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On the geometrical representation and interconnection of infinite dimensional port controlled Hamiltonian systems

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Abstract— **This contribution is dedicated to the geometrical representation of infinite dimensional port controlled Hamiltonian systems. After an introduction of the used mathematical framework, a review on a well established geometrical representation of finite dimensional port controlled Hamiltonian systems is given. These results are in the subsequent analysis extended to the infinite dimensional case. After that the interconnection properties of the proposed description is under investigation. Additionally the developed theory is applied to the derivation of a PCH representation of a membrane interconnected with a string. Finally some concluding remarks are given and future interests are defined.**

I. INTRODUCTION

Port controlled Hamiltonian systems with dissipation, or PCHD systems [6] for short, have turned out to be a versatile tool for the mathematical modeling in control theory. This class of systems comes along with a mathematical description, that separates structural properties, storage elements and dissipative parts. Thus a network description of such plants, which is very useful for simulation and control, becomes available.

This contribution presents an extension of the PCH approach to the infinite-dimensional case. It is shown, which differential geometric objects have to be introduced and how boundary conditions come into play. Additionally the key property of PCHD systems – their behavior with respect to interconnection – is investigated for domain and boundary interconnections.

In the first section a short summary of the used mathematical notation is given. After that, some well known results for finite-dimensional PCHD systems are presented. The third section is dedicated to the introduction of a possible extension of the approach to the infinite-dimensional case. Here special attention is paid on the interconnection of two infinite-dimensional PCHD systems via power conserving interconnections. The developed representation is applied to a mechanical plant, consisting of a membrane with a boundary string in the fourth section. Finally, a summary of the achieved results is given and remarks on extensions of the introduced approach close this contribution.

II. NOTATIONS

This contribution uses the concept of smooth manifolds and bundles [3], [5]. A bundle is a triple $(\mathcal{E},\pi,\mathcal{B})$ with the total manifold $\mathcal{E}$, the base manifold $\mathcal{B}$ and the surjective submersion $\pi:\mathcal{E}\to\mathcal{B}$. For each point $p\in\mathcal{B}$, the subset $\pi^{-1}(p)=\mathcal{E}_p$ is called the fibre over p. We can introduce the adapted coordinates (X^i,x^α) to $\mathcal{E}$ at least locally with the independent coordinates X^i, $i=1,\ldots,p$ and the dependent ones x^α, $\alpha=1,\ldots,q$. Often, we will write $\mathcal{E}$ instead of $(\mathcal{E},\pi,\mathcal{B})$, whenever the projection π and the base manifold $\mathcal{B}$ follow from the context. Bundles, whose fibres are vector spaces, are referred to as vector bundles. A section σ of $\mathcal{E}$ is a map $\sigma:\mathcal{B}\to\mathcal{E}$ such that $\pi\circ\sigma=\mathrm{id}_{\mathcal{B}}$ is met, where $\mathrm{id}_{\mathcal{B}}$ denotes the identity map on $\mathcal{B}$. We do not require that a section σ exists globally and write for the set of all sections $\Gamma(\mathcal{E})$. From now on we use Latin indices for the independent and Greek indices for the dependent variables. Additionally a domain of integration is defined as an orientable, bounded manifold $\mathcal{D}$ with coherently oriented boundary manifold $\partial\mathcal{D}$.

Let $\mathcal{M}$ be a smooth m-dimensional manifold, then its tangent and cotangent bundles are denoted by $\mathcal{T}(\mathcal{M})$ and $\mathcal{T}^*(\mathcal{M})$. Using local coordinates, we write $\dot{x}^\alpha\partial_\alpha\in\Gamma(\mathcal{T}(\mathcal{M}))$, $\dot{x}_\alpha\mathrm{d}x^\alpha\in\Gamma(\mathcal{T}^*(\mathcal{M}))$, $\alpha=1,\ldots,m$ for sections of $\mathcal{T}(\mathcal{M})$, $\mathcal{T}^*(\mathcal{M})$, where we applied already the Einstein convention for sums to keep the formulas short and readable. From these vector bundles one derives further bundles, like the exterior k-form bundle $\wedge_k(\mathcal{T}^*(\mathcal{M}))$ or other tensor bundles. We denote the exterior algebra over $\mathcal{M}$ by $\wedge(\mathcal{T}^*(\mathcal{M}))$, $\mathrm{d}:\wedge_k(\mathcal{T}^*(\mathcal{M}))\to\wedge_{k+1}(\mathcal{T}^*(\mathcal{M}))$ is the exterior derivative and $\rfloor:\mathcal{T}(\mathcal{M})\times\wedge_{k+1}(\mathcal{T}^*(\mathcal{M}))\to\wedge_k(\mathcal{T}^*(\mathcal{M}))$ is the interior product written as $\dot{x}\rfloor\omega$ with $\dot{x}\in\mathcal{T}(\mathcal{M})$ and $\omega\in\wedge_{k+1}(\mathcal{T}^*(\mathcal{M}))$. The symbol $\wedge$ denotes the exterior product of the exterior algebra $\wedge(\mathcal{T}^*(\mathcal{M}))$. The Lie derivative of $\omega\in\wedge(\mathcal{T}^*(\mathcal{M}))$ along the field $f\in\mathcal{T}(\mathcal{M})$ is written as $f(\omega)$. Additionally we will use Stokes's theorem [1]

$$\int_{\mathcal{M}}\mathrm{d}\omega=\int_{\partial\mathcal{M}}\iota^*\omega\ ,\quad \omega\in\wedge_{m-1}(\mathcal{T}^*(\mathcal{M})) \tag{1}$$

whereby the manifold and its boundary is related using the inclusion mapping $\iota:\partial\mathcal{M}\to\mathcal{M}$.

Let γ be a smooth section of a bundle $(\mathcal{E},\pi,\mathcal{B})$ with adapted coordinates (X^i,x^α), $i=1,\ldots,p$, $\alpha=1,\ldots,q$. The k^{th} order partial derivatives of γ^α will be denoted by

$$\frac{\partial^k}{\partial_1^{j_1}\cdots\partial_p^{j_p}}\gamma^\alpha=\partial_{[J]}\gamma^\alpha=\gamma^\alpha_{[J]}\ ,\quad \partial_i=\frac{\partial}{\partial X^i}$$

with $J=j_1,\ldots,j_p$, and $k=\#J=\sum_{i=1}^p j_i$. J is nothing else than an ordered multi-index [4]. The special index $J=j_1,\ldots,j_p$, $j_i=\delta_{il}$, $l\in\{1,\ldots,p\}$ will be denoted by 1_l and $J+1_l$ is a shorthand notation for $j_i+\delta_{il}$ with

