Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Global Practical Output Regulation of a Class of Nonlinear Systems by Output Feedback

Qi Gong and Chunjiang Qian

Abstract— **In this paper we solve the global practical output regulation problem for a class of nonlinear systems by dynamic output feedback control. Unlike most of the existing results where the unmeasurable states in the nonlinear vector field can only grow linearly, we allow higher-order growth of unmeasurable states. The proposed controller makes the tracking error arbitrarily small and demonstrates nice properties such as robustness to the disturbances and universality to any continuously differentiable references.**

I. Introduction and Problem Statement

Consider the output regulation problem of the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + \phi_1(x, u, d(t)) \\ \dot{x}_2 &= x_3 + \phi_2(x, u, d(t)) \\ &\vdots \\ \dot{x}_n &= u + \phi_n(x, u, d(t)) \\ y &= x_1 - y_r(t) \end{aligned} \qquad (1)$$

where $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system state, input and measurement output respectively. $d(t) \in \mathbb{R}^s$ represents the unknown disturbances. For $i = 1, 2, \ldots, n$, $\phi_i(x, u, d(t))$ are unknown nonlinear functions of the states, input and disturbances. The goal is to regulate the output y to zero.

Unlike the tracking case, where the reference as well as its derivatives is assumed to be measurable, the only measurable signal in system (1) is the error between the first state x_1 and the reference y_r. There are several reasons for such a consideration. First, in a practical system, it is not unusual that the error signal is the one to be directly measured. For example, in some missile systems, instead of measuring the absolute position of the moving target, i.e, $y_r(t)$, the onboard radar keeps measuring the distance between the missile and the target. In other words, it is the error been measured. Assuming only error signal also makes the actuator design simple, since the controller does not depend on the signal to be tracked explicitly. In this way, the controller is more adaptive to different kinds of references.

In traditional output regulation theory [13], [8], the only measurable signal is the error between the state and the reference, which is the same as our problem setting. But the reference and the disturbance, in traditional output regulation theory, are assumed to be generated by a neutrally stable exosystem. In our problem setting, we remove such restrictions; instead, assume the reference signal $y_r(t)$ satisfying the following assumption.

Assumption 1: The reference signal $y_r(t)$ is continuously differentiable. Moreover, there is a known constant $M > 0$, such that

$$|y_r(t)| + |\dot{y}_r(t)| \quad \leq \quad M, \qquad \forall t \in [0, \infty).$$

The condition on the disturbance $d(t)$ is implicitly contained in Assumption 2, which basically allows $d(t)$ to be any bounded disturbances.

Such relaxations do not come free. The price been paid is the solvability to achieve *asymptotic* tracking and *asymptotic* disturbance rejection. For example, in the case of linear systems, the celebrated *internal model principle* [4], [13], [8], [9] indicates that any regulator that solves the asymptotic tracking problem must incorporate a suitable internal model of the exosystem which generates the disturbance and the reference. In our case, since the disturbance $d(t)$ and the reference $y(t)$ are assumed to be unknown and do not belong to any prescribed class of signals, we do not know what kind of exosystem can generate them. The lack of the information on the exosystem makes the asymptotic tracking extremely difficult. Being aware of aforementioned difficulties, we pursue a less ambitious goal and focus on global practical output regulation instead of asymptotic one.

The global practical output regulation problem: For any given tolerance $\epsilon > 0$, design a dynamic output feedback controller u of the form

$$\begin{aligned} \dot{\xi} &= \alpha(\xi, y), \quad \xi \in \mathbb{R}^{m} \\ u &= u(\xi, y) \end{aligned} \qquad (2)$$

such that

1) the state of the closed-loop system (1)-(2) is well defined on $t \in [0, \infty)$ and globally bounded;
2) for any initial condition $(x(0), \xi(0))$, there is a finite time $T > 0$, such that

$$|y(t)| \ = |x_1(t) - y_r(t)| \ \leq \epsilon, \ \forall t > T.$$

For most practical control systems, such a relaxation does not sacrifice too much, since the stable error ϵ can be rendered arbitrarily small.

When the reference and the disturbance are generated by suitable exosystem, the global asymptotic output regulation

The research was supported in part by the U.S. National Science Foundation under Grant ECS-023915.

