_\omega S(\Omega_0)I_M+\frac{\rho_\omega}{4}I_M$$

It can be shown that Eq.(23) is satisfied, if

$$M+\begin{bmatrix}C_\sigma I_3 & 0\\ 0 & C_\omega I_M\end{bmatrix}\left(G+G^T\right)<0,\quad \forall G\in S_G \quad (24)$$

where M is also a function matrix of C_σ and C_ω. To guarantee the negativeness of (24), the maximal eigenvalue of (24) has to be negative. Define the nonlinear μ value:

$$\mu_r \underset{=}{\Delta} \min_{\substack{C_\sigma>0\\ C_\omega>0}}\max_{G\in S_G}\lambda_{\max}\left(M+\begin{bmatrix}C_\sigma I_3 & 0\\ 0 & C_\omega I_M\end{bmatrix}\left(G+G^T\right)\right) \quad (25)$$

where $\lambda_{\max}(.)$ stands for the maximal eigenvalue. Hence, we have the following property.

Theorem 1. *If $\mu_\gamma<0$, then the nonlinear closed-loop flight system is internally stable and has L_2-gain lower than γ, i.e.,*

$$\int_0^\infty z^T z\,dt<\gamma^2\int_0^\infty w^T w\,dt \quad \text{for all } G\in S_G$$

His condition 2 is important to guarantee that the nonlinear flight control system in total varying aerodynamic range has L_2-gain lower than γ. It's so called performance robustness. Furthermore, since the solution of HJPDI is a qualified Lyapunov function, the close loop system (with disturbance $d=0$) is stable in the sense of Lyapunov.

III. The Calculation of μ_γ

The performance robustness of the closed-loop flight system can be always kept if the value of μ_γ is negativeness. However, the performance can be refined by considering lower quantity of γ. The minimal γ can be obtained by

$$\gamma_{\min}=\min\left\{\gamma:\mu_\gamma<0\right\} \quad (26)$$

It is clear that μ_γ is affected by S_G. It implies that we can relate μ_γ to be $\mu_\gamma(S_G)$. S_G stands for the range of the uncertainty and γ denotes the ability of disturbance rejection (i.e., performance.) Therefore, $\mu_\gamma(S_G)$ increasing follows S_G and decreasing follows γ. Obviously, the calculation of $\mu_\gamma(S_G)$ is difficult. It is necessary for the largest maximal eigenvalue to be found numerically by searching all $G\in S_G$ to determine the optimal C_σ and C_ω to minimize (25).

To overcome the difficulty of calculating $\mu_\gamma(S_G)$, we have consider the following theorem:

Theorem 2. *Define*

$$\lambda_0=\max_{G\in S_G}\lambda_{\max}\left(G+G^T\right),\quad \overline{M}=\begin{bmatrix}C_\sigma^{-1}M_{11} & C_\sigma^{-1}M_{12}\\ C_\omega^{-1}M_{12}^T & C_\omega^{-1}M_{22}\end{bmatrix}$$

then $\mu_\gamma(S_G)<0$ is satisfied, if there exists C_σ and C_ω such that

$$\lambda_{\max}\left(\overline{M}\right)<-\lambda_0 \quad (27)$$

,i.e., $\overline{M}+\lambda_0 I$ *is negative definite.*

Let A be a square matrix that the following equation is true for any scalar *c*,

$$\lambda\left(A+cI\right)=\lambda\left(A\right)+c \tag{28}$$

It implies that (27) can be rewritten as

$$\lambda_{\max}\left(\overline{\mathrm{M}}+\lambda_0\mathrm{I}\right)<0 \tag{29}$$

Clearly, $\lambda_{\max}\left(\overline{M}+\lambda_0 I\right)<0 \Leftrightarrow \overline{M}+\lambda_0 I$ i. e., $\overline{M}+\lambda_0 I$ is negative definite. It should be pointed out that the solution of (27) is much easier than checking $\mu_\gamma(S_G)<0$. It's because that the control parameters C_σ and C_ω are not shown in λ_0 and the aerodynamic derivatives are not shown in $\overline{M}$.