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the Kronecker symbol δ_{il}. Using adapted coordinates we can extend γ to a map $j^1(\gamma) : x \rightarrow \left(x^i, \gamma^\alpha(x), \partial_i \gamma^\alpha(x)\right)$, the first jet of γ. One can provide the set of all first jets of sections $\Gamma(\mathcal{E})$ with the structure of a differentiable manifold, which is denoted by $J^1(\mathcal{E})$. An adapted coordinate system of $\mathcal{E}$ induces an adapted system on $J^1(\mathcal{E})$, which is denoted by $\left(X^i, x^\alpha, x^\alpha_{[1_i]}\right)$ with the $p \cdot q$ new coordinates $u^\alpha_{[1_i]}$. The manifold $J^1(\mathcal{E})$ has two natural projections, $\pi^1 : J^1(\mathcal{E}) \rightarrow \mathcal{B}$ and $\pi^1_0 : J^1(\mathcal{E}) \rightarrow \mathcal{E}$, which correspond to the bundles $\left(J^1(\mathcal{E}), \pi^1, \mathcal{B}\right)$ and $\left(J^1(\mathcal{E}), \pi^1_0, \mathcal{E}\right)$. Analogously to the first jet of a section γ, we define the n^{th} jet $j^n(\gamma)$ of γ by $j^n(\gamma) = \left(x^i, \gamma^\alpha(x), \partial_{[J]}\gamma^\alpha(x)\right)$, $\#J = 1, \ldots, n$. The n^{th} jet manifold $J^n(\mathcal{E})$ of $\mathcal{E}$ may be considered as a container for n^{th} jets of sections of $\mathcal{E}$. Furthermore, an adapted coordinate system of $\mathcal{E}$ induces an adapted system on $J^n(\mathcal{E})$ with $\left(X^i, x^\alpha_{[J]}\right)$, $\alpha = 1, \ldots, q$, $\#J = 0, \ldots, n$.

The unique operator d_i, which meets $(d_i f) \circ j^{n+1}(\sigma) = \partial_i f(j^n(\sigma))$ for all functions $f \in C^\infty(J^n(\mathcal{E}))$ and sections $\sigma \in \Gamma(\mathcal{E})$, is the vector field $d_i \in \mathcal{T}(J^\infty(\mathcal{E}))$ and is called the total derivative with respect to the independent coordinate X^i. It is defined by

$$d_i = \partial_i + x^\alpha_{[J+1_i]}\partial^{[J]}_\alpha \;, \quad \partial_i = \frac{\partial}{\partial X^i},\ \partial^{[J]}_\alpha = \frac{\partial}{\partial x^\alpha_{[J]}} \qquad (2)$$

in adapted coordinates $\left(X^i, x^\alpha\right)$. The introduction of the total derivative d_i enables us to introduce the horizontal derivative d_h through

$$\left(j^{n+1}\sigma\right)^* (\mathrm{d}_h(\omega)) = \mathrm{d}\left(\left(j^n\sigma\right)^*(\omega)\right) \;, \quad \omega \in \wedge\left(J^n\mathcal{E}\right) \quad (3)$$

or in local coordinates $\mathrm{d}_h = \mathrm{d}X^i \wedge d_i$.

A vector field $v \in \Gamma(\mathcal{T}(\mathcal{E}))$ is said to be π-projectable, iff there exists a field $w \in \Gamma(\mathcal{T}(\mathcal{B}))$ such that $\pi_* \circ v = w \circ \pi$ is met. We say v is π-vertical in the case of $\pi_* \circ v = 0$. It is easy to show that the set of all π-vertical vector fields – the vertical tangent bundle $\mathcal{V}(\mathcal{E})$ – is a subbundle of $\mathcal{T}(\mathcal{E})$.

III. FINITE-DIMENSIONAL PCHD SYSTEMS (F-PCHD SYSTEMS)

In this section the geometrical structure and some additional properties of finite-dimensional PCHD systems are under investigation. The precise definition of the used spaces will serve as a basis for the subsequent analysis of the infinite-dimensional case.

A. Geometrical structure of F-PCHD systems

Let $\mathcal{M}$ denote the q-dimensional state manifold with coordinates (x^α), $\alpha = 1, \ldots, q$. The canonical product $\mathcal{T}(\mathcal{M}) \times \mathcal{T}^*(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ is given by the interior product $\dot{x}^\alpha\partial_\alpha \rfloor \dot{x}_\beta \mathrm{d}x^\beta = \dot{x}^\alpha \dot{x}_\alpha$. Let $\mathcal{U} = \operatorname{span}\{\mathrm{e}_\varsigma\}$ with coordinates (u^ς), $\varsigma = 1, \ldots, m$ denote the input vector space. Consequently we choose the dual vector space $\mathcal{Y} = \mathcal{U}^* = \operatorname{span}\{\mathrm{e}^\varsigma\}$ with coordinates (y_ς) as the output vector space. The structure of a PCHD-system with state (x^α), input (u^ς), output (y_ς) and Hamiltonian $H_0 \in C^\infty(\mathcal{M})$ is given by

$$\dot{x} \;=\; (J - R)\rfloor \mathrm{d}H_0 + u\rfloor B \qquad (4)$$

$$y \;=\; B\rfloor \mathrm{d}H_0 \qquad (5)$$

where $J = J^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$, $J^{\alpha\beta} = -J^{\beta\alpha}$, $R = R^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$, $R^{\alpha\beta} = R^{\beta\alpha}$, $B = B^\alpha_\varsigma e^\varsigma \otimes \partial_\alpha$ is used. Additionally the matrix $\left[R^{\alpha\beta}\right]$ is positive semidefinite and all coefficients are assumed to meet $J^{\alpha\beta}, R^{\alpha\beta}, B^{\alpha\varsigma} \in C^\infty(\mathcal{M})$. Obviously, J, R are maps $J, R : \mathcal{T}^*(\mathcal{M}) \rightarrow \mathcal{T}(\mathcal{M})$ and B is a map $B : \mathcal{U} \rightarrow \mathcal{T}(\mathcal{M})$ with its adjoint $B^* : \mathcal{T}^*(\mathcal{M}) \rightarrow \mathcal{Y}$. The exterior derivative d,

$$\mathrm{d}H_0 = \partial_\alpha H_0 \mathrm{d}x^\alpha$$

serves here as a map $\mathrm{d} : C^\infty(\mathcal{M}) \rightarrow \mathcal{T}^*(\mathcal{M})$. The circumstance, that the introduced Hamiltonian system is roughly speaking enveloped by the two linear spaces $\mathcal{U}$ and $\mathcal{Y}$ is visualized is Fig. 1.

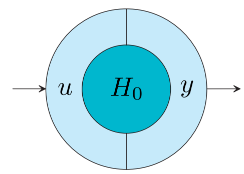

Fig. 1. A F-PCHD system

B. The Hamilton vector field and collocation

Let us introduce the Hamilton vector field[1] $v_H = \dot{x}^\alpha\partial_a$ with $\dot{x}^\alpha$ from (4). Taking into account this definition, we easily obtain the well known relation

$$v_H(H_0) = v_H \rfloor \mathrm{d}(H_0) = -\left(R\rfloor \mathrm{d}H_0\right)\rfloor \mathrm{d}H_0 + u\rfloor y \,. \qquad (6)$$

Obviously the product $u\rfloor y$ equals the external impact on the time derivative of the Hamiltonian H_0. One can often interpreted this product as the power fed into the system. It is common to say, that in this case the input u and the output y are collocated.

If the input map is given by $B = -e^\varsigma \otimes J\rfloor \mathrm{d}H_\varsigma$ with suitable functions H_ς, then from

$$v_H(H_\varsigma) = \left((J - R)\rfloor \mathrm{d}H_0 - u^\omega J\rfloor \mathrm{d}H_\omega\right)\rfloor \mathrm{d}H_\varsigma \;, \omega = 1 \ldots m$$

it follows that $v_H(H_\varsigma) = y_\varsigma$ is fulfilled for the case $\left(R\rfloor \mathrm{d}H_0\right)\rfloor \mathrm{d}H_\varsigma = \left(J\rfloor \mathrm{d}H_\omega\right)\rfloor \mathrm{d}H_\varsigma = 0$. This often applies in mechanics.

