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On the blowing-up of stably free behaviours

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***Abstract*— It is well-known that a time-varying controllable ordinary differential linear system is flat outside some singularities. In this paper, we prove that every time-varying controllable linear system is a projection of a flat system. We give an explicit description of a flat system which projects onto a given controllable one. This phenomenon is similar to a classical one largely studied in algebraic geometry and called the *blowing-up* of a singularity. These results simplify the ones obtained in [6] and generalize them to MIMO multidimensional systems. Finally, we prove that every controllable multi-input ordinary differential linear system with polynomial coefficients is flat.**

***Index Terms*— Multidimensional linear systems with varying coefficients, controllability, flatness, singularity, behaviours.**

I. INTRODUCTION AND MOTIVATIONS

A classical question in algebraic geometry is to recognize when it is possible to parametrize the points of a curve by means of rational functions. For instance, it is well-known that the unit circle $x^2 + y^2 - 1 = 0$ can be parametrized by:

$$x(t) = (1 - t^2)/(1 + t^2), \quad y(t) = 2\,t/(1 + t^2), \quad \forall\, t \in \mathbb{R}. \tag{1}$$

Such a parametrization may parametrize all but a finite number of points of the curve ($x = -1$). We also note that the parameter t can be obtained as a rational function of the coordinates of the curve. For the unit circle, we have $t = y/(1 + x)$, $x \neq -1$.

Two historical problems were leading the study of such curves:

1) Study of *Diophantine equations*, i.e., finding the rational solutions of curves. For instance, all rational solutions of $x^2 + y^2 = 1$ are obtained by substituting $t \in \mathbb{Q}$ into (1).
2) *Integral calculus*, i.e., integrating rational functions on a curve. For instance, the integration of the differential form $\omega = (y/(1 + x))\,dx$ on the unit circle gives:

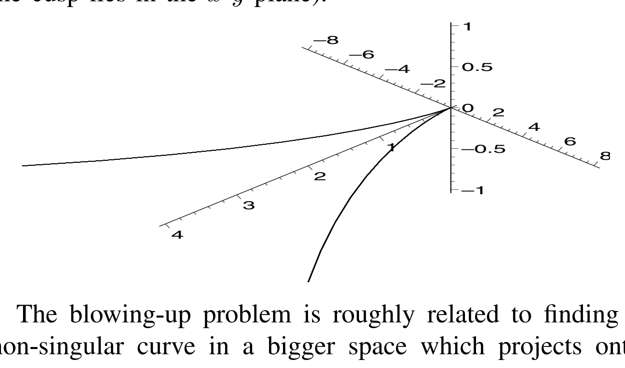
$$\int_0^1 \frac{\sqrt{1-x^2}}{(1+x)}\,dx = -\int_1^0 \frac{4\,t^2}{(1+t^2)^2}\,dt = \frac{\pi}{2} - 1.$$

A natural question is to understand if a class of systems could play a similar role in control theory as the one played by rational curves in algebraic geometry. Indeed, a curve defines an underdetermined system (e.g., one equation in two unknowns). In the nineties, the concept of *flat systems* was introduced in [4] for non-linear control systems defined by means of ordinary differential equations (ODEs). This concept has been extended since to different classes of systems (e.g., systems of differential time-delay equations/multidimensional discrete equations/partial differential equations (PDEs)) and is related to the *Monge problem* which consists in deciding whether or not it is possible to parametrize all solutions of an underdetermined (nonlinear) system of ODEs or PDEs by means of free functions. See [16] for more details and historical developments obtained by J. Hadamard, D. Hilbert, E. Cartan, E. Goursat. When the free functions can in turn be expressed in terms of the system variables, then the system is called flat [4].

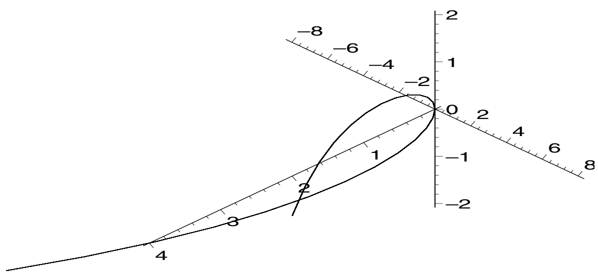
An analogon of the first problem in control theory is the *controllability problem* [9], [10]: Is it possible to patch two sets of trajectories? The main application of flat systems is the *motion planning problem* [4]: Is it possible to design an input which gives a desired output in open-loop? In both cases, the problem is to find a free parameter of the parametrization so that the corresponding trajectory satisfies certain imposed conditions [4], [9], [10].

A control-theoretic version of Problem 2 is the *optimal control problem*. Let us suppose that we want to minimize a quadratic cost under the differential constraint formed by the control system. Then, by substituting the parametrization of the system into the Lagrangian, we obtain an optimization problem without differential constraint whose solutions can be obtained by integrating the corresponding Euler-Lagrange equations and by substituting the result into the parametrization of the system. We refer to [1], [13] for more details.

In algebraic geometry, some curves are *singular* in the sense that their gradient vanishes at some particular points. For instance, the curve defined by $F(x, y) = y^2 - x^3$, called *cusp*, is *singular* at the origin $(0, 0)$ as its gradient defined by $\vec{\nabla} F = (-3\,x^2,\ 2\,y)$ vanishes at $(0, 0)$. We plot the graph of this curve (we are going to see why we use a 3D plot; the cusp lies in the x-y-plane).



