Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Embedding Polynomial Matrices: A practical perspective

Ricardo Zavala Yoé, C. Praagman, H. L. Trentelman
Research Institute for Mathematics and Computer Science
P.O. Box 800, 9700 AV Groningen, The Netherlands
rzyoe@math.rug.nl

I. Introduction

Although in [7], a theoretical solution for the unimodular embedding problem is offered , in [16], we applied embedding algorithms in order to solve the pole placement and stabilization problems from the Behavioral point of view, remarking that serious numerical problems arise when we deal with actual computer implementation practiced to 12 numerical examples. [1] In this note we try to find a convergent explanation about why this happens from different perspectives. We found out that solving numerically either the unimodular embedding problem or the stable embedding problem [16] opens a completely new field in polynomial modelling and control. A sketch of an alternative solution is given as well at the end of this note.

II. Embedding a polynomial matrix into a unimodular or stable ones

For notational convenience we denote the class of non constant polynomial matrices $P(\xi)$ that have full row rank for all λ in $\mathbb{C}$ by $\mathcal{U}$. A *unimodular* matrix is a square matrix in $\mathcal{U}$. The class of non constant polynomial matrices $P(\xi)$ that have full row rank for all λ in $\overline{\mathbb{C}^+}$ (the closed right half plane) will be denoted by $\mathcal{M}$. A stable or *Hurwitz* matrix is a square matrix in $\mathcal{M}$. Let $P(\xi)$ be an $m \times n$ (with $n > m$) polynomial matrix of degree d:

$$P(\xi) = P_0 + P_1\xi + P_2\xi^2 + ... + P_d\xi^d$$

where each P_i is a real $m \times n$ matrix. The goal is to construct another $(n-m) \times n$ polynomial matrix

$$Q(\xi) = Q_0 + Q_1\xi + Q_2\xi^2 + ... + Q_d\xi^{d_q}, \quad d_q \leq d-1$$

such that the square stacked matrix $W(\xi) = [P(\xi); Q(\xi)] \in \mathcal{U}$ or at least, $W(\xi) \in \mathcal{M}$. Since a unimodular (resp. Hurwitz) matrix is invertible for all λ in $\mathbb{C}$ (resp. $\overline{\mathbb{C}^+}$), the embedding (resp. stable embedding [16]) problem can only have a solution if $P(\lambda)$ has full row rank m for all λ in $\mathbb{C}$ (resp. $\overline{\mathbb{C}^+}$). To construct a $Q(\xi)$, $P(\xi)$ is linearized ([11], [14]) in order to get an equivalent non square pencil $\xi E - A$ to work with. This means that there exist unimodular matrices $U(\xi), V(\xi)$ such that

$$U(\xi)(\xi E - A)V(\xi) = diag(I, P(\xi))$$

where the identity matrix is of order $(d-1)m$ and matrices $U(\xi)$, $V(\xi)$ are of orders dm and $(d-1)m+n$, resp. The $m_p = dm \times n_p = (d-1)m + n$ matrices E and A are defined as

$$E = \begin{bmatrix} 0 & & & -P_d \\ I & & & -P_{d-1} \\ & \ddots & & \vdots \\ & \ddots & 0 & -P_2 \\ & & I & -P_1 \end{bmatrix}, A = \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & P_0 \end{bmatrix}$$

Afterwords, constant orthogonal transformations M, N are applied to $\xi E - A$, which produces a block staircase form $\xi\widehat{E} - \widehat{A} = M(\xi E - A)N$ (the latter is a perturbation of $\xi E - A$ as we shall see). From $\xi\widehat{E} - \widehat{A}$, a $Q(\xi)$ is computed ([7]) as

$$Q(\xi) = K_d - \sum_{i=1}^{d-1} K_i \sum_{j=0}^{i-1} \xi^{i-j} P_{d-j}$$

where $K = [K_1|K_2|\cdots|K_{d-1}|K_d]$ is a matrix in terms of the orthogonal transformations M and N. K which can be chosen constant ([7]) embeds the structured $\xi\widehat{E} - \widehat{A}$ into a unimodular or stable pencil. This means that we can associate a matrix pencil to a polynomial matrix of arbitrary degree d, such that the property of full row rank of $P(\lambda)$ for all $\lambda \in \mathbb{C}$ translates to the same property for the pencil. The special solution for the pencil then leads to a solution of the general problem, i.e., if we can find a polynomial matrix Q of degree $d-1$ such that $W(\xi) \in \mathcal{U}$ or $\mathcal{M}$, then the respective problem is solved. According to [7], [16] embedding a polynomial matrix into a unimodular (resp. stable) one always has a solution, however, it was found out in [16] that in spite of the embedding problem is solved by means of equivalence between two polynomial matrices (one with degree d and the second with degree one), such a linearization process [11] implies a loss of numerical information of our original system, i.e., it renders a loss of the full row rank property in $\xi E - A$ for $d > 2$, mainly (but even for $d = 1$ if the pencil is too wide $m \ll n$ [2]). We shall try to show in this note, that the latter will imply uncontrollability and instability of $\xi E - A$, yielding a not suitable $Q(\xi)$. To illustrate this facts, we provide a table (see appendix and [16]). It contains information obtained from twelve examples of systems given as polynomial matrices of degrees $d = 1, 2, 3, 4, 6, 7$ and concerns geometric parameters (a lower bound for the distance to uncontrollability ($||\delta E, \delta A||$) and a distance to instability $f(\lambda)$ for $\xi E - A$. The last column displays $\sigma_{min}(\delta Y)$, term linked to N and explained later on. We shall deduce from this, that even before designing or programming any algorithm to solve the embedding problems, a suitable linearization has to be designed in such a way the controllability and stability of $\xi E - A$ can be guaranteed.