Qi Gong, Department of Mechanical and Astronautical Engineering, Naval Postgraduate School, Monterey, CA 93943, qgong@nps.edu

Chunjiang Qian, Department of Electrical Engineering, The University of Texas at San Antonio, San Antonio, TX 78249, cqian@utsa.edu

has been solved for the nonlinear systems in the output feedback form [1], [2], [3], [6]. A unique feature for the output feedback form is that the nonlinearities of the system can only depend on the measurement y. When nonlinear systems satisfy global Lipschitz or linear growth type of conditions, the asymptotic tracking can be tackled by the methods presented in [5], [16], [17]. To the best of our knowledge, most existing results of global output feedback tracking problems impose restrictive requirements on the unmeasurable states, and can not allow the unmeasurable states growing faster than linearly. The main focus of this paper is to solve tracking problem while allowing higher-order growing nonlinearities of the unmeasurable states, for instance, system (3).

One particular difficulty imposed by higher-order growing nonlinearities is the lack of observer design tool. So far, many of the global output feedback design methods are fundamentally based on Luenberger type of observer. The linear nature of the observers limits their ability to handle the higher-order growing nonlinearities of the unmeasurable states. Nonlinear observer design methods are proposed in [10], [12]. But they are locally convergent; therefore are not suitable for the global output tracking problems. Recently, a novel homogeneous nonlinear observer design is introduced in [15]. This observer is inherently nonlinear and provides the ability to handle higher-order growing unmeasurable states. It has been shown in [15] that the global stabilization of (1) can be solved by output feedback under suitable growth conditions. In this paper, we extend the result in [15] to solve the global output tracking problem. To this end, the following condition is introduced.

Assumption 2: There are constants $\tau \geq 0$ and $c \geq 0$ such that, for $i = 1, \ldots, n$

$$|\phi_i(x, u, d(t))| \leq c\left(|x_1|^{i\tau+1} + |x_2|^{\frac{i\tau+1}{\tau+1}} + \cdots + |x_i|^{\frac{i\tau+1}{(i-1)\tau+1}}\right) + c.$$

One particular example that satisfies A2 (with $\tau = 2$) is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + x_2 \ln(1 + x_2^2) \\ \dot{x}_3 &= u + d(t)x_2^2 + x_3^{\frac{4}{3}} + x_3^{\frac{1}{3}} + d(t) \end{aligned} \tag{3}$$

where $|d(t)| \leq 1$ is a bounded disturbance. From this example, it is easy to see A2 covers nonlinear systems with higher-order growing, as well as linear and lower-order growing, unmeasurable states. It is in sharp contrast to many existing output feedback design methods, where the nonlinear vector field needs to be Lipschitz or linear growth in the unmeasurable states [5], [16]. On the other hand, the counter examples in [14] indicate that, due to the finite escape time phenomenon, the global output feedback stabilization of systems (1) can not be solved if the nonlinear functions $\phi_i(\cdot)$ growing too fast. From this point of view, A2 is very tight already; see [15] for further explanations.

In the remaining of the paper, we will show that the global practical tracking of system (2) can be solved under A1-A2.

II. Preliminaries

A. A Brief Review of Stabilization Result

In this section, we briefly review a new output feedback stabilization result presented in [15]. Based on homogeneous theory, the result provides a systematic design tool for the construction of dynamic compensators, and is essential in solving our practical tracking problem. Consider the linear system

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \qquad i = 1, \ldots, n-1 \\ \dot{z}_n &= v \\ y &= z_1 \end{aligned} \tag{4}$$

where v is the input and y is the output. For system (4), one can easily design a linear observer plus a linear feedback controller to globally stabilize the system. This method has been extended to nonlinear system (1) with linearly growth condition on the nonlinear vector field [16]. However, the linear nature of this type of design makes it inapplicable to inherently nonlinear systems. For instance, when the nonlinear vector field has higher-order growth terms such as those satisfying Assumption 2, linear dynamic output feedback controller fails to globally stabilize the system.

For the output feedback design of highly nonlinear systems, a genuinely nonlinear observer design method is needed. Recently, a systematic nonlinear observer design tool is proposed in [15] based on homogeneous theory. The method provides a powerful tool for the construction of dynamic output feedback controller of systems with higher-order nonlinearities. Throughout the paper the homogeneity is defined as follows:

Weighted Homogeneity [7], [11], [15]: For fixed coordinates $(x_1, \ldots x_n) \in \mathbb{R}^n$ and real numbers $r_i > 0$, $i = 1, \cdots, n$,

1). the dilation $\Delta_\epsilon(x)$ is defined by

$$\Delta_\epsilon(x) = (\epsilon^{r_1} x_1, \cdots, \epsilon^{r_n} x_n), \quad \forall \epsilon > 0$$

with r_i being called as the weights of the coordinates (For simplicity of notation, we define dilation weight $\Delta = (r_1, \cdots, r_n)$).