The blowing-up problem is roughly related to finding a non-singular curve in a bigger space which projects onto the given singular curve. For the cusp, we consider the relation $y = t\,x$ with a new parameter t. By substituting into $F(x, y) = 0$, we obtain $x = t^2$, and thus, $y = t^3$. If we consider the curve in $\mathbb{R}^3$ defined by $(x,\, y,\, t) = (t^2,\, t^3,\, t)$, we easily show that it is non-singular and its projection onto the x-y-plane is the cusp:



We consider now the *behaviour* $\mathcal{B}$ [2], [10] of a (OD, PD, differential time-delay...) linear system with variable coefficients, namely, the set of solutions of a linear system over an *Ore algebra* D [2] in a *signal space* $\mathcal{F}$ having a left D-module structure.

Problem A: Is it possible to find a flat linear system over D whose behaviour projects onto the behaviour $\mathcal{B}$?

If we ask the projections to be of the form $\pi((\eta_1, \ldots, \eta_p, \ldots, \eta_r)) = (\eta_1, \ldots, \eta_p)$, then we can consider the following new problem:

Problem B: Is it possible to find $r, s \in \mathbb{Z}_+$ such that we have

$$\mathcal{B} \oplus \mathcal{F}^s \cong \mathcal{F}^r, \tag{2}$$

where $\cong$ (resp., $\oplus$) denotes the isomorphism (resp., direct sum) of abelian groups?

In order to motivate Problems A and B, we consider the analytic time-varying OD system $\dot{x}(t) = t\,u(t)$. As the controllability matrix $C(t) = (B(t) = t,\; A(t)\,B(t) - \dot{B}(t) = -1)$ has rank 1 at $t = 0$, we know that the system is controllable at $t = 0$ [15]. But, we shall prove in Example 3 that it is not flat. This result is non-trivial as we must show that there exists no injective parametrization of the system. Intuitively, this result can be understood if we examine the following parametrization:

$$x(t) = \xi(t), \quad u(t) = \dot{\xi}(t)/t, \quad \forall\, \xi \in \mathcal{F}.$$

But, $t = 0$ is a singularity of this parametrization, showing that we cannot deduce the flatness of the system in the neighbourhood of 0. We have the following parametrization of all the system trajectories without singularities

$$\begin{cases} x(t) = t^2\,\xi_1(t) + t\,\dot{\xi}_2(t) - \xi_2(t), \\ u(t) = t\,\dot{\xi}_1(t) + 2\,\xi_1(t) + \ddot{\xi}_2(t), \end{cases} \tag{3}$$

where ξ_1 and ξ_2 are two arbitrary smooth functions [1], [2], [11]. However, we cannot obtain ξ_1 and ξ_2 in terms of x, u and their derivatives as it would imply that the rank of the system, which is equal to the number of inputs, is 2, i.e., (3) is not injective.

It was shown in [6] that a dynamic compensator of the form $\dot{v}(t) = -u(t)$ can be used in order to obtain the following flat system

$$\begin{cases} \dot{x}(t) - t\,u(t) = 0, \\ \dot{v}(t) + u(t) = 0, \end{cases} \Leftrightarrow \begin{cases} x(t) = -t\,\dot{\xi}(t) + \xi(t), \\ u(t) = -\ddot{\xi}(t), \\ v(t) = \dot{\xi}(t), \end{cases}$$

where $\xi(t) = x(t) + t\,v(t)$ is a flat output of the system. For analytic time-varying single-input controllable linear systems, a general algorithm is given in [6] in order to construct the dynamic compensator which allows to obtain a flat system.

Solving Problem B, we show that we can get rid of the dynamic compensator used in [6] by giving a new interpretation of controllability: a controllable time-varying linear system is a projection of a flat system. Then, we give a simple formula to compute a flat system that projects onto the controllable one.

Moreover, we generalize the previous result to multidimensional linear systems over Ore algebras with variable coefficients [2] such as differential time-delay systems, multidimensional discrete systems or PD systems.

Finally, K. B. Datta proposes in [3] that an interesting problem is to extend the results of [6] to analytic time-varying controllable linear systems having multi-inputs. In the case of polynomial coefficients, we prove that this problem is theoretically solved as such linear systems are shown to be flat.

II. A MODULE-THEORETIC APPROACH

Reformulating Problem B within module theory, we show how we can solve it in the different situations we are interested in.

We recall that a ring D is said to be a *domain* if the product of non-zero elements of D is non-zero [14].

Definition 1: 1) D is called a *left noetherian ring* if every left ideal of D is *finitely generated*, namely, generated by a finite number of elements of D.

2) A domain D has the *left Ore property* if for every pair (a_1, a_2) in D^2, there exists a non-trivial pair $(b_1,\, b_2)$ in D^2 satisfying $b_1\,a_1 = b_2\,a_2$.

In what follows, we shall only consider a left noetherian domain D. Then, D has the left Ore property [8].

Let us now consider a matrix $R \in D^{q \times p}$ and the left D-morphism (i.e., left D-linear map) defined by

$$.R : D^{1 \times q} \longrightarrow D^{1 \times p}, \quad (a_1, \ldots, a_q) \longmapsto (a_1, \ldots, a_q)\,R.$$

Then, we define the *cokernel* of the left D-morphism $.R$ as the left D-module $M = D^{1 \times p}/(D^{1 \times q}\,R)$.