III. Embedding a polynomial matrix of degree one: The almost trivial case

In this section we compute the unimodular embedding of the example 1 of degree $d = 1$ (see appendix and [16]).

Example. Let us consider the following polynomial matrix:

$$P(\xi) = \begin{matrix} 11\xi + 1 & 9.5\xi + 2 & 3\xi + 3 \\ 1.4\xi + 2.5 & 3\xi + 1.7 & 2.7\xi + 7.6 \end{matrix}$$

[1] Actually two of them are real physical systems: $Ex_1, d = 6, 7$ represent an electric motor and a rotary cement kiln, resp.

[2] Naturally, a key factor in this phenomenon is the size of the resultant pencil.

Running the algorithm described in [16], we get a unimodular embedding $W(\xi)$. Since $d = 1$, $Q(\xi) = K_F$.

$$W(\xi) = \frac{P(\xi)}{Q(\xi)} = \left[\begin{array}{ccc} 11.\xi + 1. & 9.500\xi + 2. & 3.\xi + 3. \\ 1.400\xi + 2.500 & 3.\xi + 1.700 & 2.700\xi + 7.600 \\ \hline -.7549 & -.6310 & -.1787 \end{array}\right]$$

and $det(W(\xi)) = -(0.8854 \times 10^{-15})\xi^2 - (0.4066 \times 10^{-15})\xi - 6.972 \approx -6.972$.

Although it may seem that solving the embedding problem for polynomial matrices of degree one, (a pencil) is trivial, we realized it is not. Nothing seemed to be rare... Up to now. However, let's glimpse briefly what is happening with the linearization proposed in [7]. Since $A = diag(I, P_0)$ after linearizing the examples of the appendix, it was noticed that as higher the degree d and as bigger the size of their corresponding pencils, as many singular values of A equal to 1 were appearing [3]. As a consequence, as many $\sigma_i(A) = 1$ a pencil has, as worse the embedding become. The following section will make the latter clearer.

IV. Loss of controllability: Geometric point of view

First, we have to introduce some concepts from Elmroth ([9], and references therein). There, the author developed a nice theoretical/practical point of view to study the structure of matrix pencils. Some definitions are borrowed from there. Such concepts, concerns the geometry of matrix pencils. First, we recall that the set of all matrices similar to a matrix A, defines the orbit o of a matrix A

$$o(A) = \{M^{-1}AM : det(M) \neq 0\}$$

In this sense, it is also possible to extend the latter definition to matrix pencils, saying that the set of all equivalent pencils to $\xi E - A$ defines the equivalence orbit of that pencil:

$$o(\xi E - A) = \{M(\xi E - A)N : det(M), det(N) \neq 0\}$$

More concretly, any matrix pencil $\xi E - A$ with real or complex entries defines a manifold of strictly equivalent pencils in the $2mn$ dimensional space. Hence, it is possible to say that an orbit of matrix pencils is a set of pencils with the same Kronecker canonical form. In fact, if we find that for some pencil $\xi E - A$, $m_p \neq n_p$, then for for almost all (E, A) it will have the same Kronecker structure, depending only on its size. This case is called the *generic case* when the pencil has full rank $\forall \lambda \in \mathbb{C}$. In contrast, when the pencil at hand has no full rank for $\forall \lambda \in \mathbb{C}$, it is called *non generic*. Since the dimension of the orbit of $\xi E - A$ is equal to the dimension of the tangent space to the orbit at that point $(\xi E - A)$, it is possible to say that the tangent space is the range space of the following $2m_p n_p \times n_p^2 + m_p^2$ matrix T ($\otimes$ denotes (right) Kronecker product):

$$T = \begin{array}{cc} -A^T \otimes I_{m_p} & -I_{n_p} \otimes -A \\ -E^T \otimes I_{m_p} & -I_{n_p} \otimes -E \end{array}$$

[3] As a matter of fact, this number of $\sigma_i(A) = 1$ *equals the codimension* of A , i.e., $cod(A)$=no. of $\sigma_i(A) = 1$. It is given by the number of singular values of $cod(A) = (I_{n_p} \otimes -A^T) - (-A^T \otimes I_{n_p})$ equal to zero. It follows that the codimension of A also warns us about problems. It says that if $cod(\xi E - A) = 0$, the dimension of the corresponding complementary space has to be $2m_p n_p$. In this case, $\xi E - A$ spans the whole $2m_p n_p$ space. This is the case for $d = 1$ but not for $d > 1$. The latter implies that the $2m_p n_p$ pencil space can not be spanned now and we fall into numerical problems. In contrast their corresponding generic pencils (roughly speaking, defined as pencils of the same size as $\xi E - A$ but built up only by Kronecker blocks: Jordan blocks (for finite or infinite eigenvalues) and singular blocks (for columns or rows)) i.e., full rank $\forall \lambda \in \mathbb{C}$ have codimension always equal to zero. In fact, we are near of loosing this spanning property in $Ex_2, d = 1$. This characteristic is definitely lost for $d > 1$. See [9], [16].