2). a function $V \in C(\mathbb{R}^n, \mathbb{R})$ is said to be homogeneous of degree τ if there is a real number $\tau \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R}^n \setminus \{0\}, \epsilon > 0, \; V(\Delta_\epsilon(x)) = \epsilon^\tau V(x_1, \cdots, x_n).$$

3). a vector field $f \in C(\mathbb{R}^n, \mathbb{R})$ is said to be homogeneous of degree τ if there is a real number $\tau \in \mathbb{R}$ such that $\forall x \in \mathbb{R}^n \setminus \{0\}$, and $\epsilon > 0$

$$f_i(\Delta_\epsilon(x)) = \epsilon^{\tau + r_i} f_i(x_1, \cdots, x_n), \; i = 1, \cdots, n.$$

According to [15], one can construct a reduced order homogeneous observer for system (4) as follows

$$
\begin{aligned}
\dot{\eta}_2 &= f_{n+1}(z_1,\eta_2) \ = -l_1\hat{z}_2\\
\hat{z}_2 &= \text{sign}(\eta_2+l_1z_1)|\eta_2+l_1z_1|^{r_2/r_1}\\
\dot{\eta}_3 &= f_{n+2}(z_1,\eta_2,\eta_3) \ = -l_2\hat{z}_3\\
\hat{z}_3 &= \text{sign}(\eta_3+l_2\hat{z}_2)|\eta_3+l_2\hat{z}_2|^{r_3/r_2}\\
&\vdots\\
\dot{\eta}_n &= f_{2n-1}(z_1,\eta_2,\cdots,\eta_n) \ = -l_n\hat{z}_n\\
\hat{z}_n &= \text{sign}(\eta_n+l_{n-1}\hat{z}_{n-1})|\eta_n+l_{n-1}\hat{z}_{n-1}|^{r_n/r_{n-1}} \quad (5)
\end{aligned}
$$

where $(\hat{z}_2,\ldots,\hat{z}_n)$ are the estimations of the unmeasurable states $(z_2,\ldots,z_n)$. The controller can be constructed as

$$v = -\text{sign}(\xi_n)\,|\xi_n|^{(r_n+\tau)/r_n}\,\beta_n \quad (6)$$

with

$$
\begin{aligned}
&z_1^*=0, && \xi_1=\hat{z}_1-z_1^*\\
&z_k^*=-\text{sign}(\xi_{k-1})\,|\xi_{k-1}|^{r_k/r_{k-1}}\,\beta_{k-1}, && \xi_k=\hat{z}_k-z_k^*
\end{aligned}
$$

where $\hat{z}_1=z_1$ and $k=2,\ldots,n$.

In (5) and (6), $l_i>0$ and $\beta_i>0$, $i=1,\ldots,n$, are constant gains to be specified and $r_i=(i-1)\tau+1$ for any constant $\tau\geq 0$. Denote

$$
\begin{aligned}
Z &= (z_1,z_2,\ldots,z_n,\eta_2,\ldots,\eta_n)^T \quad (7)\\
F(Z) &= (z_2,\ldots,z_n,v,f_{n+1},\ldots,f_{2n-1})^T. \quad (8)
\end{aligned}
$$

The closed-loop system (4)-(5)-(6) can be written down in a compact form $\dot{Z}=F(Z)$. Moreover, it can be verified that $F(Z)$ is homogeneous of degree τ with dilation

$$
\begin{aligned}
\Delta = \ &(1,\tau+1,2\tau+1,\cdots,(n-1)\tau+1,\\
&1,\tau+1,\cdots,(n-2)\tau+1). \quad (9)
\end{aligned}
$$

Lemma 1: There exist constant gains $l_i>0$, $\beta_i>0$, $i=1,\ldots,n$, such that the closed-loop system (4)-(5)-(6) admits a Lyapunov function $V(Z)$ with the following properties

1) V is positive definitive and proper with respect to Z;
2) V is homogeneous of degree $2r_n-\tau$ with dilation (9);
3) the derivative of $V(Z)$ along (4)-(5)-(6) satisfies

$$\dot{V}(Z(t)) = \frac{\partial V}{\partial Z}F(Z) \ \leq\ -C\|Z\|_\Delta^{2r_n}$$

where $\|Z\|_\Delta=\sqrt{\sum_{i=1}^{2n-1}\|Z_i\|^{\frac{2}{r_i}}}$ and $C>0$ is a constant.

The detailed proof of Lemma 1 can be found in [15]. For the sake of simplicity, we omit it.

Remark 1: If one set $\tau=0$, it is easy to see $r_i=1$ for all $1\leq i\leq n$. In this case, (5)-(6) reduce to a linear dynamic output feedback controller; and Lemma 1 is simply linear Lyapunov stability theory with $V(\cdot)$ being a quadratic function.