In terms of generators and relations, the left D-module M is generated by $z_1, \ldots, z_p$, where z_i denotes the class in M of the row vector e_i defined by 1 in the i^{th} entry and 0 elsewhere, and $z = (z_1, \ldots, z_p)^T$ satisfies the system $R\,z = 0$ and all left D-linear combinations of these equations [2], [12]. As the left D-module M is defined by means of a finite linear system over D, we say that M is a *finitely presented* left D-module [14].

If $\mathcal{F}$ is a left D-module and $\hom_D(M, \mathcal{F})$ denotes the abelian group of left D-morphisms (i.e., left D-linear maps) from M to $\mathcal{F}$, then we have the following standard isomorphism of abelian groups:

$$\mathcal{B} = \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \cong \hom_D(M, \mathcal{F}).$$

In other words, if $\eta = (\eta_1, \ldots, \eta_p)^T$ is an element of the behaviour $\mathcal{B}$, then we can define a unique left D-morphism f of $\hom_D(M, \mathcal{F})$ by $f(z_i) = \eta_i$ for $i = 1, \ldots, p$ [2], [12].

Now, using the following trivial isomorphisms [14]

$$\begin{cases} \hom_D(M \oplus P, \mathcal{F}) \cong \hom_D(M, \mathcal{F}) \oplus \hom_D(P, \mathcal{F}), \\ \hom_D(D^{1\times r}, \mathcal{F}) \cong \mathcal{F}^r, \end{cases}$$

we can write (2) as:

$$\hom_D(M \oplus D^{1\times s}, \mathcal{F}) \cong \hom_D(D^{1\times r}, \mathcal{F}).$$

Therefore, it is natural to consider the following problem:

Problem C: Is it possible to find $r, s \in \mathbb{Z}_+$ such that:

$$M \oplus D^{1\times s} \cong D^{1\times r}. \quad (4)$$

Let us recall a few definitions of module theory [14].

Definition 2: Let M be a finitely generated left module over a left noetherian domain D. Then, M is said to be:

- *free* if there exists $r \in \mathbb{Z}_+$ such that $M \cong D^{1\times r}$,
- *stably free* if there exist two integers $r, s \in \mathbb{Z}_+$ such that we have $M \oplus D^{1\times s} \cong D^{1\times r}$,
- *projective* if there exist $r \in \mathbb{Z}_+$ and a left D-module P such that $M \oplus P \cong D^{1\times r}$,
- *torsion-free* if the left D-submodule $t(M) = \{m \in M \mid \exists\, 0 \neq a \in D,\ a\,m = 0\}$ of M is the zero module.
- *torsion* if $t(M) = M$.

Therefore, we obtain the following lemma.

Lemma 1: Problem C is solvable iff the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is stably free.

It is clear that a free module is stably free (take $s = 0$) and a stably free module is projective (take $P = D^{1\times s}$). Moreover, we can prove that a projective module is torsion-free [14].

Theorem 1:
1) *[14] If $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring over a field k, then every projective D-module is free.*
2) *[8] If D is a* (left) principal ideal domain*, namely, every (left) ideal of D can be generated by means of one element, (e.g., $D = \mathbb{R}\left[\frac{d}{dt}\right]$, $\mathbb{R}(t)\left[\frac{d}{dt}\right]$), then every torsion-free (left) D-module M is free.*
3) *[8] If D is a (left)* hereditary ring*, namely, every left ideal of D is a projective (left) D-module, (e.g., $D = k[t]\left[\frac{d}{dt}\right]$), then every torsion-free (left) D-module is projective.*

A sequence of left D-modules and left D-morphisms of the form

$$\ldots \xrightarrow{d_{i+2}} P_{i+1} \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} P_{i-1} \xrightarrow{d_{i-1}} P_{i-2} \xrightarrow{d_{i-2}} \ldots \quad (5)$$

is called *exact* at P_i if the *defect of exactness* at P_i defined by $H(P_i) = \ker d_i / \operatorname{im} d_{i+1}$ vanishes, i.e., $\ker d_i = \operatorname{im} d_{i+1}$. By extension, we say that (5) is *exact* if it is exact at every P_i. See [14] for more details.

A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0, \quad (6)$$

i.e., f is injective, $\ker g = \operatorname{im} f$ and g is surjective.

Definition 3: [14] The short exact sequence (6) *splits* if one of the following equivalent conditions holds:

1) There is a left D-morphism $h : M'' \longrightarrow M$ such that $g \circ h = id_{M''}$.
2) There is a left D-morphism $k : M \longrightarrow M'$ such that $k \circ f = id_{M'}$.
3) There are two left D-morphisms $h : M'' \longrightarrow M$ and $k : M \longrightarrow M'$ such that $f \circ k + h \circ g = id_M$.
4) M is isomorphic to $M' \oplus M''$, i.e., $M \cong M' \oplus M''$.

Using a classical result in homological algebra saying that the *functor* $\hom_D(\cdot, \mathcal{F})$ transforms split exact sequences of left D-modules into split exact sequences of abelian groups [14], we find the following relationship between Problems B and C.