In this sense, we can define as well the normal space $nor(\xi E - A)$, as the space perpendicular to $tan(\xi E - A)$. The dimension of the normal space is also known as the codimension of the orbit, $cod(\xi E - A)$ which can be computed as the *number of zero singular values of* T. Finally, with all this, it is possible to compute for a given pencil a lower bound on the distance to the closest non generic pencil $\xi(E + \delta E) - (A + \delta A)$ by means of

$$||(\delta E, \delta A)|| \geq \frac{\sigma_{min}(T)}{\sqrt{m_p + n_p}} = \frac{\sigma_{min}(T)}{\sqrt{n - m}}$$

Now, we explain briefly the content of our table shown in the appendix. The first column includes the set of examples for different degrees d. Next to it, we give the sizes of the corresponding polynomial matrices $P(\xi)$ under study (denoted as (m, n)). The next column (3rd) gives a lower bound to the closest non-generic pencil (the closest pencil which looses rank). Evidently, if such a distance is near of zero, our "equivalent" pencil becomes a non-generic one, i.e., the linearized equivalent polynomial matrix $\xi E - A$ becomes uncontrollable because the distance the the closest non controllable equivalent system of degree 1 is practically zero. The 4th column will be commented in part V (section A), but at the moment, we can argue that it shows the distance of $\xi E - A$ to instability, given by $f(\lambda) = min_{\forall \lambda}(\sigma_{min}(\lambda E - A))$ (global minimum of $f(\lambda)$). This column says also that, as wider the pencil ($m \ll n$) as closer it is to be unstable. This phenomenon is explained by the fact that removing columns of the pencil will increase the value of $\sigma_{min}(\lambda E - A)$ whereas removing rows will decrease it. An example of this are the third degree examples ($d = 3$), Ex_1 and Ex_2. Comparing their sizes we see the way $f(\lambda)$ changes. Column 5th, will be explained in part V (section B) where M and N will be considered as *pencils* as well. A graphic version of column 3, is the figure shown below. It gives an idea of how corrupted T is if we apply it a QR factorization. The example considered is Ex_1,$d = 7$ for which $T \in \mathbb{R}^{476 \times 485}$. The black squares represent non zero entries and blank squares are zero entries. We observe that white zones are "invading" the black one. Many rows are - practically - blank. That represents that T is very close to loose rank, which is equivalent to say that $\xi E - A$ has practically lost controllability and hence it is close to be rank deficient. This QR factorization was applied to all the examples. All of them shown the latter pattern.

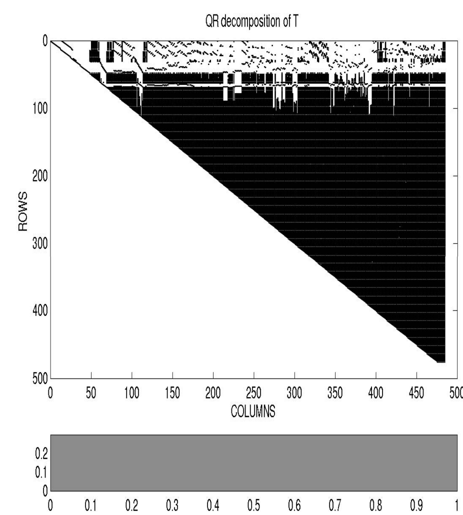

Fig. 1. *Corruption of controllability.*

V. Why does this linearization $\xi E - A$ fail?

In the past section we gave an explanation of why $\xi E - A$ becomes uncontrollable, which is equivalent to not solving the unimodular embedding problem. When this phenomenon occurs, the instability of $\xi E - A$ is "activated" also. It forbid us of solving the stable embedding problem as well. Such a pencil will be called here *corrupted* . We shall try to explain here why either loosing controllability or stability in $\xi E - A$ is equivalent to not solving neither the embedding nor the stable embedding problems with the linearization described in [7] (section II) [4]. The first explanation is based in *pseudospectra* (section A). The second one is geometry - based (section B). A third argument (section C) considers the *polynomial eigenvalue problem (PEP)* point of view. As might be expected, serious numerical errors arise within this perspective as well. A relational perspective is used also as fourth part (in part D).

A. Loss of stability in the frequency domain: The pseudospectrum of $\xi E - A$

As we mentioned before, it was found out that there exists a relation between the $cod(A)$ and the $\epsilon-$*pseudospectrum* of $\xi E - A$. A pseudospectrum (or ϵ *pseudospectrum* [13], [15])for a rectangular matrix is a generalization of the spectrum for square matrices. There also exists a generalization for non-square matrix pencils ([4], [15])

$$\Lambda(E, A) = \{\lambda \in \mathbb{C} : ||(\lambda E - A)|| \leq \epsilon\}$$

Linked to the latter definition is the following one ([4]) which is often used to construct the pseudospectra of square matrices:

$$f(\lambda) = \sigma_{min}(\lambda E - A)$$

$f(\lambda)$ was obtained and shown in column 4 (table). As we see, the pattern is that $f(\lambda)$ is smaller as d increases. We shall come back to this at the end of section B (part V), where we study the conditioning of this problem. That introduces unstable eigenvalues of $W(\xi)$. Deeper explanations about this and plots are provided in [16].

B. Geometric point of view

As we said before and from [16], once our matrix A has got singular values $\sigma(A)$=1 (or equivalently, the codimension of $\xi E - A$ almost becomes a non zero constant), the distance $||(\delta E, \delta A)||$ is close to zero. Such a distance is derived from the deformation [9] shown below, where the term $O(\delta^2)$ has been neglected. Hence, we can say we have a deformed pencil.

$$(I_{mp} + \delta X)(\xi E - A)(I_{np} - \delta Y) = \xi E - A + \delta(\xi T_E - T_A)$$

We shall denote $\Pi(\xi) = \xi E - A$, $\xi T_E - T_A = \xi(XE - EY) - (XA - AY)$. In fact, we can consider that the embedding algorithm *deforms* our original pencil $\xi E - A$ when we obtain its corresponding staircase form, i.e., its corresponding generalized Schur form:

$$\xi E - A + \delta(\xi T_E - T_A) = \xi(E + \delta T_E) - (A + \delta T_A) =$$

$$= \xi \widehat{E} - \widehat{A}$$

Since the lower bound for the distance to the closest non generic pencil $||(\delta E, \delta A)||$ was computed from the first order deformation $\xi T_E - T_A$, let's calculate the instability distance of $\xi T_E - T_A$, i.e., $\sigma_{min}(\xi T_E - T_A)$. Then the above equation becomes

$$\sigma_{min}(X\Pi(\xi)) - \sigma_{max}(\Pi(\xi)Y) \leq \sigma_{min}(\xi T_E - T_A) \leq$$

$$\leq \sigma_{min}(X\Pi(\xi)) + \sigma_{max}(\Pi(\xi))Y)$$

After applying inequalities for singular values, we obtain

$$\sigma_{min}(\Pi(\xi)) \leq \frac{\sigma_{min}(\xi T_E - T_A) + \sigma_{max}(\Pi(\xi))\sigma_{max}(Y)}{\sigma_{min}(X)}$$