IV. INFINITE-DIMENSIONAL PCHD SYSTEMS (I-PCHD SYSTEMS)

To extend the PCHD approach from the finite- to the infinite-dimensional case, we have to replace the state manifold $\mathcal{M}$, its tangent bundle $\mathcal{T}(\mathcal{M})$, its cotangent bundle $\mathcal{T}^*(\mathcal{M})$ and $C^\infty(\mathcal{M})$ by new spaces. Furthermore, the Hamiltonian H_0, the maps J, R, B and the exterior derivative d have to be substituted by new functions and operators.

[1] The introduced Hamilton vector field is no vector field on $\mathcal{M}$ because of its dependence on the input u. In fact it is a submanifold of $\mathcal{T}(\mathcal{M})$ parametrized by u.

A. Geometrical structure of I-PCHD systems

First we introduce the bounded base manifold $\mathcal{D}$ with local coordinates $\left(X^i\right)$, $i=1,\ldots,p$. Commonly these coordinates will represent the independent spatial coordinates according to the analyzed plant. Additionally let $(\mathcal{E},\pi,\mathcal{D})$ be the state bundle with local coordinates $\left(X^i,x^\alpha\right)$, $\alpha=1,\ldots,q$, where x^α represents the dependent coordinates. From $\mathcal{E}$ we derive four important structures. The n^{th} jet manifold $J^n(\mathcal{E})$ with adapted coordinates $\left(X^i,x^\alpha,x^\alpha_{[J]}\right)$, the vertical tangent bundle $\mathcal{V}(\mathcal{E})$ with coordinates $\left(X^i,x^\alpha,\dot{x}^\alpha\right)$, and the exterior bundles $\wedge^0_p(\mathcal{T}^*(\mathcal{E}))=\operatorname{span}\{\mathrm{dX}\}$, $\wedge^1_p(\mathcal{T}^*(\mathcal{E}))=\operatorname{span}\{\mathrm{d}x^\alpha\wedge\mathrm{dX}\}$ with coordinates $\left(X^i,x^\alpha,r\right)$, $\left(X^i,x^\alpha,\dot{r}_\alpha\right)$ and the volume form $\mathrm{dX}=\mathrm{d}X^1\wedge\cdots\wedge\mathrm{d}X^p$. The interior product $\dot{x}^\alpha\partial_\alpha\rfloor\dot{r}_\alpha\mathrm{d}x^\alpha\wedge\mathrm{dX}=\dot{x}^\alpha\dot{r}_\alpha\mathrm{dX}$ induces the canonical product $\mathcal{V}(\mathcal{E})\times\wedge^1_p(\mathcal{T}^*(\mathcal{E}))\rightarrow\wedge^0_p(\mathcal{T}^*(\mathcal{E}))$. Now, we replace $\mathcal{T}(\mathcal{M})$, $\mathcal{T}^*(\mathcal{M})$, $C^\infty(\mathcal{M})$ of the Section III by $\mathcal{V}(\mathcal{E})$, $\wedge^1_p(\mathcal{T}^*(\mathcal{E}))$, $\wedge^0_p(\mathcal{T}^*(\mathcal{E}))$ and introduce the first order Hamiltonian density $H_0\mathrm{dX}$, $H_0\in C^\infty\left(J^1(\mathcal{E})\right)$. With the n^{th} jet bundle $(J^n(\mathcal{E}),\pi^n_0,\mathcal{E})$ we see that $H_0\mathrm{dX}\in\pi_0^{1,*}\left(\wedge^0_p(\mathcal{T}^*(\mathcal{E}))\right)$ (see[2]) is met.

Now, we replace the exterior derivative of the Section III by the variational derivative $\delta:\wedge^0_p(\mathcal{T}^*(\mathcal{E}))\rightarrow\wedge^1_p(\mathcal{T}^*(\mathcal{E}))$, and substitute the tensors J,R by suitable maps $\mathfrak{J},\mathfrak{R}:\wedge^1_p(\mathcal{T}^*(\mathcal{E}))\rightarrow\mathcal{V}(\mathcal{E})$, which are differential operators (see [4]) in general. As input space we choose a vector bundle $(\mathcal{U},\pi_\mathcal{U},\mathcal{D})$ with local coordinates $\left(X^i,u^\varsigma\right)$, $\varsigma=1,\ldots,m$ and basis $\{e_\varsigma\}$. Of course, the output space $\mathcal{Y}=\mathcal{U}^*$ is given by the dual vector bundle, where we use the coordinates $\left(X^i,y_\varsigma\right)$ and the basis $\{e^\varsigma\otimes\mathrm{dX}\}$. Furthermore, we conclude that there exists a bilinear map $\mathcal{U}\times_\mathcal{D}\mathcal{Y}\rightarrow\wedge^0_p(\mathcal{T}(\mathcal{E}))$ given by $u^\varsigma e_\varsigma\rfloor y_\varsigma e^\varsigma\otimes\mathrm{dX}$. With the input map $\mathfrak{B}:\mathcal{U}\rightarrow\mathcal{V}(\mathcal{E})$, $u^\varsigma e_\varsigma\rightarrow u^\varsigma e_\varsigma\rfloor B^\alpha_\varsigma e^\varsigma\otimes\partial_\alpha$ we propose the structure of an infinite-dimensional port controlled Hamiltonian system by

$$\dot{x} = (\mathfrak{J}-\mathfrak{R})(\delta(H_0\mathrm{dX}))+\mathfrak{B}(u) \quad (7)$$
$$y = \mathfrak{B}^*(\delta(H_0\mathrm{dX}))\ , \quad (8)$$

where $\mathfrak{B}^*$ denotes the adjoint map $\mathfrak{B}^*:\wedge^1_p(\mathcal{T}^*(\mathcal{E}))\rightarrow\mathcal{Y}$, $\delta_\beta H_0\mathrm{d}x^\beta\wedge\mathrm{dX}\rightarrow B^\alpha_\varsigma e^\varsigma\otimes\partial_\alpha\rfloor\delta_\beta H_0\mathrm{d}x^\beta\wedge\mathrm{dX}$. Here we confine ourselves to the case, where $\mathfrak{J},\mathfrak{R},\mathfrak{B}$ are linear maps and thus no differential operators. The map $\mathfrak{J}$ is assumed to be skew symmetric i.e. $\mathfrak{J}^{\alpha\beta}=-\mathfrak{J}^{\beta\alpha}$ and $\mathfrak{R}$ to be a symmetric positive semidefinite map.

B. Infinite-dimensional Hamilton vector field and collocation

Let the π-vertical vector field[3] $v_H=\dot{x}^\alpha\partial_\alpha$ with $\dot{x}^\alpha$ from (7) denote the Hamilton vector field. Thus we are able to depict v_H in Fig.2, whereby its fibre preserving property is stressed.