IV. Suboptimal Nonlinear μ Analytic Solution

To satisfy $\overline{M}+\lambda_0 I<0$ and achieve the minimum of $C_\sigma^2+C_\omega^2$, the values of C_σ, C_ω is required to be determined. If we can obtain the ranges of C_σ and C_ω first, it will be convenient to predict the optimal solution of the system. (24) can be applied to find the suboptimal solution and modified as

$$\begin{bmatrix} M_{11}^* & M_{12}^* \\ M_{12}^* & M_{22}^* \end{bmatrix}<0 \tag{30}$$

where

$$M_{11}^*=\frac{1}{2}\left(\frac{1}{\gamma^2}-\frac{1}{\rho_u^2}\right)C_\sigma^2 I_3+\frac{m_s\rho_\sigma}{4}I_3+C_\sigma\overline{G_{11}}$$

$$M_{12}^*=\frac{1}{2}C_\sigma m_s S\left(\Sigma_0\right)+C_\sigma\overline{G_{12}}$$

$$M_{21}^*=\frac{-1}{2}C_\sigma m_s S\left(\Sigma_0\right)+C_\omega\overline{G_{21}},$$

$$M_{22}^*=\frac{1}{2}\left(\frac{1}{\gamma^2}-\frac{1}{\rho_u^2}\right)C_\omega^2 I_3-\frac{1}{2}C_\omega I_M S^T\left(\Omega_0\right)$$

$$-\frac{1}{2}C_\omega S\left(\Omega_0\right)I_M+\frac{1}{4}\rho_\omega I_M+C_\omega\overline{G_{22}}$$

We apply the following property of matrix inequality:

$$\begin{bmatrix} M_{11}^* & M_{12}^* \\ M_{21}^* & M_{22}^* \end{bmatrix}<0 \text{ if and only if } M_{11}^*<0,\ M_{22}^*-M_{21}^*M_{11}^{*-1}M_{12}^*<0 \tag{31}$$

Now, let

$$\left\|G_{11}^+\right\|=\max_{\overline{G_{11}}\in S_G}\left\|\overline{G_{11}}\right\|,\quad \left\|G_{12}^+\right\|=\max_{\overline{G_{12}}\in S_G}\left\|\overline{G_{12}}\right\|,$$

$$\left\|G_{21}^+\right\|=\max_{\overline{G_{21}}\in S_G}\left\|\overline{G_{21}}\right\|,\quad \left\|G_{22}^+\right\|=\max_{\overline{G_{22}}\in S_G}\left\|\overline{G_{22}}\right\|$$

It can be shown that if the following equation exists, then $M_{11}^*<0$,

$$\frac{1}{2}\left(\frac{1}{\gamma^2}-\frac{1}{\rho_u^2}\right)C_\sigma^2 I_3+\frac{m_s\rho_\sigma}{4}I_3+C_\sigma\left\|G_{11}^+\right\|I_3<0$$

$$\Rightarrow C_\sigma>\frac{\left\|G_{11}^+\right\|+\sqrt{\left\|G_{11}^+\right\|^2+\frac{1}{2}\left(\frac{1}{\rho_u^2}-\frac{1}{\gamma^2}\right)m_s\rho_\sigma}}{\left(\frac{1}{\rho_u^2}-\frac{1}{\gamma^2}\right)} \tag{32}$$

In a similar manner, we can shown that if the following equation exists, $M_{22}^*-M_{21}^*M_{11}^{*-1}M_{12}^*<0$ is satisfied:

$$\begin{aligned}&\frac{C_\omega^2}{2}\left(\frac{1}{\gamma^2}-\frac{1}{\rho_u^2}\right)-C_\omega\left(\left\|G_{22}^+\right\|+\left\|G_{21}^+\right\|\frac{\frac{1}{2}C_\sigma m_s\left\|S(\Sigma_0)\right\|+C_\sigma\left\|G_{21}^+\right\|}{\frac{1}{2}\left(\frac{1}{\gamma^2}-\frac{1}{\rho_u^2}\right)C_\sigma^2+\frac{m_s\rho_\sigma}{4}+C_\sigma\left\|G_{11}^+\right\|}\right)\\&-\left(\frac{1}{4}\rho_\omega\left\|I_M\right\|-\frac{\frac{1}{4}C_\sigma^2 m_s^2\Sigma_0^T\Sigma_0+\frac{1}{2}C_\sigma^2 m_s\left\|S(\Sigma_0)\right\|\left\|G_{21}^+\right\|}{\frac{1}{2}\left(\frac{1}{\gamma^2}-\frac{1}{\rho_u^2}\right)C_\sigma^2+\frac{m_s\rho_\sigma}{4}+C_\sigma\left\|G_{11}^+\right\|}\right)<0\end{aligned} \tag{33}$$

This leads to the range of C_ω. Compare (32) with the solution of C_σ for nonlinear H_∞ controller in [7]:

$$C_\sigma\geq\sqrt{\frac{m_s\rho_\sigma w_\sigma^2\gamma^2}{2\left(\gamma^2-w_\sigma^2\right)}} \tag{34}$$