Remark 2: In the case that $\tau>0$, both the observer (5) and the controller (6) are nonlinear. And Lemma 1 guarantees the global asymptotic stability of the closed-loop system (4)-(5)-(6). Of cause, for simple linear system (4), a nonlinear dynamic compensator is unnecessary. But as shown in [15], the nonlinear nature of the observer and the controller makes it possible to deal with system (1) with higher-order nonlinearities.

B. Some Technical Lemmas

In this section we list several useful lemmas. The first two present some interesting properties of homogeneous functions; see [7], [11], [15] for the details on the homogeneous theory and the proofs of these two lemmas.

Lemma 2: Given a dilation weight $\Delta=(r_1,\ldots,r_n)$, suppose $V_1(x)$ and $V_2(x)$ are homogeneous functions of degree τ_1 and τ_2, respectively. Then $V_1(x)V_2(x)$ is also homogeneous with respect to the same dilation Δ. Moreover the homogeneous degree of $V_1(x)V_2(x)$ is $\tau_1+\tau_2$.

Lemma 3: Suppose $V:\mathbb{R}^n\to\mathbb{R}$ is a homogeneous function of degree τ with respect to the dilation weight Δ. Then the following holds:

1). $\dfrac{\partial V}{\partial x_i}$ is still homogeneous of degree $\tau-r_i$ with r_i being the homogeneous weight of x_i.

2). There is a constant $\bar{c}$ such that

$$V(x)\leq \ \bar{c}\|x\|_\Delta^\tau$$

Moreover, if $V(x)$ is positive definite,

$$\underline{c}\|x\|_\Delta^\tau \ \leq\ V(x)$$

for a positive constant $\underline{c}$.

The next two lemmas provide important tools for the nonlinear domination design.

Lemma 4: For $x\in\mathbb{R}$, $y\in\mathbb{R}$, $p\geq 1$ is a constant, the following inequality holds:

$$|x+y|^p \ \leq\ 2^{p-1}\,|x^p+y^p|$$

Lemma 5: Let c, d be positive constants. Given any positive number $\gamma>0$, the following inequality holds:

$$|x|^c|y|^d \ \leq\ \frac{c}{c+d}\gamma|x|^{c+d}+\frac{d}{c+d}\gamma^{-\frac{c}{d}}|y|^{c+d}$$

III. Practical Tracking by Output Feedback

In this section we show how to extend the output feedback stabilization results in [15] to achieve practical tracking of system (1).

Theorem 1: Under Assumptions 1-2, the global practical output regulation problem of system (1) can be solved by a dynamic output feedback controller of the form (2).

Proof: Define $(e_1,e_2,\ldots,e_n)=(y,x_2,\ldots,x_n)$. Then

$$
\begin{aligned}
\dot{e}_i &= e_{i+1}+\hat{\phi}_i(e,u,d(t)), \qquad i=1,\cdots,n-1\\
\dot{e}_n &= u+\hat{\phi}_n(e,u,d(t)) \quad (10)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\phi}_1(\cdot) &= \phi_1(e_1+y_r(t),e_2,\ldots,e_n,u,d(t))-\dot{y}_r(t)\\
\hat{\phi}_i(\cdot) &= \phi_i(e_1+y_r(t),e_2,\ldots,e_n,u,d(t)),
\end{aligned}
$$

$i=2,\ldots,n$. Note that, in the definition of the error signal $(e_1,\cdots,e_n)$, we only change the coordinate of the first state x_1. It is different to the common definition used in solving asymptotic tracking, where the error is defined as the difference between all the states and their steady values.

By Assumptions 1-2 and Lemma 4, it is readily to show that, for $i = 1, \ldots, n$,

$$
\begin{aligned}
|\hat{\phi}_i(\cdot)| &\leq c\left(|e_1 + y_r(t)|^{i\tau+1} + |e_2|^{\frac{i\tau+1}{\tau+1}} + \cdots \right.\\
&\quad \left. +|e_i|^{\frac{i\tau+1}{(i-1)\tau+1}}\right) + |\dot{y}_r(t)| + c \\
&\leq c\left(2^{i\tau}|e_1|^{i\tau+1}| + |e_2|^{\frac{i\tau+1}{\tau+1}} + \cdots + |e_i|^{\frac{i\tau+1}{(i-1)\tau+1}}\right) \\
&\quad +2^{i\tau}|y_r(t)|^{i\tau+1} + |\dot{y}_r(t)| + c \\
&\leq c_1\left(|e_1|^{i\tau+1} + \cdots + |e_i|^{\frac{i\tau+1}{(i-1)\tau+1}}\right) + c_1 \qquad (11)
\end{aligned}
$$

where $c_1 > 0$ is a constant only depending on c (in Assumption 2) and M (in Assumption 1).