Next we shall compute some parameters which shall allow us to explain (from the latter inequality), why when a pencil $\xi E - A$ under study becomes uncontrollable ($||(\delta E, \delta A)|| \approx 0$) it becomes unstable at the same time.
More precisely, let's consider the following row and column "deformations" defined as $M = I + \delta X$ and $N = I - \delta Y$. Immediately, we realize that, actually, our row and column transforming matrices M and N, respectively, are *square pencils in* δ . Since the ϵ-pseudospectrum can be computed for both of them as $\sigma_{min}(I + \delta X) = 1$ and $\sigma_{min}(I - \delta Y) = 1$, it will be interesting to get some information from the *purest* deformation of $\xi E - A$, i.e., δX and δY. In order to accomplish such a goal, we have computed the condition number of $\delta X, \delta Y$ for all the examples given in our table (not shown here because of space), as well as their maximum and minimum singular values. A partial but meaningful result is given in the 5th column of the table. The latter information says that although our transformation matrices M, N are always orthogonal, something crisp happens within them. As we see, if sometime we have to deal with $\delta X, \delta Y$ directly, we shall have to take into account their smallest and biggest singular values (sometimes $O(10^{-16}), O(10^{-17})$ [5]) which can affect some computations (as we shall see later on).
It was discovered that practically, $\sigma_{max}(\delta X) \approx \sigma_{max}(\delta Y) \approx 2$. The latter and the fact that the distance from $\xi E - A$ to instability is almost zero for all the examples (see last column of the table) implies that

$$0 \approx \sigma_{min}(\Pi(\xi)) \leq \frac{\sigma_{min}(\xi T_E - T_A)}{\sigma_{min}(X)} + \frac{2\sigma_{max}(\Pi(\xi))}{|\delta|\sigma_{min}(X)}$$

$$\sigma_{min}(\xi T_E - T_A) + \frac{2\sigma_{max}(\Pi(\xi))}{|\delta|} \approx 0$$

Hence $|\delta| \to \infty$ and $\sigma_{min}(\xi T_E - T_A) \approx 0$ produces what we have observed in the experiments described in the past section. As it is natural, having $|\delta| \to \infty$ implies that if we want to obtain a generalized Schur form $\xi \widehat{E} - \widehat{A}$ from $\xi E - A$ what we get is a big deformation of $\xi E - A$ as its staircaseform,i.e.,

$$\xi \widehat{E} - \widehat{A} \approx \delta(\xi T_E - T_A)$$

Moreover, that $\delta \to \infty$ can be interpreted graphically. We need to define ([8]) the distance between two pencils $\xi E_1 - A_1$ and $\xi E_2 - A_2$ in the following way:

$$dist(\xi E_1 - A_1, \xi E_2 - A_2) \stackrel{\triangle}{=} \sigma_{max}(E_1 - E_2) + \sigma_{max}(A_1 - A_2)$$

Noticing that for $I + \delta X$ and $I - \delta Y$ the corresponding most generic square pencil $\delta E_{\square_g} - A_{\square_g}$ will be composed of only one Jordan block

$$\delta E_{\square_g} - A_{\square_g} = J_n(\delta)$$

[4]However, the stable embedding problem was proposed and solved in [16] after relaxing the unimodular restrictions

[5]These terms are linked with the machine precision $\epsilon \approx 2.22 \times 10^{-16}$ and with the existence of zero and infinite eigenvalues of $\xi E - A$, N, and M.

where $J_n(\delta)$ denotes a Jordan block J of size n with eigenvalue δ, it is possible to compute the distance between $\Pi_M(\delta) \triangleq I+\delta X$, $\Pi_N(\delta) \triangleq I-\delta Y$, and $\Pi_{\square_g} \triangleq \delta E_{\square_g} - A_{\square_g}$ which yields

$$dist(\Pi_M(\delta), \Pi_{\square_g}) = dist(\Pi_N(\delta), \Pi_{\square_g}) \leq 1 + \frac{2}{|\delta|}$$

Similarly, we can find the distance between $\Pi_N(\delta)$ and its corresponding zero pencil $\Pi_{\square_0}$ which will be

$$dist(\Pi_N(\delta), \Pi_{\square_0}) = 1 + \frac{2}{|\delta|}$$

Noteworthy is the the distance between the most generic square pencil $\delta E_{\square_g} - A_{\square_g}$ and the square zero pencil:

$$dist(\Pi_{\square_g}, \Pi_{\square_0}) = 2 = \sigma_{max}(\delta Y)$$

This interesting fact expresses that when a square pencil as $I + \delta X$ or $I - \delta Y$, as special case behaves as orthogonal transformation for some δ, the corresponding $\sigma_{max}(\delta X)$ or $\sigma_{max}(\delta Y)$ are *equal to distance between a square generic pencil* $\delta E_g - A_g$ *and the null pencil* $\delta 0 - 0$ (which is always equal to 2). We realize that actually, watching the "internal" behavior of N (in terms of its distance to the worst uncontrollability, $dist(\Pi_N(\delta), \Pi_{\square_g})$ we can predict how far $\xi E - A$ from instability is. Naturally, since the calculation of N depends on $\xi E - A$, δ is a good indicator of how far we are from loosing controllability and stability at the same time. Actually, if our pencils become corrupted, reviewing deeply what happens within the algorithm won't be too useful. We realize that M, N will do their job doing orthogonal transformations on $\xi E - A$, however, since $\sigma_{min}(M)$=$\sigma_{min}(N) = 1$ and they are orthogonal during the process (and up to the end of the staircase computing) we won't notice anything wrong within the algorithm's running. On the other hand, it is interesting to mention that all the eigenvalues of δX have a negative real part and all the eigenvalues of δY have a positive real part.