The former is conservative obviously. The fact is that (32) contains an extra term $\left\|G_{11}^+\right\|$ to increase the range of C_σ. It can be also observed in the solution of C_ω. Up to this stage, we have shown that $E(\sigma,\omega)$ given in (21) is truly a qualified solution of the HJPDI, and those constants C_σ, C_ω, in $E(\sigma,\omega)$ can be determined analytically in (32) and (33). After having obtained the solution of $E(\sigma,\omega)$, we can compute the desired control forces and moments from (16):

$$\begin{aligned}u_\sigma^u&=\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}=-\frac{1}{\rho_\sigma^2 m_s}\frac{\partial E}{\partial\sigma}=-\frac{1}{\rho_\sigma^2}C_\sigma\sigma=-\frac{1}{\rho_\sigma^2}C_\sigma\begin{bmatrix} u \\ v \\ w \end{bmatrix}\\ u_\omega^u&=\begin{bmatrix} l \\ m \\ n \end{bmatrix}=-\frac{1}{\rho_\omega^2}I_M^{-1}\frac{\partial E}{\partial\omega}=-\frac{1}{\rho_\omega^2}C_\omega\omega=-\frac{1}{\rho_\omega^2}C_\omega\begin{bmatrix} p \\ q \\ r \end{bmatrix}\end{aligned} \tag{35}$$

Fig. 1 shows the closed loop system after the application of controller (35) to the open loop system (14). Note that the resulting controller ensures more than Lyapunov's stability. This simple proportional feedback control can guarantee that the nonlinear flight control system is stable in the sense of Lyapunov and has L_2 -gain lower than γ as well. Inspecting the penalized output z in Eq.(19) with control u^c given by (35), we can find that the only state for *z*=0 is the equilibrium state $\left(\sigma,\omega\right)=\left(0,0\right)$, i.e., *z* is zero-state observable. Therefore, from the LaSalle's theorem the closed loop system with *d*=0 is asymptotically stable. The states $(\alpha,\beta,\sigma,\omega)$ are always located over the flight envelope with the upper-bounds computation of u^a, the stability of the closed-loop system is also with respect to the range.

V. Control Surface Inverse Algorithm

Since general flight vehicle has only four or five control surfaces, it is not possible to exactly implement the control commands u^c which has six independent components. Let u^b be the actual forces and moments that can be generated by flight vehicle control surfaces. We will determine the best deflection angles of the control surfaces such that u^b can be close to u^c. The mechanism of determining the best deflection angles is based on the minimization of the following command tracking error

$$J_{error} = \left(Q_u u^c - u^b\right)^T \left(Q_u u^c - u^b\right) \quad (36)$$

where weighting Q_u is used to filter the commands of forces and moments u^c such that control surfaces can operate in their unsaturated regions. The required forces and moments u^b created by control surfaces can be expressed as a compact matrix form with control surface deflections δ, δ∈[δ^*, δ^{*+1}]:

$$\begin{bmatrix} u_1^b \\ u_2^b \\ \vdots \\ u_6^b \end{bmatrix} = \begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_6^* \end{bmatrix} + \begin{bmatrix} \frac{\Delta u_1^1}{\Delta \delta_1} & \cdots & \frac{\Delta u_1^1}{\Delta \delta_m} \\ \vdots & \ddots & \vdots \\ \frac{\Delta u_6^1}{\Delta \delta_1} & \cdots & \frac{\Delta u_6^1}{\Delta \delta_m} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{bmatrix} \quad (37)$$

or in an abbreviated form:

$$u^b = u^* + A\delta \quad (38)$$

u^* consists of gravity, forces and moments with respect to δ^*, etc. A is the derivative matrix about δ^*. In this paper, we explicitly characterize the two matrices u^* and A using F-16 aerodynamic data. The optimal control surface deflection can be found by Moore Penrose inverse formulas as

$$\delta^+ = \left(A^T A\right)^{-1} A^T \left(Q_u u^c - u^*\right) \quad (39)$$

The details can be referred to [11]. It is worth noting that the linearity assumed here is only limited to the way generating aero data. There may exist some fitting errors by using piecewise linear interpolation, but the global aerodynamic model is still nonlinear, and the nonlinear equations of motion can cover all the flight envelope of the vehicle. On the contrary, in the conventional linearized model, aero data are assumed to be fixed at some trim point, and no aero data interpolation is required. Another important fact has to be mentioned here that the closed loop system is not global asymptotical stable when the control input is u^b. Inspecting the penalized output z in (14) with control u^b given by Eq. (36), we can find that the state for z=0 is not the only equilibrium state over a large flight envelope, i.e., z is not zero-state observable. Therefore, the resulting controller u^b only ensures Lyapunov's stability. On the other hand, the aerodynamic coefficients are changing with the different flight condition or the varying disturbance. The varying aerodynamic coefficients are realized as uncertainty for linear model, but are reflected to state variables and are considered as the components of u_σ and u_ω. To achieve the varying u_σ and u_ω, CSIA is applied to obtain the optimal control surface deflections to ensure stability of the nonlinear model(14).