Next, introducing the change of coordinates

$$z_i = \frac{e_i}{L^{i-1}},\ i = 1, \ldots, n, \text{ and } v = \frac{u}{L^n}$$

where $L > 1$ is a scaling constant to be determined later, system (10) is transferred to

$$
\begin{aligned}
\dot{z}_i &= Lz_{i+1} + \hat{\phi}_i(\cdot), \qquad i = 1, \cdots, n-1 \\
\dot{z}_n &= Lv + \frac{\hat{\phi}_n(\cdot)}{L^{n-1}}\ . \qquad (12)
\end{aligned}
$$

Following the homogeneous observer and controller design proposed in [15], we construct a dynamic compensator for (12) as

$$
\begin{aligned}
\dot{\eta}_2 &= -Ll_1\hat{z}_2 \\
\hat{z}_2 &= \text{sign}(\eta_2 + l_1 z_1)|\eta_2 + l_1 z_1|^{r_2/r_1} \\
\dot{\eta}_3 &= -Ll_2\hat{z}_3 \\
\hat{z}_3 &= \text{sign}(\eta_3 + l_2\hat{z}_2)|\eta_3 + l_2\hat{z}_2|^{r_3/r_2} \qquad (13)\\
&\vdots \\
\dot{\eta}_n &= -Ll_n\hat{z}_n \\
\hat{z}_n &= \text{sign}(\eta_n + l_{n-1}\hat{z}_{n-1})|\eta_n + l_{n-1}\hat{z}_{n-1}|^{r_n/r_{n-1}}
\end{aligned}
$$

and a controller

$$u = -L^n \text{sign}(\xi_n)\,|\xi_n|^{(r_n+\tau)/r_n}\,\beta_n \qquad (14)$$

with

$$
\begin{aligned}
&z_1^* = 0, && \xi_1 = \hat{z}_1 - z_1^* \\
&z_k^* = -\text{sign}(\xi_{k-1})\,|\xi_{k-1}|^{r_k/r_{k-1}}\,\beta_{k-1}, && \xi_k = \hat{z}_k - z_k^*
\end{aligned}
$$

where $\hat{z}_1 = z_1$ and $k = 2, \ldots, n$. In (13) and (14), $l_i > 0$ and $\beta_i > 0$, $i = 1, \ldots, n$ are constant gains specified as in Lemma 1 and $L > 1$ is a constant to be determined later.

Note that (13)-(14) are in the dynamic output feedback form (2), because the only information used in the construction of (13) and (14) is the measurement $y(t)$. In the next, we will determine the gain L such that the global practical output tracking is achieved.

By using notations (7) and (8), the closed-loop system (12)-(13)-(14) can be written down in a compact form

$$\dot{Z} = LF(Z) + \left(\hat{\phi}_1(\cdot), \frac{\hat{\phi}_2(\cdot)}{L}, \ldots, \frac{\hat{\phi}_n(\cdot)}{L^{n-1}}, 0, \ldots, 0\right)^T.$$

By Lemma 1, there exist constants l_i, β_i, $i = 1, \ldots, n$ and a Lyapunov function $V(Z)$, such that

$$\frac{\partial V}{\partial Z}F(Z) \leq -C\|Z\|_\Delta^{2r_n}.$$

Furthermore, $V(Z)$ is homogeneous of degree $2r_n - \tau$ with dilation (9). Therefore, with these choice of l_i, β_i, the derivative of V along the trajectory of (12)-(13)-(14) satisfies

$$
\begin{aligned}
\dot{V}(Z) &\leq -LC\|Z\|_\Delta^{2r_n} + \frac{\partial V(Z)}{\partial Z}\left[\hat{\phi}_1(\cdot), \frac{\hat{\phi}_2(\cdot)}{L}, \ldots, \right. \\
&\qquad \left. \frac{\hat{\phi}_n(\cdot)}{L^{n-1}}, 0, \ldots, 0\right]^T. \qquad (15)
\end{aligned}
$$