[2]Here $\pi_0^{1,*}(\wedge^0_p(\mathcal{T}^*(\mathcal{E})))$ denotes the pullback bundle of $\wedge^0_p(\mathcal{T}^*(\mathcal{E}))$ by the map $\pi_0^{1,*}$. In the following several bundles are pull back bundles by the maps π^n_0. To keep the notation as simple as possible, we will suppress the pull back, whenever it is clear from the context.

[3]Again, this field is not a vector field, but a submanifold of $\mathcal{V}(\mathcal{E})$ parametrized in u and $x^\alpha_{[J]}$, $\#J>0$.

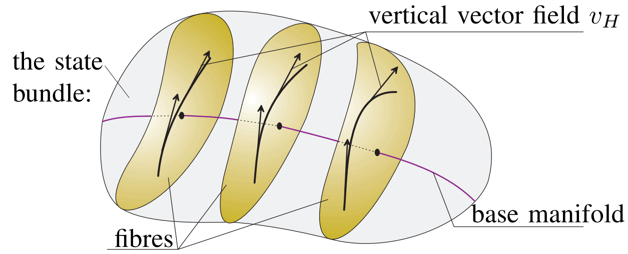

Fig. 2. The π-vertical Hamiltonian vector field v_H (for $u=u(X^i,x^\alpha,x^\alpha_{[J]})$).

The central object of interest along the solution of an I-PCHD system is the Hamiltonian functional $\mathcal{H}(\sigma)=\int_\mathcal{D}\left(j^1\sigma\right)^*(H_0\mathrm{d}X)$, whereat the first prolongation of the section $\sigma\in\Gamma(\mathcal{E})$ is applied.

The time derivative of $\mathcal{H}$ along the solution of the corresponding I-PCHD system leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}=\int_\mathcal{D}\left(j^2\sigma\right)^*\left(j^1(v_H)(H_0\mathrm{dX})\right), \quad (9)$$

where the first prolongation of the Hamilton vector field $j^1(v_H)$ has to be introduced. Fortunately it is possible to apply the identity

$$\mathrm{d}(H_0\mathrm{dX})=\delta_\alpha H_0\mathrm{d}x^\alpha\wedge\mathrm{dX}-\mathrm{d}_h\left(\partial_\alpha^{[1_i]}(H_0)\,\mathrm{d}x^\alpha\wedge\partial_i\rfloor\mathrm{d}X\right) \quad (10)$$

in (9) and consequently it follows from (3) and (1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}=-\int_\mathcal{D}\left(j^2\sigma\right)^*(\mathfrak{R}(\delta(H_0\mathrm{dX}))\rfloor\delta(H_0\mathrm{dX})+(u^\varsigma y_\varsigma)\,\mathrm{dX})$$
$$+\int_{\partial\mathcal{D}}\iota^*\left(\left(j^2\sigma_\partial\right)^*\left(v_H\rfloor\partial_\alpha^{[1_i]}(H_0)\,\mathrm{d}x^\alpha\wedge\partial_i\rfloor\mathrm{dX}\right)\right) \quad (11)$$

is met. It is worth mentioning that equation (10) defines the splitting of the exterior derivative in a variational derivative and an exact form with respect to d_h. Additionally it justifies the previous definition of the variational derivative. Equation (11) states, that the dissipative operator $\mathfrak{R}$, the pairing $u^\varsigma y_\varsigma$ on the domain, and the boundary term

$$\int_{\partial\mathcal{D}}\iota^*\left(\left(j^2\sigma\right)^*\left(v_H\rfloor\partial_\alpha^{[1_i]}(H_0)\,\mathrm{d}x^\alpha\wedge\partial_i\rfloor\mathrm{dX}\right)\right) \quad (12)$$
$$=\int_{\partial\mathcal{D}}\left(j^2\sigma_\partial\right)^*\underbrace{\left(\left(\dot{x}^\alpha\circ\iota\right)\partial_\alpha\rfloor\left(\partial_\alpha^{[1_p]}(H_0)\circ\iota\right)\mathrm{d}x^\alpha\wedge\mathrm{dX}_\partial\right)}_{\lambda_\partial}$$

determine the time derivative of the Hamiltonian functional $\mathcal{H}$. Here the boundary section $\sigma_\partial:\partial\mathcal{D}\rightarrow\iota^*\mathcal{E}$ and the $p-1$ boundary volume form $\mathrm{dX}_\partial=\partial_p\rfloor\mathrm{dX}=\mathrm{d}X^1_\partial\wedge\ldots\wedge\mathrm{d}X^{p-1}_\partial$ are introduced[4].

The form λ_∂ stated in equation (12) is now assumed to equal the natural pairing of the boundary in- and outputs

$$\lambda_\partial=u_\partial{}^\gamma\,y_{\partial\gamma}\,\mathrm{dX}_\partial=\bar{y}_\partial{}^\gamma\,\bar{u}_{\partial\gamma}\,\mathrm{dX}_\partial\ .$$

In contrary to the determination procedure of the collocated output y on the domain, as stated in equation (8), it is no

[4]Here the inclusion map ι is assumed to be given by $\iota:(X_\partial{}^j)\rightarrow(X^j=X_\partial{}^j,X^p=\text{const.})$, $j=1,\ldots,p-1$.

more possible to give a unique separation of the in- and output variables at the boundary (visualized by the use of (u_∂, y_∂) and $(\bar{y}_\partial, \bar{u}_\partial)$). In order to overcome this problem we investigate two cases of boundary pairings on vector bundles.

The first pair is given by the boundary input vector bundle $(\mathcal{U}_\partial, \pi_{\mathcal{U}_\partial}, \partial\mathcal{D})$ with local coordinates $\left(X_\partial{}^j, u_\partial{}^\gamma \right)$, $j = 1, \ldots, (p-1)$, $\gamma = 1, \ldots, m_\partial$ and the basis $\left\{ e_\partial{}_\gamma \right\}$ and its dual – the boundary output vector bundle $\left(\mathcal{Y}_\partial, \eta_{\mathcal{Y}_\partial}, \partial\mathcal{D} \right)$ with local coordinates $\left(X_\partial{}^j, y_\partial{}_\gamma \right)$ and basis $\{ e_\partial{}^\gamma \otimes \mathrm{dX}_\partial \}$. The second pair is given by the boundary input vector bundle $\left(\bar{\mathcal{U}}_\partial, \pi_{\bar{\mathcal{U}}_\partial}, \partial\mathcal{D} \right)$ with local coordinates $\left(X_\partial{}^j, \bar{u}_\partial{}_\gamma \right)$, $j = 1, \ldots, (p-1)$, $\gamma = 1, \ldots, \bar{m}_\partial$ and the basis $\{ \bar{e}_\partial{}^\gamma \}$ and its dual – the boundary output vector bundle $\left(\bar{\mathcal{Y}}_\partial, \eta_{\bar{\mathcal{Y}}_\partial}, \partial\mathcal{D} \right)$ with local coordinates $\left(X_\partial{}^j, \bar{y}_\partial{}^\gamma \right)$ and basis $\left\{ \mathrm{dX}_\partial \otimes \bar{e}_\partial{}_\gamma \right\}$.