VI. Case Study: F-16

The main objectives of the simulations are to validate the robustness of nonlinear μ flight control against wind gust and compare with nonlinear H_∞ control. It is important to note that, during the simulation, the aerodynamic forces and moments are not constant, but are recursively computed according to Mach number, altitude, angular velocity, angle of attack, and side slip angle. Moreover, there is no reducing process for the physical model of the F-16 aircraft plant. The computations of S_G, $\mu_\gamma(S_G)$, and λ_0 can be referred to the aero data of F-16 [13]. F-16 has five control surfaces δ_h, δ_a, δ_r, δ_{sb}, δ_r to produce aerodynamic control forces and moments. CSIA is applied to get the five control surfaces. Considering the difficulty of computing $\mu_\gamma(S_G)$ and λ_0, we solve the suboptimal solution of C_σ and C_ω by (32) and (33).To find the control forces and moments from (35), we need the mass and moments of inertia for F-16 aircraft: m_s=636.6 slug, I_{xx}=9496 slug・ft^2, I_{yy}=55814 slug・ft^2, I_{zz}=63100 slug・ft^2, and I_{xz}=982 slug・ft^2. The trim condition is set to$\Sigma_0=[U_0\ V_0\ W_0]^T=[800\ 0\ 0]^T$(ft/sec), and $\Omega_0=[P_0\ Q_0\ R_0]^T=[0\ 0\ 0]^T$(rad/sec) at altitude=*3,000*ft. The upper bound of the L_2-gain is selected to γ=*2*; the weighting coefficients ρ (see (19)) are all set to *1* and command prefilter $Q_u=\rho_u W$, ρ_u=1. To get the ranges of force derivatives, the flight envelops are chosen as 600 ≦V_t (aircraft resultant velocity)≦800 (ft/s), $-10^0 \leqq \alpha \leqq 35^0$, $-15^0 \leqq \beta \leqq 10^0$, $-1 \leqq p \leqq 1$ (rad/s), $-3 \leqq q \leqq 3$ (rad/s), $-1 \leqq r \leqq 1$ (rad/s). After searching in the flight envelops, we have

$$\left\|G_{11}^+\right\| = 1.135\times10^4,\ \left\|G_{12}^+\right\| = 3.1749\times10^4,$$
$$\left\|G_{21}^+\right\| = 3.125\times10^4,\ \left\|G_{22}^+\right\| = 2.143\times10^4$$

The nonlinear inequality solution of (30) is given by (32) and (33) wherein one of the normalized solutions can be chosen as C_σ=1.66 and C_ω=12.9. Fig.2 shows the comparison of state responses between nonlinear μ controller and nonlinear H_∞ controller. It can be expected that the performance with nonlinear μ controller is better than that of nonlinear H_∞ controller, as can be seen in Fig.2. It is recognized that the control gain C_σ=1.66 and C_ω=12.9 for nonlinear μ controller is larger than the control gain C_σ=2.47 and C_ω=2.81 for nonlinear H_∞ controller (see [11]). As a consequence, the airplane more robust to wind gust in nonlinear μ case. Fig.3 shows the control surface deflection angles for both control cases. The observation from Fig.3 reveals that the variations of control surface deflections for nonlinear H_∞ control are larger than nonlinear μ control. Fig.4 depicts the variation of L_2-gain due to change of turbulence scale length. It's well known that a physical constraint on control surface movement gives rise to the performance deviation from the theoretically predicted value. For example, the commanded δ_{sb}^+ may be round –30 degree, but the actual deflection angle of speed braker δ_{sb}^{a+} is limited to a positive angle in [*0,60*]. This also explains that the maximal L_2-gain=2.4 is greater than the specified value L_2-gain=2.

VII. Conclusion

In this paper we have demonstrated the possibility of applying nonlinear μ flight control theory to flight vehicle whose six degree-of-freedom nonlinear motions with coupled longitudinal and lateral dynamics and varying aerodynamic coefficients are considered directly. The suboptimal nonlinear μ controller derived from the Hamilton-Jacobi partial differential inequality is with simple proportional feedback structure and is very easy to implement. Another key feature is to obtain control surface angles from force and moment commands by control surface inverse algorithm. The simulation results illustrate the robustness property of nonlinear μ flight control, being able to maintain stability and performance in the presence of exogenous disturbance over large flight envelope. The simulation also reveals that nonlinear μ controller is better than nonlinear H_∞ in stability and robustness. While the former with larger gain has to pay more control energy.