From (11) and the fact $L > 1$, it is readily to deduce that

$$
\begin{aligned}
\left|\frac{\hat{\phi}_i(\cdot)}{L^{i-1}}\right| &\leq c_1\left(|z_1|^{i\tau+1} + |Lz_2|^{\frac{i\tau+1}{\tau+1}} + \cdots + \right. \\
&\qquad \left. |L^{i-1}z_i|^{\frac{i\tau+1}{(i-1)\tau+1}}\right) + \frac{c_1}{L^{i-1}} \\
&\leq c_1 L^{1-\frac{1}{(i-1)\tau+1}}\left(|z_1|^{i\tau+1} + |z_2|^{\frac{i\tau+1}{\tau+1}} + \cdots \right. \\
&\qquad \left. +|z_i|^{\frac{i\tau+1}{(i-1)\tau+1}}\right) + \frac{c_1}{L^{i-1}}. \qquad (16)
\end{aligned}
$$

By Lemma 3 and Lemma 1, $\frac{\partial V}{\partial Z_i}$ is homogeneous of degree $2r_n - \tau - r_i$ for all i. Hence, from Lemma 2,

$$\left|\frac{\partial V}{\partial Z_i}\right|(|z_1|^{i\tau+1} + |z_2|^{\frac{i\tau+1}{\tau+1}} + \cdots + |z_i|^{\frac{i\tau+1}{(i-1)\tau+1}}) \qquad (17)$$

is homogeneous of degree $2r_n$. Now using Lemma 3, we can find a constant $\rho_i > 0$ such that

$$\text{equation (17)} \leq \rho_i\|Z\|_\Delta^{2r_n}. \qquad (18)$$

Substituting (18) into (15) leads to

$$
\begin{aligned}
\dot{V}(Z) &\leq -L(C - c_1\sum_{i=1}^n \rho_i L^{-\frac{1}{(i-1)\tau+1}})\|Z\|_\Delta^{2r_n} + \\
&\qquad c_1\sum_{i=1}^n \left|\frac{\partial V(Z)}{\partial Z_i}\right|\frac{1}{L^{i-1}}. \qquad (19)
\end{aligned}
$$

On the other hand, note that, for $1 \leq i \leq n$, $\left|\frac{\partial V(Z)}{\partial Z_i}\right|$ is homogeneous of degree $2r_n - \tau - r_i$. Therefore, by Lemma 3 and Lemma 5 there are positive constants c_2, c_3, such that

$$
\begin{aligned}
c_1\left|\frac{\partial V(Z)}{\partial Z_1}\right| &\leq c_2\|Z\|_\Delta^{2r_n-\tau-r_1} \\
&= c_2(L^{\frac{1}{2r_n}}\|Z\|_\Delta)^{2r_n-\tau-r_1}(L^{-\frac{2r_n-\tau-r_1}{2r_n(\tau+r_1)}})^{\tau+r_1} \\
&\leq \frac{C}{2}L\|Z\|_\Delta^{2r_n} + c_3 L^{-\frac{2r_n-\tau-r_1}{\tau+r_1}}.
\end{aligned}
$$

Similarly, for $2 \leq i \leq n$,

$$
\begin{aligned}
c_1\left|\frac{\partial V(Z)}{\partial Z_i}\right|\frac{1}{L^{i-1}} &\leq c_2\|Z\|_\Delta^{2r_n-\tau-r_i}(L^{-\frac{i-1}{\tau+r_i}})^{\tau+r_i} \\
&\leq \|Z\|_\Delta^{2r_n} + c_4 L^{-\frac{2(i-1)r_n}{\tau+r_i}}
\end{aligned}
$$

for some constant $c_4 > 0$. Hence,

$$c_1 \sum_{i=1}^{n} \left| \frac{\partial V}{\partial Z_i} \right| \frac{1}{L^{i-1}} \leq L(\frac{C}{2} + (n-1)L^{-1}) \|Z\|_\Delta^{2r_n} + c_3 L^{-\frac{2r_n - \tau - r_1}{\tau + r_1}} + c_4 \sum_{i=2}^{n} L^{-\frac{2(i-1)r_n}{\tau + r_i}}. \quad (20)$$

Estimations (19) and (20) lead to

$$\dot{V}(Z) \leq -L \left(\frac{C}{2} - c_1 \sum_{i=1}^{n} \rho_i L^{-\frac{1}{(i-1)\tau+1}} - (n-1)L^{-1} \right) \cdot \|Z\|_\Delta^{2r_n} + c_3 L^{-\frac{2r_n - \tau - r_1}{\tau + r_1}} + c_4 \sum_{i=2}^{n} L^{-\frac{2(i-1)r_n}{\tau + r_i}}.$$

It can be written down in a compact form

$$\dot{V}(Z) \leq -L[\frac{C}{2} - K_1(L)] \|Z\|_\Delta^{2r_n} + K_2(L) \quad (21)$$

with the notation $K_1(L) = c_1 \sum_{i=1}^{n} \rho_i L^{-\frac{1}{(i-1)\tau+1}} + (n-1)L^{-1}$ and $K_2(L) = c_3 L^{-\frac{2r_n-\tau-r_1}{\tau+r_1}} + c_4 \sum_{i=2}^{n} L^{-\frac{2(i-1)r_n}{\tau+r_i}}$. It is easy to see both K_1 and K_2 are positive and monotone decreasing to zero as L increases. Next, we will show that (21) implies the existence of a gain L to achieve the global practical tracking of system (1).