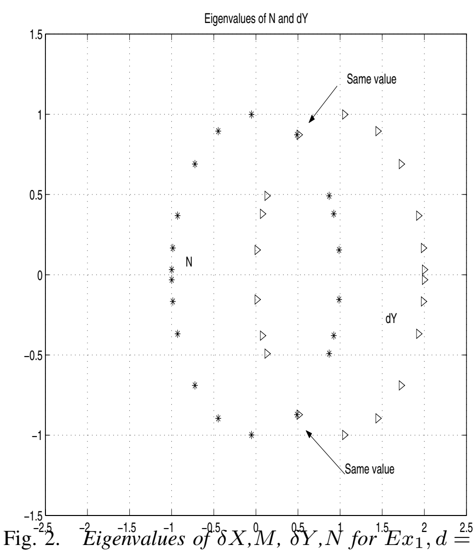

Fig. 2. *Eigenvalues of* $\delta X, M, \delta Y, N$ *for* $Ex_1, d = 4$

Moreover, the eigenvalues of δX, M, N, and δY lie on circles of radius one, centered in $(-1, j0)$,$(0, j0)$, $(0, j0)$, and $(1, j0)$, resp. They have certain symmetric patterns as we may expect if we notice that N and δY are complementaries. In figure 2, we appreciate that two pairs of eigenvalues of δY and N almost coincide at the same place of $\mathbb{C}$. This behavior is exhibited when $d > 4$ but also when the pencil is too wide (Ex_1, $d = 2$). It implies that as higher the d, as more common eigenvalues δY and N have (as it was observed during the running of the programs). Since, $I = N + \delta Y$, the second member of this equation is - obviously - diagonalizable, not defective (one eigenvalue for one eigenvector) and hence, they don't share eigenvalues. Nevertheless, as we said, this fact starts to be violated when d is big (which implies handling very big pencils) and eventually will reveal serious numerical errors. All of this has to be taken into account in order to study what happen when $W(\xi)$ is computed. In order to determine an expression for $W(\xi)$ in terms of δY let us take a look of K_F, the matrix we compute $Q(\xi)$ from:

$$K_F = \widehat{K}_F N^T = \widehat{K}_F(I - \delta Y^T) = [0|I](I - \delta Y^T) =$$
$$= [0|I] - \delta[Y_{11}^T|Y_{12}^T] = [-\delta Y_{11}^T|I - \delta Y_{12}^T]$$

From above and from the definition of K_F, we can write

$$K_F = [K_1|K_2|...|K_{d-1}|K_d] = [-\delta Y_{11}^T|I - \delta Y_{12}^T]$$

Partitioning $K_d = [K_{d_{11}}|K_{d_{12}}]$ and taking

$$-\delta Y_{11}^T = [K_1|K_2|...|K_{d-1}|K_{d_{11}}]$$

which will be subpartitioned as

$$-\delta Y_{11}^T = [-\delta Y_{11_{11}}^T| - \delta Y_{11_{12}}^T]$$

where

$$-\delta Y_{11_{11}}^T = [K_1|K_2|...|K_{d-1}], \quad -\delta Y_{11_{12}}^T = K_{d_{11}}$$

Besides, we define

$$I - \delta Y_{12}^T = K_{d_{12}}$$

In consequence K_F has as equivalent representation the matrix given below:

$$K_F = [-\delta Y_{11_1}^T| - \delta Y_{11_2}^T|...| - \delta Y_{11_{d-1}}^T| - \delta Y_{11_{12}}^T|I - \delta Y_{12}^T]$$

Computing $Q(\xi)$ is equivalent to

$$Q(\xi) = [-\delta Y_{11_{12}}^T|I - \delta Y_{12}^T] - \sum_{k=1}^{d-1} \xi^k \sum_{i=k}^{d-1} (-\delta Y_{11_i}^T) P_{d-i+k}$$

As a result, the embedding of $P(\xi)$ is

$$W(\xi) = \frac{P_0}{-\delta Y_{11_{12}}^T| \quad I - \delta Y_{12}^T} +$$
$$+ \sum_{k=1}^{d-1} \xi^k \frac{P_i}{\sum_{i=k}^{d-1} (-\delta Y_{11_i}^T) P_{d-i+k}} + \frac{P_d}{0}$$

The latter has to be interpreted adequately. We have realized that the linearization $\xi E - A$ is closer and closer to uncontrollability and instability as d (mainly) grows . As well, after j iterations done by the algorithm, the final transformations practiced on $\xi E - A$ are given by $M = M_j M_{j-1} \cdots M_1 M_0$ and $N = N_0 N_1 \cdots N_{j-1} N_j$, resp. In spite of they remain being orthogonal transformations during the running of the algorithm, these matrices M and N will contain the effect of having been applied to $\xi E - A$ to get $\xi \widehat{E} - \widehat{A}$. Hence, we can measure how "deficient" was our pencil $\xi E - A$ in terms of N. This is done by checking the derivative of N:

$$\frac{dN}{d\delta} = \frac{d(I - \delta Y)}{d\delta} = -Y$$

Recalling that a numerical problem can be conceived as a mapping from an input (data) space to an output (solution) space, condition numbers are upper bounds of the derivatives of these mappings.