Lemma 2: Problem C implies Problem B, i.e., if the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is stably free, then there exist $r, s \in \mathbb{Z}_+$ such that we have (2).

If $M = D^{1\times p}/(D^{1\times q}\,R)$ is stably free, then we shall say that $\mathcal{B} = \ker_{\mathcal{F}}(R.)$ is a *stably free behaviour*. We can constructively parametrize [2], [11] all solutions of a stably free behaviour in any signal space $\mathcal{F}$ which has a left D-module structure (see Example 5).

The free left D-module $D^{1\times m}$ defines a flat behaviour as we have $\mathcal{B} \cong \hom_D(D^{1\times m}, \mathcal{F}) \cong \mathcal{F}^m$ and $\mathcal{F}^m$ is a flat behaviour (the identity map is an injective parametrization of $\mathcal{F}^m$). Conversely, let us consider a behaviour defined by $R \in D^{q\times p}$. If $\mathcal{B} \cong \hom_D(M, \mathcal{F})$ is flat, then there exist $Q \in D^{p\times m}$ and $T \in D^{m\times p}$ such that we have the following exact sequence

$$\begin{array}{ccccccc} \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \xleftarrow{Q.} & \mathcal{F}^m & \longleftarrow 0, \\ R\,\eta & \longleftarrow & \eta & & & \\ & & Q\,\xi & \longleftarrow & \xi & \end{array}$$

and $T\,Q = I_m$. Then, we obtain $\mathcal{B} = \ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m \cong \mathcal{F}^m$, showing that a flat behaviour is isomorphic to $\mathcal{F}^m$.

Proposition 1: [4], [11] A behaviour $\mathcal{B} \cong \hom_D(M, \mathcal{F})$ is flat iff the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is free.

Then, we have the following corollary.

Corollary 1: The condition that M is a free left D-module of the form $D^{1\times m}$ is a sufficient condition for the existence of a solution to Problem B of the form $r = m$ and $s = 0$. Then, the projection π can be chosen to be id.

By Lemma 2, it is important to check whether or not the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is stably free.

Proposition 2: [2], [11], [12] Let us consider $R \in D^{q\times p}$ and the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$.

1) *M is free iff there exist $Q \in D^{p\times m}$ and $T \in D^{m\times p}$ such that $T\,Q = I_m$ and:*

$$\ker_D(.Q) = \{\lambda \in D^{1\times p} \mid \lambda\,Q = 0\} = D^{1\times q}\,R.$$

2) *M is stably free iff there exist $A \in D^{s\times r}$, $B \in D^{r\times s}$ such that $M = D^{1\times r}/(D^{1\times s}\,A)$ and $A\,B = I_s$.*

We recall that a matrix $R \in D^{q\times p}$ has *full row rank matrices* if its rows are left D-linearly independent.

Proposition 3: [11] If R has full row rank, then we have:

1) *$M = D^{1\times p}/(D^{1\times q}\, R)$ is a free left D-module iff there exist $S \in D^{p\times q}$, $Q \in D^{p\times (p-q)}$ and $T \in D^{(p-q)\times p}$ such that we have the following Bézout identities:*

$$\left(\begin{array}{c} R \\ T \end{array}\right) (S, \quad Q) = I_p, \quad (S, \quad Q) \left(\begin{array}{c} R \\ T \end{array}\right) = I_p.$$

2) *$M = D^{1\times p}/(D^{1\times q}\, R)$ is a stably free left D-module iff there exists $S \in D^{p\times q}$ such that $R\,S = I_q$.*

Without loss of generality, we can assume in what follows that every stably free left D-module M is defined by a matrix R which admits a right-inverse S, i.e., we have $R\,S = I_q$.

A left D-module $\mathcal{F}$ is called *injective cogenerator* if the *functor* $\hom_D(\cdot, \mathcal{F})$ transforms exact sequences of left D-modules into exact sequences of abelian groups [14] and $\hom_D(M, \mathcal{F}) = 0$ implies $M = 0$ for all left D-modules $\mathcal{F}$ [2], [9], [12].

Proposition 4: If $\mathcal{F}$ is an injective cogenerator, then Problem B is equivalent to Problem C. Therefore, Problem B is solvable iff $M = D^{1\times p}/(D^{1\times q}\, R)$ is a stably free left D-module.

Example 1: Let us consider the ring $D = \mathbb{R}[d_1, \ldots, d_n]$ of partial differential operators in $d_i = \partial/\partial x_i$ with constant coefficients and the D-module $\mathcal{F} = C^\infty(\Omega)$ (resp., $\mathcal{F} = \mathcal{D}'(\Omega)$) of smooth functions (resp., distributions) in an open convex subset Ω of $\mathbb{R}^n$. It is known that $\mathcal{F}$ is an injective cogenerator [9]. Therefore, by Proposition 4, we obtain that Problems B and C are equivalent. Hence, there exist r and $s \in \mathbb{Z}_+$ such that we have (2) iff the D-module M is stably free. But, by the Quillen-Suslin theorem (see 1 of Theorem 1), every stably free module over a commutative polynomial ring $k[x_1, \ldots, x_n]$ with coefficients in k is free. Hence, we have $M \cong D^{1\times r}$ for a certain $r \in \mathbb{Z}_+$ equal to the *rank* of M. We recall that $\mathrm{rank}_D(M)$ is the dimension over the field of rational functions $\mathbb{R}(d_1, \ldots, d_n)$ of the $\mathbb{R}(d_1, \ldots, d_n)$-vector space generated by M. Therefore, $\mathcal{B} \cong \mathcal{F}^r$ is a flat behaviour and Corollary 1 holds with $s = 0$ and $r = \mathrm{rank}_D(M)$, which solves Problem B with $\pi = id$.