Since $V(Z)$ is homogeneous of degree $2r_n - \tau$ and positive definite, by Lemma 3, there are two positive constants α_1, α_2 such that

$$\alpha_1 \|Z\|_\Delta^{2r_n - \tau} \leq V(Z) \leq \alpha_2 \|Z\|_\Delta^{2r_n - \tau}. \quad (22)$$

Therefore

$$\dot{V}(Z) \leq -L[\frac{C}{2} - K_1(L)] \alpha_2^{-\frac{2r_n}{2r_n - \tau}} V(Z)^{\frac{2r_n}{2r_n - \tau}} + K_2(L).$$

Using monotone decreasing property of K_1, one can find a sufficiently large L, which renders

$$\dot{V}(Z) \leq -V(Z)^{\frac{2r_n}{2r_n - \tau}} + K_2(L).$$

From here it is not difficult to show that, there is a finite time T such that

$$V(Z) \leq (2K_2(L))^{\frac{2r_n - \tau}{2r_n}} \quad (23)$$

for all $t \geq T$. By (23) and (22), for all $t \geq T$

$$|y(t)| \leq \|Z\|_\Delta \leq \alpha_1^{\frac{-1}{2r_n - \tau}} (2K_2(L))^{\frac{1}{2r_n}}.$$

Now, from monotone decreasing property of K_2, for any given tolerance ϵ there is a sufficiently large L such that

$$\alpha_1^{\frac{-1}{2r_n - \tau}} (2K_2(L))^{\frac{1}{2r_n}} \leq \epsilon.$$

That is $|y(t)| \leq \epsilon$, for all $t \geq T$. ■

Remark 3: In the observer and the controller design, the gain L needs to be assigned as a sufficiently large number to achieve the given tracking accuracy. The value of L depends on the bounds of the reference and its first order derivative. In other words, once the bound of $|y_r(t)| + |\dot{y}_r(t)|$ and desired accuracy ϵ are given, the gain L can be determined. However, to calculate the precise value of L could be tedious and most likely conservative. In practice, one can simply choose a large L such that the closed-loop system is stable; then keep increasing L until the given tracking accuracy is achieved.

Remark 4: Note that the controller (14) and the observer (13) are constructed only based on the nominal system (4). No precise information of the nonlinearities is needed. In other words, the same dynamic controller (13)-(14) can be applied to different nonlinear systems as long as they satisfy Assumption 2. This property makes it possible to deal with nonlinear systems with unknown disturbances. Also note that, there are only three set of parameters l_i, β_i and L need to be determined in our dynamic compensator. The choice of l_i and β_i only depends on the nominal system (4). Therefore, they can be pre-fixed even for different nonlinear systems. The gain L can be determined according to Remark 3. This advantage greatly reduces the design complexity normally associated with the dynamic output feedback design.

Remark 5: In the proposed dynamic output feedback controller (13)-(14), no precise information of the reference is needed. Therefore the controller remains the same even the reference is changed (assuming A1 is satisfied with a fixed bound M). This property not only simplifies the design procedure, but also provides the ability to handle uncertainties in the reference. For example, in output regulation case, the reference is generated by a neutrally stable exosystem [13]

$$\dot{w} = s(w) \quad (24)$$

and the measurement is

$$y = x_1 - q(w). \quad (25)$$

Since the initial condition of the exosystem is unknown, the trajectory of the reference $q(w)$ is not available for the design of the controller. However, if we assume to know the bound of $w(t)$, or equivalently the bound of the initial state $w(0)$, it is easy to see Assumption 1 holds with $y_r = q(w)$. Therefore, our dynamic controller can be applied to system (1) with exosystem (24) and measurement (25) to achieve the practical output regulation.

In the analysis in Theorem 1, we assume the reference signal $y_r(t)$ to be continuously differentiable. Actually, this assumption can be easily relaxed to cover some reference that is only continuous. For example, let $y_r(t)$ be continuous, periodic and of bounded variation on each period. Then, for any given accuracy ϵ, there is a smooth approximation $\hat{y}_r(t)$ satisfying Assumption 1 and

$$|y_r(t) - \hat{y}_r(t)| \leq \frac{\epsilon}{2}, \quad \forall t \in [0, \infty). \quad (26)$$

One choice of such smooth approximation is the Fourier extension of the reference. Then, by Theorem 1, one can design a dynamic output feedback controller to practically tracking $\hat{y}_r(t)$ with the error bound $\frac{\epsilon}{2}$. The same controller will guarantee the error between x_1 and the real reference $y_r(t)$ to be less than ϵ after a finite time. Based on this analysis, the following corollary can be easily proved.