Taking condition numbers in the equation given above produces the following:

$$\kappa \left(\frac{dN}{d\delta} \right) = \kappa \left(\frac{d(I-\delta Y)}{d\delta} \right) = \kappa(Y) = \kappa(\delta Y)$$

As we know, the condition number of a problem measures the sensivity of the solution to small changes in the input. Hence, although N is a constant matrix, $\kappa(\delta Y)$ will reflect its sensivity against a corrupted pencil, $\xi E - A$. Hence, owing $\kappa(\delta Y)$ (and $\kappa(\delta X)$) becomes very large in some of the examples we provide (see appendix), this ill conditioned problem tends to be ill posed. As a result, the matrix coefficients of $W(\xi)$ will be quite innacurate and very large (or very small) in κ. This fact will avoid to give a suitable unimodular or stable embedding for $P(\xi)$. Hence, we can explain why the embedding problems are close to be ill posed if we see that when $|\delta|$ goes to infinity, $\kappa(\delta Y) = \frac{2}{\sigma_{min}(\delta Y)}$ goes to infinity as well, which implies $|\delta|\sigma_{min}(Y) \to 0$, and $\sigma_{min}(Y) \to 0$. That's why $W(\xi)$ shows strange behaviors as the following one. Let us recall examples Ex_1 and Ex_2 of degree 1 of our table. From there, we obtain that $det(W_0) = O(10^4)$, $det(W_1) = 0$ for both of them. Even worse is the case for $Ex_1, d = 2$ where $det(W_0) = O(10^3)$ but $det(W_1) = O(10^{-303})$, and as always $det(W_2) = 0$. Similarly, for $Ex_2, d = 2$ we get $det(W_0) = -365.8883$, $det(W_1) = O(10^{-30})$ and $det(W_3) = 0$. There is no doubt about these serious numerical errors are caused by the high sensitivity (reflected in all the condition numbers) of all the variables with this linearization.

We want to add an extra comment about our pencils N and M. We can see that something like $||I - \delta Y||$ [6] and $||I + \delta X||$ reveals a measure of how alike I and N (resp. M) are. In fact, our pencils M and N are constituted (linearly in δ) by orthogonal matrices: N by $I \perp Y$ and M by $I \perp X$. Actually the family of pencils $||I - \delta Y||$=$||I + \delta X||$=1 is member of a less restrictive family of orthogonal matrices given by

$$||\Upsilon + z\Phi|| \geq ||\Upsilon||$$

for all $z \in \mathbb{C}$, Υ, Φ square matrices. Υ is said to be orthogonal to Ψ ([2]). Moreover, the latter allows us to calculate the derivative of N in terms of this orthogonality:

$$\frac{d(||I-\delta Y||}{d\delta}\bigg|_{\delta=0} = \langle Yu|u\rangle$$

where u is a unitary vector in $\mathbb{R}^{n_p}$. Since $Y = (1/|\delta|)(I - N)$ and $\langle Nu|u\rangle = ||Nu||||u||cos(\theta)$, we have

$$\frac{d(||I-\delta Y||}{d\delta}\bigg|_{\delta=0} \geq \frac{2}{|\delta|} = ||Y|| = dist(\Pi_N(\delta), \Pi_{\square_0}) - 1$$

This expression reveals again that having $|\delta| \to \infty$ will produce numerical problems because then the distance between N and the corresponding null pencil reaches its minimum value.

Thinking beforehand that the linearization applied to solve the unimodular and the stable embedding problems had no inconvenients, avoided us of checking the condition number of $A, E,, \widehat{A}, \widehat{E}$ (mainly) and even the pencil's one, considering the latter as a relation (see section D, part V). Another set of computations was done (not shown because of space), where the condition number of the twelve sets of matrices $A, \widehat{A}, E, \widehat{E}$ and their lower bound to the closest non generic pencil for $\xi\widehat{E} - \widehat{A}$, $||(\delta\widehat{E}, \widehat{A})||$ are also obtained. Their distance to instability, $\widehat{f}(\lambda)$ was, as well, calculated. From the latter information, we realize that in general, the distance to the corresponding non generic pencil still is practically zero for $d > 1$ (excluding the last example of $d = 1$ where the codimension of $\xi E - A$ is almost equal to one). Since the condition numbers are increasing exponentially as higher d is, we see that as higher d as closer the problem to become ill posed (in fact, for $d > 3$, we can consider that the corresponding examples are ill conditioned) [7] This fact explains many things because as closer a condition number to infinity as closer its matrix to loose rank. This fact supports the fact, that we can not have stable eigenvalues around zero because since marices A and E are close of being not full row rank, $\sigma_{min}(A), \sigma_{min}(E)$ become smaller and smaller as higher d is. Naturally, if such full row rank property is lost, the condition number becomes infinity, but this fact does not occur crisply, but rather it becomes worse according to the size of the corresponding pencils. Actually, we can not realize that something wrong is happening with the controllability and stability of some computed embedding if we just check the full row rank property of $\xi E - A$ and $\xi\widehat{E} - \widehat{A}$. We have learned that the geometric point of view has to be taken into account as well as the numerical issues in order to establish a well posed embedding problem.