Example 2: If we consider an open interval Ω of $\mathbb{R}$, the ring $D = K(\Omega)\left[\frac{d}{dt}\right]$ of OD operators with coefficients in

$$K(\Omega) = \{n/d \in \mathbb{C}(t) \mid 0 \neq d,\, n \in \mathbb{C}[t],\, \forall\, s \in \Omega,\, d(s) \neq 0\}$$

and the left D-module $\mathcal{F} = \mathcal{B}(\Omega)$ of *hyperfunctions* in Ω [5], then we know that $\mathcal{F}$ is an injective cogenerator [5]. By Proposition 4, this shows that Problems B and C are equivalent. But, contrary to what happened for $\mathbb{R}[d_1, \ldots, d_n]$, stably free left D-modules are not necessarily free ones [5].

An Ore algebra of OD/PD, time-delay or shift operators, with polynomial or rational coefficients, satisfies that finitely generated projective left D-modules are stably free [2], [8].

Proposition 5: If D is any Ore algebra defined in Proposition 4.8 of [2] and $\mathcal{F}$ is an injective cogenerator left D-module, then Problem B is also equivalent to the problem of finding $r \in \mathbb{Z}_+$ and a behaviour $\mathcal{B}'$ such that $\mathcal{B} \oplus \mathcal{B}' \cong \mathcal{F}^r$.

Let $\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t)$ be an analytic linear system on an open interval Ω of $\mathbb{R}$. Then, it is shown in [15] that the system is *controllable* on every non-trivial subinterval of Ω iff, for any fixed t_0 in Ω, there exists $k \in \mathbb{Z}_+$ such that the rank of the controllability matrix $C(k, t_0)$ at t_0, defined by

$$C(k, t_0) = (B_0(t_0), B_1(t_0), \ldots, B_k(t_0)),$$

where $B_{i+1} = A\, B_i - \frac{d}{dt}\, B_i$, $B_0 = B$, is equal to the number of states n in the system. We have the following result.

Proposition 6: The analytic ordinary differential linear system $\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t)$ is controllable on an open interval Ω of $\mathbb{R}$ iff the left $D = H(\Omega)\left[\frac{d}{dt}\right]$-module

$$M = D^{1\times(n+m)} / \left(D^{1\times n} \left(\frac{d}{dt}\, I_n - A(t), \quad -B(t) \right) \right)$$

is stably free, where $H(\Omega)$ denotes the ring of analytic functions in Ω.

Proof: It is shown in [11] that the left D-module M is stably free iff the left D-module $\widetilde{N} = D^{1\times n}/(D^{1\times(m+n)}\, \widetilde{R})$, where the matrix $\widetilde{R}$ is defined by (m is the number of inputs)

$$\widetilde{R} = \left(-\frac{d}{dt}\, I_n - A^T(t), \quad B^T(t) \right)^T \in D^{(n+m)\times n},$$

is the zero module. As we have previously seen, $\widetilde{N}$ is defined by the OD linear system

$$\left\{ \begin{array}{l} \dot{\lambda} = -A^T(t)\, \lambda, \\ B^T(t)\, \lambda = 0, \end{array} \right. \tag{7}$$

where $\dot{\lambda}$ denotes the time-derivative of λ. Hence, we obtain that $\widetilde{N} = 0$ iff (7) implies $\lambda = 0$. Differentiating the zero-order equation of (7) and substituting the result into the first equation of (7), we find the new zero-order equation defined by $(B^T(t)\, A^T(t) - \dot{B}(t)^T)\, \lambda = 0$. Repeating recursively the same procedure with the last zero-order equation obtained, we get:

$$\forall\, k \in \mathbb{Z}_+, \quad \left\{ \begin{array}{l} \dot{\lambda} = -A^T(t)\, \lambda, \\ C(k, t)^T\, \lambda = 0. \end{array} \right.$$

Therefore, (7) implies $\lambda = 0$ on every non-trivial subinterval of Ω iff, for any fixed t_0 in Ω, there exists $k \in \mathbb{Z}_+$ such that $\mathrm{rank}\, C(k, t_0) = n$. ■

Using 2 of Theorem 1 and Proposition 6, we obtain that a time-invariant Kalman system is controllable iff the $\mathbb{R}\left[\frac{d}{dt}\right]$-module M is torsion-free [2], [4], [11]. More generally, controllability of multidimensional systems with constant coefficients (see Example 1) in terms of the possibility to patch two solutions on open subsets of $\mathbb{R}^n$ with disjoint closures was proved to be equivalent to the torsion-freeness of the D-module M [9].

We now prove that $\dot{x}(t) = t\, u(t)$ is not a flat system.