At first we consider the bundle pairing $\mathcal{U}_\partial$ and $\mathcal{Y}_\partial$ and formulate the boundary input map $\mathfrak{B}_\partial$ to determine the vector part of λ_∂ by

$$
\begin{aligned}
\mathfrak{B}_\partial \left(u_\partial \right) &= u_\partial{}^\gamma \, e_\partial{}_\gamma \rfloor \, B_\partial{}_\zeta^\alpha \, e_\partial{}^\zeta \otimes \partial_\alpha \\
&= \left(\dot{x}^\alpha \circ \iota \right) \partial_\alpha \, .
\end{aligned}
$$

Consequently we can reformulate λ_∂ and obtain

$$
\lambda_\partial = u_\partial{}^\gamma \, e_\partial{}_\gamma \rfloor \, B_\partial{}_\zeta^\alpha \, e_\partial{}^\zeta \otimes \partial_\alpha \rfloor \left(\left(\partial_\alpha^{[1_p]} H_0 \circ \iota \right) \mathrm{d}x^\alpha \wedge \mathrm{dX}_\partial \right) \, .
$$

This leads directly to the adjoint map given by

$$
\begin{aligned}
\mathfrak{B}_\partial^* \left(\partial_\alpha^r H_0 \circ \iota \right) &= B_\partial{}_\zeta^\alpha \, e_\partial{}^\zeta \otimes \partial_\alpha \rfloor \left(\left(\partial_\alpha^{[1_p]} H_0 \circ \iota \right) \mathrm{d}x^\alpha \wedge \mathrm{dX}_\partial \right) \\
&= y_\partial{}_\zeta \, e_\partial{}^\zeta \otimes \mathrm{dX}_\partial \, .
\end{aligned}
$$

We see that this port configuration is fully defined by the tensor $B_\partial{}_\zeta^\alpha \, e_\partial{}^\zeta \otimes \partial_\alpha$. Now we consider the bundles $\bar{\mathcal{U}}_\partial$, $\bar{\mathcal{Y}}_\partial$ and formulate the boundary input $\bar{\mathfrak{B}}_\partial$ map to determine the form part of λ_∂ by

$$
\begin{aligned}
\bar{\mathfrak{B}}_\partial \left(\bar{u}_\partial \right) &= \bar{B}_\partial{}_\alpha^\gamma \, \mathrm{d}x^\alpha \wedge \mathrm{dX}_\partial \otimes \bar{e}_\partial{}_\gamma \rfloor \, \bar{u}_\partial{}_\zeta \, \bar{e}_\partial{}^\zeta \\
&= \left(\partial_\alpha^{[1_p]} H_0 \circ \iota \right) \, \mathrm{d}x^\alpha \wedge \mathrm{dX}_\partial \, .
\end{aligned}
$$

This definition of the input map results in

$$
\lambda_\partial = \left(\dot{x}^\alpha \circ \iota \right) \partial_\alpha \rfloor \, \bar{B}_\partial{}_\alpha^\gamma \, \mathrm{d}x^\alpha \wedge \mathrm{dX}_\partial \otimes \bar{e}_\partial{}_\gamma \rfloor \, \bar{u}_\partial{}_\zeta \, \bar{e}_\partial{}^\zeta
$$

and consequently the adjoint map is given by

$$
\begin{aligned}
\bar{\mathfrak{B}}_\partial^* \left(\dot{x}^\alpha \circ \iota \right) &= \left(\dot{x}^\alpha \circ \iota \right) \partial_\alpha \rfloor \, \bar{B}_\partial{}_\alpha^\gamma \, \mathrm{d}x^\alpha \wedge \mathrm{dX}_\partial \otimes \bar{e}_\partial{}_\gamma \\
&= \bar{y}_\partial{}^\gamma \, \mathrm{dX}_\partial \otimes \bar{e}_\partial{}_\gamma \, .
\end{aligned}
$$

We see that this port configuration is purely defined by the tensor $\bar{B}_\partial{}_\alpha^\gamma \, \mathrm{d}x^\alpha \wedge \mathrm{dX}_\partial \otimes e_\partial{}_\gamma$.

Remark 1: If one vector or form part of λ_∂ vanishes i.e. $\dot{x}^\alpha \circ \iota = 0$ or $\partial_\alpha^r H_0 \circ \iota = 0$ for a certain α, then the corresponding pairing does not represent a port anymore. Consequently the tensor entries $B_\partial{}_\zeta^\alpha$ resp $\bar{B}_\partial{}_\alpha^\gamma$ do not exist – these entries must not be set to zero, as this could violate the equations of motion.

Now we are able to conclude, that a first order I-PCHD system is given by the domain equations (7), (8), and the boundary equations

$$
\begin{aligned}
\dot{x}^\alpha \circ \iota &= \mathfrak{B}_\partial \left(u_\partial \right) \\
y_\partial &= \mathfrak{B}_\partial^* \left(\partial_\alpha^{[1_p]} H_0 \circ \iota \right)
\end{aligned}
\quad \text{resp.} \quad
\begin{aligned}
\partial_\alpha^{[1_p]} H_0 \circ \iota &= \bar{\mathfrak{B}}_\partial \left(\bar{u}_\partial \right) \\
\bar{y}_\partial &= \bar{\mathfrak{B}}_\partial^* \left(\dot{x}^\alpha \circ \iota \right) .
\end{aligned}
$$

Consequently the attached boundary systems determine, whether the first or the second case of boundary conditions apply for the I-PCHD system. This behavior of infinite-dimensional systems is well known in mechanics and there referred to as dynamical and geometrical boundary conditions.

The introduced representation of I-PCHD systems uses also an envelope of linear spaces built by the vector bundles $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{U}_\partial$, $\mathcal{Y}_\partial$, $\bar{\mathcal{U}}_\partial$, $\bar{\mathcal{Y}}_\partial$. Thus we are again able to give a representative illustration of an I-PCHD system depicted in Fig. 3. Another consequence of the proposed structures is

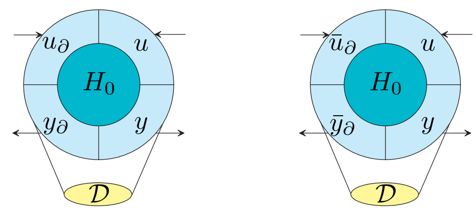

Fig. 3. I-PCHD systems with 1st order Hamiltonian.

that F-PCHD and I-PCHD systems cannot be subdivided in several PCHD subsystems in general, because one must be able to introduce subsystems interacting through linear spaces.

One of the most important properties of F-PCHD systems is their structural invariance with respect to power conserving interconnections. Thus we investigate in the subsequent section the behavior of I-PCHD systems with respect to domain and boundary interconnections.