C. Polynomial eigenvalue problem perspective

Up to now, we have not considered that having $W(\xi) = W_0 + \xi W_1 + \xi^2 W_2 + \cdots + \xi^d W_d$ defines the *nonlinear* [8] *eigenvalue problem*, which consists of finding eigenvalues λ and corresponding right eigenvectors $x \neq 0$ such that

$$W(\lambda) = (W_0 + \lambda W_1 + \lambda^2 W_2 + \cdots + \lambda^d W_d)x = 0 \quad (1)$$

for $W_i \in \mathbb{R}^{n\times n}, i = 0 \ldots d, x \in \mathbb{R}^n$ [13] . We realize that the linear eigenvalue problem, $W(\lambda) = W_0 + \lambda W_1$ is a special case of general one given above. If there , $d = 1$, and $W_1 = I$ we have the *standard eigenvalue problem (SEP)* . If $d = 1$, but $W_1 \neq I$ we have to deal with the *generalized eigenvalue problem (GEP)* . The quadratic case, (the *quadratic eigenvalue problem (QEP)*), $W(\lambda) = W_0 + \lambda W_1 + \lambda^2 W_2$ arises from the solution of Euler - Lagrange's modelling equations for mechanical and electro-mechanical systems, for instance. Now, if $W(\lambda)$ is a matrix polynomial, $W(\lambda) = W_0 + \lambda W_1 + \lambda^2 W_2 + \cdots + \lambda^d W_d$ we refer to as a *polynomial eigenvalue problem (PEP)* . However, it is remarkable that the number of algorithms for solving the PEP is rather scanty. That is why when d is *small* [9] the PEP can be reformulated as a linear problem [10] in dn dimensions, provided that, W_d or W_0 is nonsingular . The most applied linearizations are the so called canonical ones [14]. The linearization used in [7] belongs to this kind.

If either one, but not both of W_0 and W_d is singular the problem is well posed but some of the eigenvalues might be zero or infinite. If both of them are singular, then the problem is potentially ill posed [11], hence, we shall notice that existence and accuracy of solutions become a serious problem. If a solution exists, it might be inaccurate.

When W_d is nonsingular, $W(\lambda)$ is said to be *regular* and has dn finite eigenvalues. When $r = rank(W_d) < n$, $W(\lambda)$ have r finite and $dn - r$ infinite eigenvalues. Knowing all of this, we see that a non regular leading coefficient polynomial matrix $W(\xi)$ may cause some numerical difficulties. In order to support this, let us apply these background to the examples given in the table. Since $det(W_d) = 0$ we may expect some eigenvalues at infinity.

[6] We use $||\ \ ||$ instead of $\sigma_{max}(\ \ )$ to simplify notation in this part.

[7] We can glimpse this, computing $\kappa(\delta Y)$. Indeed, $\kappa(\delta X)$ behaves alike. Both $\to \infty$.

[8] This is the most general name for the eigenvalue problem $W(\lambda)x = 0$ where the entries of $W(\lambda)$ are analytical functions of the parameter λ. When $W(\lambda)$ is a matrix polynomial, we refer to as the *polynomial eigenvalue problem* .

[9] Although relative, this term implies we may select d taking into account the content of this note.

[10] In fact, a GEP.

[11] We recall [1] that a problem is called *ill posed* if its condition number is infinite and *ill conditioned* if its condition number is large.

For instance, let us take $Ex_0, d = 1$. The determinants of the coefficients of $W(\xi)$ are $det(W_0) = -6.9720$ and $det(W_1) = 0$. The latter says that we may have either zero or infinite eigenvalues. In fact, solving this GEP for $W(\xi)$, yields[12]:

$$Vecs = \begin{bmatrix} 0.6529 & 0.6529 & 0.6529 \\ -1.0000 & -1.0000 & -1.0000 \\ 0.7725 & 0.7725 & 0.7725 \end{bmatrix}, \Lambda = 1 \times 10^{14} \begin{bmatrix} -\infty \\ -2.1685 \\ -\infty \end{bmatrix}$$

Since $det(W_d) = 0$ we shall note this difference between both sets of the *same* eigenvalues, the ones produced by $det(W(\xi))$ and the ones obtained from solving a GEP with Matlab ©. This is characteristic of this way of embedding a polynomial matrix into a unimodular one. This phenomenon always was observed for all the values of d considered in the table. Moreover, some $det(W_i(\xi)), 1 \leq i \leq d-1$ are extremely high, which reflects a lot of sensivity produced by the almost lost property of having full row rank. Hence, as far as this part concerns, $Q(\lambda)$ should have the same degree d as $P(\lambda)$ in order to get regularity for $W(\lambda)$. Finally, we might - roughly -pose the embedding problem as follows: *Given an $m \times n$ matrix polynomial $P(\xi) = P_0 + \xi P_1 + ... + \xi^{d-1} P_{d-1} + \xi^d P_d$ finding an $n - m \times n$ matrix polynomial $Q(\xi) = Q_0 + \xi Q_1 + ... + \xi^{d-1} Q_{d-1} + \xi^d Q_d$ in such a way that the resultant stacked matrix $W(\xi) = W_0 + \xi W_1 + ... + \xi^{d-1} W_{d-1} + \xi^d W_d$ (where $W_i = [P_i; Q_i], i = 1..d$) is regular and well posed, i.e., $det(W_0) \neq 0$ and $det(W_d) \neq 0$.*

D. Matrix pencils as mathematical relations

It is known [3] that a pencil can be defined as a relation.
A matrix pencil $\xi E - A, E, A \in \mathbb{R}^{m_p \times n_p}$ defines the relation $\Xi \subset \mathbb{R}^{n_p} \times \mathbb{R}^{n_p}$

$$\Xi = \{(x, y) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_p} | Ey = Ax\} = null([A - E])$$

for $\Xi \in \mathbb{R}^{2n_p \times 2n_p - m_p}$. The latter has as special case to $\xi E - A$ for $y = \xi x$. This relation has an advantage. It can consider the joined effect of A and E in only one matrix (independent of ξ), $null([A - E])$. By constructing all these relations for the known twelve examples, and applying a QR factorization on each of them, it is possible to appreciate how Ξ looses rank for big values of d. Although it is not surprising (after having seen the material of the latter sections) Ξ can show also that the pencil (constructed from those two matrices) looses rank. We got figures where this is evident. They look like figure 1.