Example 3: Let $D = k[t]\left[\frac{d}{dt}\right]$ be the Weyl algebra and $R = (\frac{d}{dt}, \quad -t) \in D^{1\times 2}$. Then, $M = D^{1\times 2}/(D\, R)$ corresponds to $\dot{x}(t) = t\, u(t)$. If we denote by $S = (t, \quad \frac{d}{dt})^T$, then we check that we have $R\, S = 1$. Hence, the left D-morphism $.S : D^{1\times 2} \longrightarrow D$ defined by $(.S)(\lambda) = \lambda\, S$

satisfies the relation $(.S) \circ (.R) = id_D$. Thus, the following exact sequence

$$0 \longrightarrow D \xrightarrow{.R} D^{1\times 2} \longrightarrow M \longrightarrow 0$$

splits (see Definition 3) and we obtain $M \oplus D \cong D^{1\times 2}$, showing that M is a stably-free left D-module by Definition 2.

We can check that we have the following exact sequence

$$0 \longrightarrow D \xrightarrow{.R} D^{1\times 2} \xrightarrow{.P} D \longrightarrow L \longrightarrow 0,$$

where $P = \left(t^2, \quad t\,\frac{d}{dt} + 2\right)^T$ and $L = D/(D^{1\times 2}\, P)$ is a torsion left D-module, i.e., P is a *minimal parametrization* of M. See [1], [2] for more details. Thus, we have

$$M = \operatorname{coker}_D(.R) \cong D^{1\times 2}\, P = D\, t^2 + D\left(t\,\frac{d}{dt} + 2\right) \subseteq D.$$

Only principal left ideals of D are free left D-submodules. Hence, M is a free left D-module iff the left ideal of D defined by $J = D\,t^2 + D\left(t\,\frac{d}{dt} + 2\right)$ is principal. Let us study whether or not J is a principal left ideal of D.

We define by $L(a) = a_m(t) \neq 0$ (resp., $\operatorname{ord}(a) = m$) the *leading term* (resp., *order*) of $a = \sum_{i=0}^{m} a_i(t)\,\frac{d^i}{dt^i} \in D$ and we denote by J_m the family of ideals of $k[t]$ given by:

$$J_m = \{L(a) \,|\, a \in J,\, \operatorname{ord}(a) = m\} \cup \{0\}.$$

We easily check that $J_m \subseteq J_{m+1}$. Now, if J were principal, then we would obtain $J_m = J_{m+1}$ for all $m \geq 0$ as we have $L(\frac{d}{dt}\, a) = L(a)$ and $L(t\, a) = t\, L(a)$. But we easily check that $J_0 = (t^2) \subsetneq J_1 = (t^2, t) = (t)$, which proves that J is not principal, and thus, M is a stably free but not a free left D-module. In particular, D is a left hereditary but not a left principal ideal domain. Therefore, by Propositions 1 and 6, we obtain that $\dot{x}(t) = t\, u(t)$ is a controllable but not a flat system.

III. Main results

We now give an explicit construction of a free left D-module which projects onto the stably free left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$, and thus, a construction of a flat behaviour which projects onto the stably free behaviour $\ker_{\mathcal{F}}(R.)$ for the projection defined by $\pi((\eta_1, \ldots, \eta_p, \ldots, \eta_{p+q})) = (\eta_1, \ldots, \eta_p)$.

Proposition 7: Let $R \in D^{q\times p}$ be a full row rank matrix which admits a right-inverse $S \in D^{p\times q}$ and let us define the matrix $R' = (R \quad 0) \in D^{q\times (p+q)}$. Then, we have the following split short exact sequence

$$0 \longrightarrow D^{1\times q} \underset{\xleftarrow{.S'}}{\xrightarrow{.R'}} D^{1\times (p+q)} \underset{\xleftarrow{.T'}}{\xrightarrow{.Q'}} D^{1\times p} \longrightarrow 0, \qquad (8)$$

with the following notations:

$$\begin{cases} S' = \begin{pmatrix} S \\ -I_q \end{pmatrix} \in D^{(p+q)\times q},\ T' = (I_p,\ S) \in D^{p\times (p+q)}, \\ Q' = \begin{pmatrix} I_p - S\,R \\ R \end{pmatrix} \in D^{(p+q)\times p}. \end{cases} \qquad (9)$$

Equivalently, we have the following Bézout identities:

$$\begin{pmatrix} R' \\ T' \end{pmatrix} (S',\ Q') = I_{p+q},\ (S',\ Q') \begin{pmatrix} R' \\ T' \end{pmatrix} = I_{p+q}. \qquad (10)$$

Proof: We easily check that we have $R'\, S' = R\, S = I_q$, $T'\, Q' = I_p - S\,R + S\,R = I_p$, $R'\, Q' = R\,(I_p - S\,R) = R - R\,S\,R = 0$ and computing $S'\, R' + Q'\, T'$, we obtain

$$\begin{pmatrix} S\,R + I_p - S\,R & (I_p - S\,R)\,S \\ -R + R & R\,S \end{pmatrix} = I_{p+q},$$

which prove the Bézout identities (10).