C. Interconnection of I-PCHD systems

In the following the I-PCHD systems, which are generated by two interconnected I-PCHD systems, are formulated on a product bundle $(\mathcal{E}_1 \times \mathcal{E}_2, \pi_{\mathcal{E}_1} \times \pi_{\mathcal{E}_2}, \mathcal{D}_1 \times \mathcal{D}_2)$. We will investigate three different cases of interconnection – domain ⇔ domain, boundary ⇔ boundary, and boundary ⇔ domain. The considered systems are defined by

$$
\begin{aligned}
\dot{x}_1{}^\alpha &= \left(J_1 - R_1 \right)^{\alpha\beta} \, \delta_\beta H_{01} + u_1{}^\varsigma \, B_1{}_\varsigma^\alpha \\
y_1{}_\varsigma &= B_1{}_\varsigma^\beta \, \delta_\beta H_{01} \, ,
\end{aligned}
$$

with boundary condition

$$
B_{\partial 1}{}_\gamma^\alpha \, u_{\partial 1}{}^\gamma = \dot{x}_1{}^\alpha \circ \iota_1 \, , \quad y_{\partial 1}{}_\gamma = B_{\partial 1}{}_\gamma^\alpha \, \left(\partial_\alpha^{[1_p]} H_{01} \circ \iota_1 \right)
$$

and

$$
\begin{aligned}
\dot{x}_2{}^\alpha &= \left(J_2 - R_2 \right)^{\alpha\beta} \, \delta_\beta H_{02} + u_2{}^\varsigma \, B_2{}_\varsigma^\alpha \\
y_2{}_\varsigma &= B_2{}_\varsigma^\beta \, \delta_\beta H_{02} \, ,
\end{aligned}
$$

with boundary condition

$$
\bar{B}_{\partial 2}{}_\alpha^\gamma \, u_{\partial 2}{}_\gamma = \partial_\alpha^{[1_p]} H_{02} \circ \iota_2 \, , \quad y_{\partial 2}{}^\gamma = \bar{B}_{\partial 2}{}_\alpha^\gamma \, \left(\dot{x}_2{}^\alpha \circ \iota_2 \right) .
$$

It is worth mentioning, that one could also introduce the boundary maps $\bar{\mathfrak{B}}_{\partial 1}$ and $\mathfrak{B}_{\partial 2}$ in system 1 resp. 2. In all three cases the systems are linked by a I-PCH system without dynamics defined by

$$
\begin{bmatrix} y_{I1} \\ y_{I2} \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} \begin{bmatrix} u_{I1} \\ u_{I2} \end{bmatrix}
$$

with $I_{11} = -I_{11}^{\intercal}$, $I_{22} = -I_{22}^{\intercal}$, $I_{21} = -I_{12}^{\intercal}$. This system belongs to the class of power-conserving interconnections. The time derivative of the interconnected Hamiltonian functional

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{12} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{D}_1} H_{01}\mathrm{dX}_1 + \int_{\mathcal{D}_2} H_{02}\mathrm{dX}_2\right)$$

will be analyzed for all considered interconnections.

1) The domain⇔domain interconnection: This interconnection represents the case, where the domain of two I-PCHD systems coincides, see Fig. 4. Thus the introduced product bundle reduces to the special case of a fibred product bundle [5]. The domain inputs are here given by

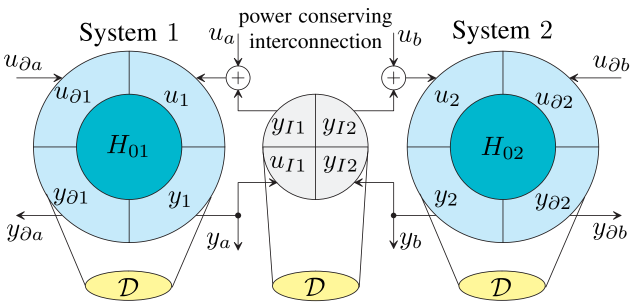

Fig. 4. The domain⇔domain interconnection

$$\begin{aligned}
u_1{}^{\varsigma} &= u_a{}^{\varsigma} + I_{11}{}^{\varsigma\alpha}\, B_1{}^{\beta}_{\alpha}\ \delta_\beta H_{01} + I_{12}{}^{\varsigma\alpha}\, B_2{}^{\beta}_{\alpha}\ \delta_\beta H_{02}\\
u_2{}^{\varsigma} &= u_b{}^{\varsigma} + I_{21}{}^{\varsigma\alpha}\, B_1{}^{\beta}_{\alpha}\ \delta_\beta H_{01} + I_{22}{}^{\varsigma\alpha}\, B_2{}^{\beta}_{\alpha}\ \delta_\beta H_{02}
\end{aligned}$$

and the boundary inputs are denoted by $u_{\partial 1} = u_{\partial a}$, $u_{\partial 2} = u_{\partial b}$. Consequently we end up with an I-PCHD system on $(\mathcal{E}_1 \times_{\mathcal{D}} \mathcal{E}_2, \pi_{\mathcal{E}_1\times_{\mathcal{D}}\mathcal{E}_2}, \mathcal{D})$ defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = (\mathfrak{J} - \mathfrak{R})\begin{bmatrix} \delta_\beta H_{01} \\ \delta_\beta H_{02} \end{bmatrix} + \mathfrak{B}\begin{bmatrix} u_a \\ u_b \end{bmatrix}$$

where

$$\begin{aligned}
\mathfrak{J} &= \begin{bmatrix} \mathfrak{J}_1 + \mathfrak{B}_1 I_{11}\mathfrak{B}_2 & \mathfrak{B}_1 I_{12}\mathfrak{B}_2 \\ \mathfrak{B}_2 I_{21}\mathfrak{B}_1 & \mathfrak{J}_2 + \mathfrak{B}_2 I_{22}\mathfrak{B}_2 \end{bmatrix}\\
\mathfrak{R} &= \begin{bmatrix} \mathfrak{R}_1 & 0 \\ 0 & \mathfrak{R}_2 \end{bmatrix},\ \mathfrak{B} = \begin{bmatrix} \mathfrak{B}_1 & 0 \\ 0 & \mathfrak{B}_2 \end{bmatrix}
\end{aligned}$$

is used and the collocated outputs are

$$\begin{aligned}
y_1 &= \mathfrak{B}_1{}^{*}\left(\delta\left(H_{01}\mathrm{dX}\right)\right) = y_a,\\
y_2 &= \mathfrak{B}_2{}^{*}\left(\delta\left(H_{02}\mathrm{dX}\right)\right) = y_b.
\end{aligned}$$

Thus the domain⇔domain interconnection preserves the structure of an I-PCHD system. The time derivative of the interconnected Hamiltonian functional leads to similar results as already shown in equation (11) for the general case.

2) The boundary⇔boundary interconnection: This interconnection represents the case, where two I-PCHD Systems are interconnected on a common boundary $\partial\mathcal{D}_{12}$ defined by $\partial\mathcal{D}_1 \supset \partial\mathcal{D}_{12} \subset \partial\mathcal{D}_2$. The boundary inputs on $\partial\mathcal{D}_{12}$ are in this case given by

$$\begin{aligned}
u_{1\partial}{}^{\zeta} &= u_{\partial a}{}^{\zeta} + I_{11}{}^{\zeta\gamma}\, y_{\partial 1\,\gamma} + I_{12}{}^{\zeta}_{\gamma}\, y_{\partial 2}{}^{\gamma}\\
u_{2\partial\,\zeta} &= u_{\partial b\,\zeta} + I_{21}{}^{\gamma}_{\zeta}\, y_{\partial 1\,\gamma} + I_{22\,\zeta\gamma}\, y_{\partial 2}{}^{\gamma}
\end{aligned}$$