E. How does all of this fit in the embedding algorithm?

After collecting the observations we have, we can conclude that within the embedding algorithm happens the following:
a) The linearization has to be changed in such a way it can satisfy geometric - numeric restrictions (to guarantee controllability).
b) $P(\xi)$ has to be stable in the sense of section V.
c) $f(\lambda) > 1$ to guarantee stability of the obtained $\xi E - A$ (and $\xi \widehat{E} - \widehat{A}$). See also [16].
d) This way of embedding is ill conditioned and potentially ill posed.
e) The lack of accuracy is explained by d).
f) In addition, a new linearization has to be aware of the leading coefficient matrix singularity (i.e when there are infinite eigenvalues), because not every linearizations preserve the partial multiplicity of the eigenvalues at infinity [12]. So, we would consider the PEP as well.
g) K_F has to be redefined. It seems that keeping it constant, the spectrum of $W(\xi)$ will be shifted to the right hand side of the complex plane.
h) The physical structure of $P(\xi)$ may be taken into account. It is well known that there exist ways to deal with such linear Euler-Lagrange systems.
i) The condition number of a pencil has to be involved.
A method which considers the items given above will be proposed in the near future. It is electrical engineering based and it is inverse eigenvalue problem -like (IPEP, [5]).

[12]The Matlab command *polyeig* was used in this section to solve the PEP corresponding to each example of the table. The syntax is *[Vecs,Lam]=polyeig(W0,...,Wd). Vecs* contains the right eigenvectors and *Lambda* records the eigenvalues.

REFERENCES

[1] Z. Bai, J. Demmel, A. McKenney, *On computing condition numbers for the nonsymetric eigenproblem*, ACM Transactions on Mathematical Software. Vol.19, No.2, June 1993, 202-223.
[2] R. Bhatia, P. Semrl, *Orthogonality of matrices and some distance problems*, Linear Algebra and its Applications. Vol. 287 (1999), pp. 77-85.
[3] P. Benner, R. Byers, *An arithmetic for rectangular matrix pencils*, In A. Beghi, L. Finesso and G. Picci, editors, Mathematical Theory for Networks and Systems, pages 573-576,Il Poligrafo, Padova, Italia, 1998.
[4] G. Bountry et al., *The generalized Eigenvalue problem for Non-Square Pencils Using a Minimal Perturbation Approach*, Accepted for publication in SIAM Journal On Matrix Analysis and Applications, May 2003.
[5] M. Chiu, *Inverse eigenvalue problems*, SIAM REV. Vol. 40, No.1, pp. 1-39, March 1998.
[6] T. Beelen, *New algorithms for computing the Kronecker canonical form of a Pencil with applications to systems and control theory*, PhD Thesis, Eindhoven Institute of Technology, 1985.
[7] T. Beelen, P. Van Dooren, *A pencil approach for embedding a polynomial matrix into a unimodular matrix*, SIAM J. Matrix Anal. Appl. Vol. 9, No. 1, January 1, 1988.
[8] J.M. García, I. de Hoyos, *Safety Neighborhoods for the Kronecker Canonical Form*
[9] E. Elmroth, Ph. D. Thesis, 1995. Department of Computing Science. Umea University, Sweden.
[10] F. R. Gantmacher, *The theory of matrices*,Vol. I, Vol. II, Chelsea Publishing Co. New York, N. Y., 1971.
[11] T. Khailath, *Linear systems*, Prentice Hall Information and Systems Science Series, Thomas Kailath editor, 1980.
[12] F. Tisseur, K. Meerbrengen, *The Quadratic Eigenvalue Problem*, 2001 SIAM Review, Vol. 43, No. 2, pp. 236-286.
[13] N. Higham, F. Tisseur, *More on pseudospectra for polynomial eigenvalue problems and applications in control theory*, Linear Algebra and its Applications 351-352 (2002) 435-453.
[14] F. Tisseur, *Backward error and condition for polynomial eigenvalue problems*, Linear Algebra and its Applications 309 (2000) 339-361.
[15] T. Wright, L. Trefethen,*Pseudospectra of rectangular matrices*, IMA Journal of Numerical Analysis (2002) 22, 501-519.
[16] R. Zavala Yoé, C. Praagman, H.L. Trentelman, *The stable embedding problem*, International Federation of Automatic Control (IFAC), Prag, Czech Republic, 2005.

Appendix

$d = 1$	$P(\xi)$	$\|\|(\delta E, \delta A)\|\|$	$f(\lambda)$	$\sigma_{min}(\delta Y)$
Ex_T	$(3, 6)$	0.3333	1	0
Ex_0	$(2, 3)$	0.2998	1.1919	1.6914×10^{-17}
Ex_1	$(5, 14)$	0.4256	4.2	3.257×10^{-17}
Ex_2	$(7, 9)$	0.0095	2.3	0.6837
$d = 2$				
Ex_1	$(5, 24)$	0.1584	1	6.3119×10^{-17}
Ex_2	$(4, 6)$	8.243×10^{-5}	0.5604	1.7230×10^{-16}
$d = 3$				
Ex_1	$(3, 4)$	0.0014	0.1990	0.0676
Ex_2	$(3, 5)$	2.2923×10^{-4}	0.4789	0.1961
Ex_3	$(4, 6)$	1.2286×10^{-4}	0.3250	2.1079×10^{-17}
$d = 4$				
Ex_1	$(4, 5)$	4.827×10^{-5}	0.1131	0.0717
$d = 6$				
Ex_K	$(3, 5)$	7.2310×10^{-18}	0.0071	0.1551
$d = 7$				
Ex_M	$(2, 5)$	3.2097×10^{-18}	1.9655×10^{-5}	$Notfrr$