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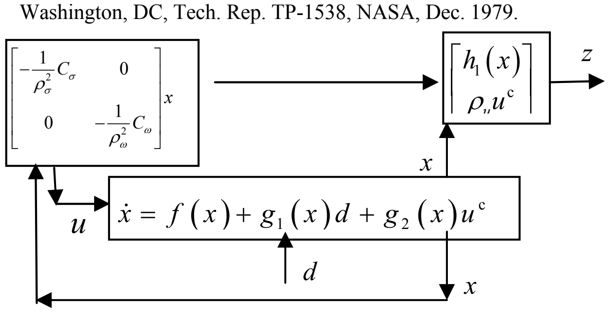

Fig. 1 The closed loop structure

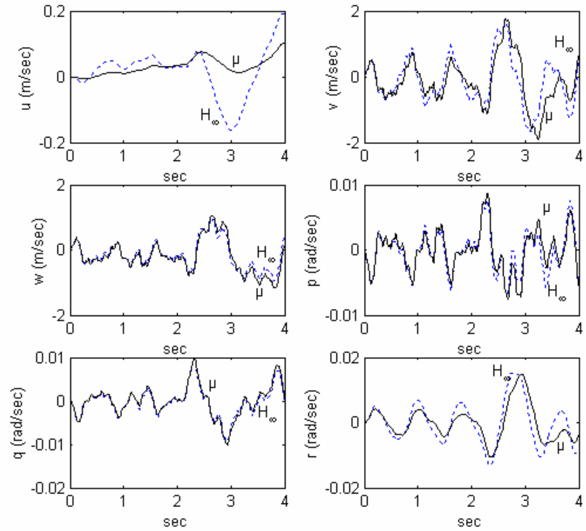

Fig. 2. Comparisons of state responses between nonlinear μ controller (solid) and nonlinear H_∞ controller (dashed).

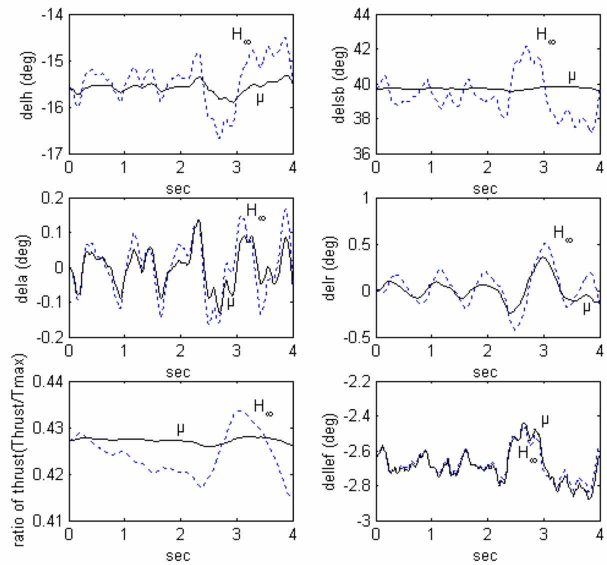

Fig. 3. comparisons of surface deflection angles between nonlinear H_∞ controller (dashed) and nonlinear μ controller (solid).

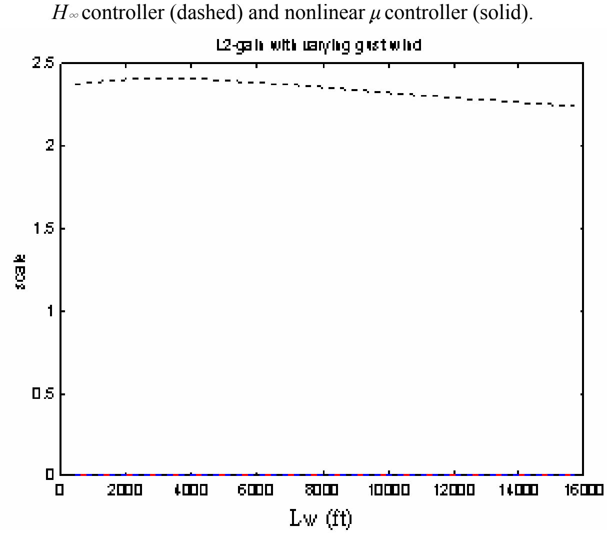

Fig. 4. Variation of L_2-gain for nonlinear μ flight control due to wind gust.