leading to

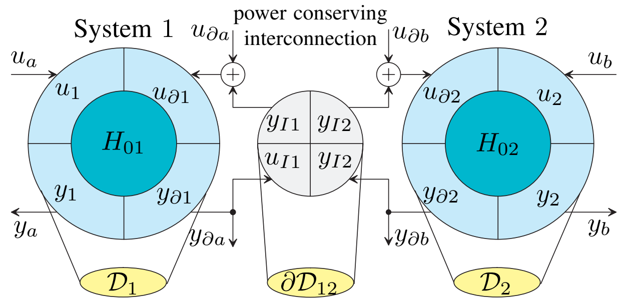

Fig. 5. The boundary⇔boundary interconnection

$$\begin{aligned}
u_{1\partial}{}^{\zeta} &= u_{\partial a}{}^{\zeta} + I_{11}{}^{\zeta\gamma} B_{\partial 1}{}^{\alpha}_{\gamma}\left(\partial_\alpha^{[1_p]} H_{01}\circ\iota_1\right) + I_{12}{}^{\zeta}_{\gamma}\bar{B}_{\partial 2}{}^{\gamma}_{\alpha}(\dot{x}_2{}^{\alpha}\circ\iota_2)\\
u_{2\partial\,\zeta} &= u_{\partial b\,\zeta} + I_{21}{}^{\gamma}_{\zeta} B_{\partial 1}{}^{\alpha}_{\gamma}\left(\partial_\alpha^{[1_p]} H_{01}\circ\iota_1\right) + I_{22\,\zeta\gamma}\bar{B}_{\partial 2}{}^{\gamma}_{\alpha}(\dot{x}_2{}^{\alpha}\circ\iota_2).
\end{aligned}$$

The time derivative of the interconnected Hamiltonian equals the sum of the individual derivatives except the $\partial\mathcal{D}_{12}$-part. Here $\int_{\partial\mathcal{D}_{12}} y_{\partial 1\,\gamma}\, u_{\partial 1}{}^{\gamma} + y_{\partial 2}{}^{\gamma}\, u_{\partial 2\,\gamma}\, \mathrm{dX}_\partial$ has to be analyzed. We get

$$\begin{aligned}
&\int_{\partial\mathcal{D}_{12}}\left(\partial_\omega^{[1_p]} H_{01}\circ\iota_1\right) B_{\partial 1}{}^{\omega}_{\zeta}\left(u_{\partial a}{}^{\zeta} + I_{12}{}^{\zeta}_{\gamma}\bar{B}_{\partial 2}{}^{\gamma}_{\alpha}(\dot{x}_2{}^{\alpha}\circ\iota_2)\right)\mathrm{dX}_\partial\\
+&\int_{\partial\mathcal{D}_{12}}(\dot{x}_2{}^{\omega}\circ\iota_2)\bar{B}_{\partial 2}{}^{\zeta}_{\omega}\left(u_{\partial b\,\zeta} + I_{21}{}^{\gamma}_{\zeta} B_{\partial 1}{}^{\alpha}_{\gamma}\left(\partial_\alpha^{[1_p]} H_{01}\circ\iota_1\right)\right)\mathrm{dX}_\partial.
\end{aligned}$$

Because of the condition $I_{12}{}^{\zeta}_{\gamma} = -I_{21}{}^{\gamma}_{\zeta}$ this integral simplifies to $\int_{\partial\mathcal{D}_{12}}\left(y_{\partial 1\,\zeta}\, u_{\partial a}{}^{\zeta} + y_{\partial 2}{}^{\zeta}\, u_{\partial b\,\zeta}\right)\mathrm{dX}_\partial$. Consequently the time derivative of the interconnected Hamiltonian functional caused on $\partial\mathcal{D}_{12}$ is purely determined by the collocation of $u_{\partial a}$ and $u_{\partial b}$ with $y_{\partial 1}$ and $y_{\partial 2}$. It is worth mentioning, that this is a simple consequence of the power-conserving interconnection.

3) The boundary⇔domain interconnection: This interconnection represents the combination of a p-dimensional I-PCHD system i.e. $\dim(\mathcal{D}_1) = p$ with a $(p-1)$-dimension system i.e. $\dim(\mathcal{D}_2) = p - 1$ along $\partial\mathcal{D}_{12}$ defined by $\partial\mathcal{D}_1 \supset \partial\mathcal{D}_{12} \subset \mathcal{D}_2$. Consequently the inputs of the systems

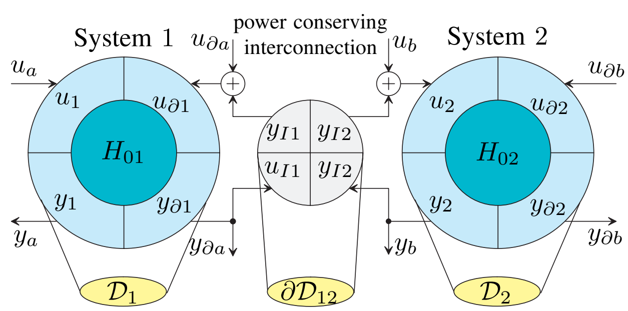

Fig. 6. The boundary⇔domain interconnection

are given by

$$\begin{aligned}
u_{\partial 1}{}^{\zeta} &= u_{\partial a}{}^{\zeta} + I_{11}{}^{\zeta\gamma} B_{\partial 1}{}^{\alpha}_{\gamma}\left(\partial_\alpha^{[1_p]} H_{01}\circ\iota_1\right) + I_{12}{}^{\zeta\gamma} B_2{}^{\beta}_{\gamma}\,\delta_\beta H_{02}\\
u_2{}^{\zeta} &= u_b{}^{\zeta} + I_{21}{}^{\zeta\gamma} B_{\partial 1}{}^{\alpha}_{\gamma}\left(\partial_\alpha^{[1_p]} H_{01}\circ\iota_1\right) + I_{22}{}^{\zeta\gamma} B_2{}^{\beta}_{\gamma}\,\delta_\beta H_{02}
\end{aligned}$$

The time derivative of $\mathcal{H}_{12}$ is again given by the sum of the individual derivatives except the $\partial\mathcal{D}_{12}$-part, plus the result of $\int_{\partial\mathcal{D}_{12}}\left(y_{\partial 1\,\gamma}\, u_{\partial 1}{}^{\gamma} + y_{2\,\varsigma}\, u_2{}^{\varsigma}\right)\mathrm{dX}_\partial$. This integral leads to

$$\int_{\partial\mathcal{D}_{12}}\left(\partial_\omega^{[1_p]}H_{01}\circ\iota_1\right)B_{\partial 1}{}^{\omega}_{\zeta}\left(u_{\partial a}{}^{\zeta}+I_{12}{}^{\zeta\gamma}B_2{}^{\beta}_{\gamma}\,\delta_\beta H_{02}\right)\mathrm{dX}_\partial$$
$$+\int_{\partial\mathcal{D}_{12}}\delta_\omega H_{02}B_2{}^{\omega}_{\zeta}\left(u_b{}^{\zeta}+I_{21}{}^{\zeta\gamma}B_{\partial 1}{}^{\alpha}_{\gamma}\left(\partial_\alpha^{[1_p]}H_{01}\circ\iota_1\right)\right)\mathrm{dX}_\partial$$

Once again the condition $I_{12}{}^{\zeta\gamma}=-\,I_{21}{}^{\zeta\gamma}$ simplifies this integral to

$$\int_{\partial\mathcal{D}_{12}}\left(y_{\partial 1}{}_{\gamma}\,u_{\partial a}{}^{\gamma}+y_{2}{}_{\varsigma}\,u_b{}^{\varsigma}\right)\mathrm{dX}_\partial. \qquad (13)$$

The time evolution of $\mathcal{H}_{12}$ is consequently determined by the individual dampings $\mathfrak{R}_1$, $\mathfrak{R}_2$ on $\mathcal{D}_1$, $\mathcal{D}_2$, the pairings $y_1\rfloor u_1$ on $\mathcal{D}_1$, $y_2\rfloor u_2$ on $\mathcal{D}_2-\partial\mathcal{D}_{12}$, $y_{\partial 1}\rfloor u_{\partial 1}$ on $\partial\mathcal{D}_1-\partial\mathcal{D}_{12}$, $y_{\partial 2}\rfloor u_{\partial 2}$ on $\partial\mathcal{D}_2$, and the quantity in equation (13).