Now, if (8) splits, then we trivially obtain (10). Conversely, suppose that we have (10). In particular, $R'\, Q' = 0$ implies $(D^{1\times q}\, R') \subseteq \ker_D(.Q')$. Moreover, if $\lambda \in \ker_D(.Q')$, we then obtain $\lambda = \lambda\,(S'\, R' + Q'\, T') = (\lambda\, S')\, R'$, which proves that $\ker_D(.Q') \subseteq (D^{1\times q}\, R')$, and thus, the exactness of (8). The other identities show that (8) is a split exact sequence. ■

We easily check that we have:

$$\begin{aligned} M' &= D^{1\times (p+q)}/(D^{1\times q}\, R') = D^{1\times (p+q)}/((D^{1\times q}\, R \quad 0)) \\ &= D^{1\times p}/(D^{1\times q}\, R) \,\oplus\, D^{1\times q} = M \,\oplus\, D^{1\times q}. \end{aligned}$$

Using Proposition 7 and 1 of Proposition 3, we obtain that M' is a free left D-module isomorphic to $D^{1\times p}$.

Theorem 2: Let $\mathcal{F}$ be a left D-module, $R \in D^{q\times p}$ admitting a right-inverse $S \in D^{p\times q}$ and let us define the stably free behaviour $\mathcal{B} = \ker_{\mathcal{F}}(R.)$. Then, the behaviour $\mathcal{B} \oplus \mathcal{F}^q$ is flat and we have the following injective parametrization:

$$\begin{cases} R\,\eta = 0, \\ \eta \in \mathcal{F}^p,\ \zeta \in \mathcal{F}^q \end{cases} \Leftrightarrow \begin{cases} \eta = (I_p - S\,R)\,\xi, \\ \zeta = R\,\xi, \end{cases} \qquad (11)$$

where ξ is any arbitrary element of $\mathcal{F}^p$. Moreover, a flat output of the behaviour $\mathcal{B} \oplus \mathcal{F}^q$ is defined by $\xi = \eta + S\,\zeta$.

In particular, the behaviour $\mathcal{B} = \pi(\mathcal{B} \oplus \mathcal{F}^q)$ is the projection of the flat behaviour (11), where $\pi : \mathcal{F}^{p+q} \longrightarrow \mathcal{F}^p$ is defined by $\pi((\eta_1, \ldots, \eta_p, \zeta_1, \ldots, \zeta_q)) = (\eta_1, \ldots, \eta_p)$.

Proof: Applying $\hom_D(\cdot, \mathcal{F})$ to the split exact sequence (8), we obtain the following split exact sequence:

$$0 \longleftarrow \mathcal{F}^q \underset{\xrightarrow{S'.}}{\xleftarrow{R'.}} \mathcal{F}^{p+q} \underset{\xrightarrow{T'.}}{\xleftarrow{Q'.}} \mathcal{F}^p \longleftarrow 0.$$

Therefore, we have $\ker_{\mathcal{F}}(R'.) = \{Q'\,\xi \,|\, \xi \in \mathcal{F}^p\}$. But, since $R'\,(\eta^T, \quad \zeta^T)^T = (R, \quad 0)\,(\eta^T, \quad \zeta^T)^T = R\,\eta$ and ζ is an arbitrary element of $\mathcal{F}^q$, we have $\ker_{\mathcal{F}}(R'.) = \mathcal{B} \oplus \mathcal{F}^q$. Then, using (9), we obtain (11). Now, applying the matrix T' on the left of $(\eta^T, \quad \zeta^T)^T = Q'\,\xi$, we then obtain $T'\,(\eta^T, \quad \zeta^T)^T = T'\, Q'\,\xi$ and, using the identity $T'\, Q' = I_p$, we get $\xi = (I_p, \quad S)\,(\eta^T, \quad \zeta^T)^T = \eta + S\,\zeta$. ■

Let us illustrate Theorem 2 on two examples.

Example 4: We consider again Example 3 with $\mathcal{F} = C^\infty(\mathbb{R})$. The embedding of $\mathcal{B} = \ker_{\mathcal{F}}(R.)$ into $\mathcal{F}^3$ allows us to "blow-up" the singularity at $t = 0$ as we have

$$\begin{cases} \dot{x} - t\,u = 0, \\ v \in \mathcal{F}, \end{cases} \Leftrightarrow \begin{cases} x(t) = -t\,\dot{\xi}_1(t) + \xi_1(t) + t^2\,\xi_2(t), \\ u(t) = -\ddot{\xi}_1(t) + t\,\dot{\xi}_2(t) + 2\,\xi_2(t), \\ v(t) = \dot{\xi}_1(t) - t\,\xi_2(t), \end{cases} \qquad (12)$$

where ξ_1 and ξ_2 are two arbitrary functions in $\mathcal{F}$ and we have $\xi_1(t) = x(t) + t\,v(t)$ and $\xi_2(t) = u(t) + \dot{v}(t)$. Then, the behaviour $\mathcal{B} = \pi(\mathcal{B} \oplus \mathcal{F})$ is the projection of the flat behaviour (12), where $\pi : \mathcal{F}^3 \longrightarrow \mathcal{F}^2$ is defined by $\pi((x, u, v)) = (x, u)$.