Corollary 1: For any given continuous, periodic reference $y_r(t)$ with bounded variation on each period, the global practical output regulation problem of system (1) can be solved by a dynamic output feedback controller of the form (2) under Assumption 2.

IV. An Illustrative Example

Consider the following nonlinear system which describes a particle moving under nonlinear viscous friction

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u - \text{sign}(x_2)|x_2|^{\alpha} \\ y &= x_1 - \sin(t) \end{aligned} \tag{27}$$

where x_1 is the displacement, x_2 is the velocity and u is the control force. The term $\text{sign}(x_2)|x_2|^{\alpha}$ represents the nonlinear viscous friction. It is assumed that $1 \le \alpha \le \frac{5}{3}$. However the precise value of α is unknown. The control objective is to force the state x_1 to track the reference $\sin(t)$ using the measurement $y(t)$ only.

Note that, although the parameter α can be estimated by experiment, it may not be a constant due to the change of the working environment. Therefore, it is quite desirable to construct a controller not depending on the precise value of α. On the other hand, the measurement of the velocity can be expensive in practice. Assuming the measurement of displacement error y is more reasonable in reality. These two limitations make the controller design very difficult by existing methods.

Clearly, system (27) satisfies Assumption 2 with $\tau = 2$. According to (13) and (14), we can construct a dynamic output feedback controller as

$$\begin{aligned} \dot{\eta} &= -L(\eta + y)^3 \\ \hat{z}_2 &= (\eta + y)^3 \\ u &= -L^2(\hat{z}_2 + 10y^3)^{\frac{5}{3}} \end{aligned} \tag{28}$$

By Theorem 1, with properly chosen L, the tracking error can be made arbitrarily small. In Fig.1 we plot out the simulation results when $\alpha = 1.5$. The gain L is chosen as 10 in Fig.1.a, and the stable error is about 0.2. In Fig.1.b, the gain is increased to 85 and the stable error reduces to 0.1.

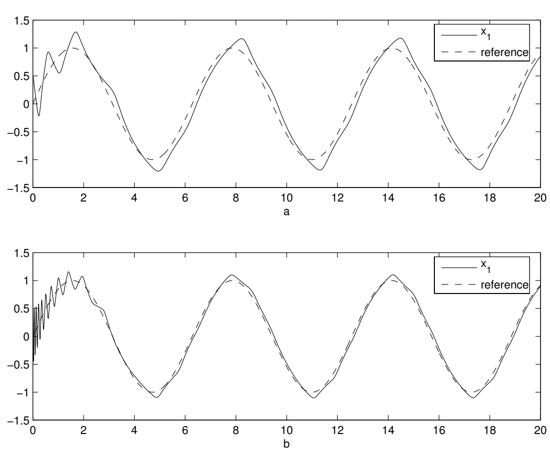

Fig. 1. Simulation results for the closed loop system (27)—(28) with $\alpha = 1.5$, $L = 10$ (Figure 1.a) and $L = 85$ (Figure 1.b).

Next we changed the reference to

$$y_r(t) = \begin{cases} \sin(\pi t) & 6k \le t \le 6k + 0.5 \\ 1 & 6k + 0.5 \le t \le 6k + 2.5 \\ \sin(\pi(t-2)) & 6k + 2.5 \le t \le 6k + 3.5 \\ -1 & 6k + 3.5 \le t \le 6k + 5.5 \\ \sin(\pi(t-4)) & 6k + 5.5 \le t \le 6(k+1). \end{cases} \tag{29}$$

where $k = 0, 1, 2, \cdots$. The observer and the controller remain the same as before. The numerical experiment demonstrates that the very same controller, without any change, achieves practical tracking for different reference (29).

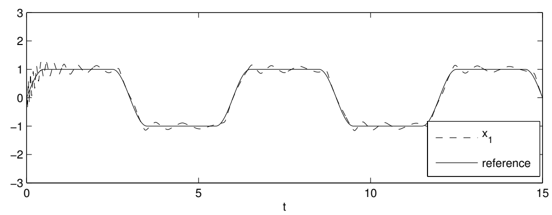

Fig. 2. Simulation results for the closed loop system (27)—(28) with reference (29), $L = 85$ and $\alpha = 1.5$.

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