V. APPLICATION

In this section we will investigate a mechanical structure, whose infinite-dimensional components can be modelled using the introduced I-PCHD description. The considered construction consists of a rectangular undamped membrane and an attached undamped string. The proposed interconnection of this systems is shown in Fig. 7. In the mathematical

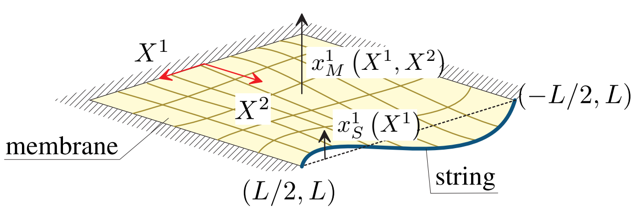

Fig. 7. The membrane-string interconnection

modelling, we assume that for both components only small vertical displacements $x_M^1\left(X^1,X^2\right)$, $x_S^1\left(X^2\right)$ appear. Consequently we are able to formulate the stored energy density of the membrane [7]

$$e_{PM}=\frac{S_M}{2}\left(\left(x^1_{M[10]}\right)^2+\left(x^1_{M[01]}\right)^2\right)\mathrm{d}X^1\wedge\mathrm{d}X^2.$$

Here the constant membrane tension S_M is introduced. Similarly we are able to define the potential energy of the string $e_{PS}=\frac{S_S}{2}\left(x^1_{S[1]}\right)^2\mathrm{d}X^1$ with the string tension S_S. The kinetic energy is given by $e_{KM}=\frac{1}{2\rho_M}\left(x_M^2\right)^2\mathrm{d}X^1\wedge\mathrm{d}X^2$ respectively $e_{KS}=\frac{1}{2\rho_S}\left(x_S^2\right)^2\mathrm{d}X^1$, where the constant mass per unit area ρ_M and mass per unit length ρ_S are used. The Hamiltonian densities $H_{0M}=\left(\frac{S_M\;\left(x^1_{M[10]}\right)^2+\left(x^1_{M[01]}\right)^2}{2}+\frac{\left(x_M^2\right)^2}{2\rho_M}\right)\mathrm{d}X^1\wedge\mathrm{d}X^2,\;H_{0S}=\left(\frac{S_S}{2}\left(x^1_{S[1]}\right)^2+\frac{1}{2\rho_S}\left(x_S^2\right)^2\right)\mathrm{d}X^1$ can now be used to define the corresponding I-PCHD representations. The membrane is described by

$$\begin{bmatrix}\dot{x}_M^1\\ \dot{x}_M^2\end{bmatrix}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}\delta_{x_M^1}H_{0M}\\ \delta_{x_M^2}H_{0M}\end{bmatrix}=\begin{bmatrix}\frac{1}{\rho_M}x_M^2\\ S_M\left(x^1_{M[20]}+x^1_{M[02]}\right)\end{bmatrix}$$

with the boundary conditions

$$\bar{B}_{\partial M}{}^{1}_{1}\,u_{\partial M}{}^{1}=\dot{x}_M^1\circ\iota_M=\frac{1}{\rho_M}x_M^2\circ\iota_M$$
$$y_{\partial M}{}_{1}=\bar{B}_{\partial M}{}^{1}_{1}\partial_1^{[01]}H_{0M}\circ\iota_M=\bar{B}_{\partial M}{}^{1}_{1}S_M x^1_{M[01]}\circ\iota_M\;,$$

where the inclusion map $\iota_M{:}(X_\partial)\rightarrow\left(X^1=X_\partial,\;X^2=L\right)$ is used. The string is described by

$$\begin{bmatrix}\dot{x}_S^1\\ \dot{x}_S^2\end{bmatrix}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}\delta_{x_S^1}H_{0S}\\ \delta_{x_S^2}H_{0S}\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}u_S=\begin{bmatrix}\frac{1}{\rho_S}x_S^2\\ S_M x^1_{S[2]}+u_S\end{bmatrix}$$

and the boundary conditions $\left(\iota_S\;:\;\left\{-\frac{L}{2},\frac{L}{2}\right\}\rightarrow\left\{X^1=-\frac{L}{2},X^1=\frac{L}{2}\right\}\right)$

$$B_{\partial S}{}^{1}_{1}\,u_{\partial S}{}_{1}\;=\;\partial_1^{[1]}H_{0S}\circ\iota_S=S_S x^1_{S[1]}\circ\iota_S$$
$$y_{\partial S}{}^{1}\;=\;B_{\partial S}{}^{1}_{1}\left(\dot{x}_S^1\circ\iota_S\right)=B_{\partial S}{}^{1}_{1}\left(\frac{1}{\rho_S}x_S^2\circ\iota_S\right)\;.$$

The power conserving interconnection is in this case given by $y_{I1}=u_{I2}$, $y_{I2}=-u_{I1}$, with $u_{I1}=y_{\partial M}{}_{1}$, $u_{I2}=y_{S}{}_{1}$ and $u_{\partial M}{}^{1}=y_{I1}$, $u_S{}^{1}=y_{I2}$. Thus we are able to derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{D}_1}H_{0M}\mathrm{d}X^1\wedge\mathrm{d}X^2+\int_{\mathcal{D}_2}H_{0S}\mathrm{d}X^1\right)=0\;.$$

Here we have taken into account, that $u_{\partial S}{}_{1}=0$ on $\partial\mathcal{D}_2$ and $u_{\partial M}{}^{1}=0$ on $\partial\mathcal{D}_1-\mathcal{D}_2$ due to the restraint support of the membrane as visualized in Fig. 7.

VI. CONCLUSIONS AND FUTURE WORKS

Based on the geometrical representation of finite-dimensional port controlled Hamiltonian systems an extension to infinite-dimensional systems was presented. The introduced mathematical concepts enabled us to define the spaces and mappings used in the proposed Hamiltonian representation. It is worth mentioning, that the stated representation is well known from the literature as e.g. [3]. The central property of port controlled Hamiltonian systems – the behavior with respect to interconnections – is investigated in detail and satisfactory results are achieved. Finally the interconnection of two infinite-dimensional mechanical systems visualizes the applicability of the proposed approach.

Future investigations will extend this approach to the case of higher order Hamiltonian densities $H_0\in C^\infty\left(J^n\left(\mathcal{E}\right)\right)$ in order to be able to handle structures like e.g. Bernoulli-Euler beams, Kirchhoff plates etc. . Additionally the limitation to non-differential operator mappings must be dropped. This will enable us to describe thermodynamics and coupled field problems like thermoelasticity, piezothermoelasticity etc. .

VII. ACKNOWLEDGMENTS

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