Example 5: Consider the differential time-delay system:

$$\dot{x}(t) = t\,u(t) + u(t-1). \tag{13}$$

We introduce $D = \mathbb{R}[t]\left[\frac{d}{dt}, \delta\right]$, $R = (\frac{d}{dt}, \quad -(t+\delta)) \in D^{1\times 2}$ and the left D-module $M = D^{1\times 2}/(D\,R)$. We can check that the matrix $S = (\delta + t, \quad \frac{d}{dt})^T$ is a right-inverse of R. Therefore, the finite free resolution of M defined by

$$0 \longrightarrow D \xrightarrow{.R} D^{1\times 2} \longrightarrow M \longrightarrow 0,$$

splits and we obtain $M \oplus D \cong D^{1\times 2}$, i.e., M is a stably-free left D-module. Using an algorithm developed in [2], we obtain the following long split exact sequence

$$0 \longrightarrow D \xrightarrow{.R} D^{1\times 2} \xrightarrow{.Q} D^{1\times 2} \xrightarrow{.P} D \longrightarrow 0, \tag{14}$$

where $P = \left(\delta + t,\ \frac{d}{dt}\right)^T \in D^2$ and Q is defined by:

$$Q = \left(\begin{array}{cc} -\delta\,\frac{d}{dt} - t\,\frac{d}{dt} + 1 & \delta^2 + (2\,t-1)\,\delta + t^2 \\ -\frac{d^2}{dt^2} & t\,\frac{d}{dt} + \delta\,\frac{d}{dt} + 2 \end{array}\right) \in D^{2\times 2}.$$

Let $\mathcal{F}$ be a left D-module (e.g., $\mathcal{F} = C^\infty(\mathbb{R})$). As (14) is a long split exact sequence, by applying the functor $\hom_D(\cdot, \mathcal{F})$, we then obtain the following exact sequence:

$$0 \longleftarrow \mathcal{F} \xleftarrow{R.} \mathcal{F}^2 \xleftarrow{Q.} \mathcal{F}^2 \xleftarrow{P.} \mathcal{F} \longleftarrow 0.$$

Thus, we obtain $\mathcal{B} = \ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^2$, i.e., we have the following explicit parametrization of all $\mathcal{F}$-solutions of (13)

$$\left\{\begin{array}{ll} x(t) = & -t\,\dot{\xi}_1(t) - \dot{\xi}_1(t-1) + \xi_1(t) + \xi_2(t-2) \\ & +(2\,t-1)\,\xi_2(t-1) + t^2\,\xi_2(t), \\ u(t) = & -\ddot{\xi}_1(t) + t\,\dot{\xi}_2(t) + \dot{\xi}_2(t-1) + 2\,\xi_2(t), \end{array}\right. \tag{15}$$

where ξ_1 and ξ_2 are two arbitrary functions in $\mathcal{F}$.

Parametrization (15) is not injective since we have:

$$Q\,\xi = 0 \Leftrightarrow \xi = P\,\phi.$$

Therefore, it is not possible to obtain $\xi(t)$ as a D-linear combination of $x(t)$ and $u(t)$. However, if we embed the behaviour

$$\mathcal{B} = \left\{(x,\ u)^T \in \mathcal{F}^2 \mid \dot{x}(t) = t\,u(t) + u(t-1)\right\}$$

into $\mathcal{F}^3$, then, by Theorem 2, we obtain the following injective parametrization of all $\mathcal{F}$-solutions of (13)

$$\left\{\begin{array}{ll} x(t) = & -t\,\dot{\xi}_1(t) - \dot{\xi}_1(t-1) + \xi_1(t) + \xi_2(t-2) \\ & +(2\,t-1)\,\xi_2(t-1) + t^2\,\xi_2(t), \\ u(t) = & -\ddot{\xi}_1(t) + t\,\dot{\xi}_2(t) + \dot{\xi}_2(t-1) + 2\,\xi_2(t), \\ v(t) = & \dot{\xi}_1(t) - t\,\xi_2(t) - \xi_2(t-1), \end{array}\right.$$

where $\xi_1(t) = x(t) + t\,v(t) + v(t-1)$ and $\xi_2(t) = u(t) + \dot{v}(t)$. Hence, $\mathcal{B}$ is a projection onto $\mathcal{F}^2$ of the flat behaviour:

$$\mathcal{B} \oplus \mathcal{F} = \{(x,\ u,\ v)^T \in \mathcal{F}^3 \mid \dot{x}(t) - t\,u(t) - u(t-1) = 0\}.$$

In his review [3] of the paper [6], K. B. Datta says "For future research, one may extend the results given here to the multi-input case". Let us recall a result of algebra.

Theorem 3: [8] If k is a field containing $\mathbb{Q}$, then any stably free left $A_n(k) = k[x_1, \ldots, x_n][d_1, \ldots, d_n]$-module M satisfying $\mathrm{rank}_{A_n(k)}(M) \geq 2$ is free.

We recall that the number of differentially independent inputs of a linear system is given by the rank of the left D-module M.

Corollary 2: Every controllable multi-input ordinary differential linear system with polynomial coefficients is flat.

IV. CONCLUSION

It is interesting to notice that the authors of [6] acknowledge B. Malgrange for motivating discussions [7] and for pointing out to them that [6] is related to the concept of stably free modules over the Weyl algebra $A_1(k)$. We hope that we have made it clear to the reader by giving a new *blowing-up interpretation* for controllable time-varying linear systems. Using such a geometric interpretation, we were able to generalize the results of [6] to MIMO multidimensional linear systems with varying coefficients. Finally, we proved that every controllable multi-input OD linear system with polynomial coefficients is flat. The extension of this result to the analytic case will be studied in the future as well as the developments of effective algorithms for the computation of bases of finitely generated modules over the Weyl algebra $A_n(k)$ and their implementations in OREMODULES.

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