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Finite Horizon LQ Optimal Control and Computation with Data Rate Constraints

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***Abstract*— This paper considers the optimal control of a linear scalar model with a finite communication rate between the sensor and the controller. We analyze the optimization of the quantizer and the controller where the latter utilizes the overall history of the received symbols to determine the control input. Making use of a tree structure representation, it is shown that the resulting optimal control problem can be reduced to a combined quantization and constrained quadratic minimization problem. We characterize the necessary conditions for the optimal control and develop numerical algorithms. A localized computational method for long time horizon is also discussed.**

I. Introduction

There has been a rapid accumulation of literature on control with communication rate constraints following the work [2]. The reader is referred to [11], [12], [13], [16] and references therein for system stabilization and the associated algebraic conditions for stabilizability. For state estimation, see [9]. On the other hand, from the practical point of view, it is desirable to consider system performance optimization with the presence of rate constraints rather than merely stabilizing a model. Compared to traditional optimal control, one then faces the task of designing both the encoder which involves quantization, and the controller. Along this line of research, the linear quadratic Gaussian (LQG) control problem was first considered in [1] with a finite digital channel described by an input-output transition matrix where the quantizer is optimal for the underlying i.i.d. innovation process and hence is essentially static. In [7], the LQG problem is studied with a noiseless digital channel and a theorem is obtained concerning the separation of estimation and control under the assumption that the control depends only on the most recently received symbol. In [14], the authors obtained separation theorems for the LQG control problem where the control is a linear function of the state estimate. In [3], the authors considered stabilization of a linear system and gave the relation between controller complexity and performance in terms of the time required for driving the state into a specified region containing the origin.

Although there has been a fair amount of literature dealing with the communication rate constrained LQG control problem under different formulations, there have been few algorithmic results for the computation of the optimal encoder and controller. Due to the introduction of quantization, the computation of optimal control becomes extraordinarily challenging. In order to reduce optimization complexity, uniform quantizers are usually implemented; see, e.g., [3], [15] for either control or estimation, or both.

In this paper, we consider the optimal linear quadratic regulation (LQR) problem with data rate constraints. The main interest of this work is in analyzing the structure of the encoder and controller. Differing from most optimal control research mentioned above, we will allow the encoder and controller to possess memory as adopted in the work [10] which considers stabilization. By assuming such a structure for the encoder and controller, one can avoid potential wastage of useful information carried by the past data and may achieve better performance. In fact for many networked control systems the controller itself is a computer located away from the plant and has a certain storage capacity for past data which have been collected by the channel. Thus the control input at each time instant may be computed using all the past channel outputs.

Our work seeks to develop a framework for the parametrization and optimality specification for causal control laws, which is potentially useful for devising tractable computational methods. Specifically, we introduce a tree structure for the realization/parametrization of causal control laws where each "long branch" in the tree corresponds to a subset in the partition of the random initial state. It is worth noting that a similar tree structure is also introduced in [4] for the representation of optimal controls with memory, and certain lower and upper bounds for the optimal cost are obtained; the authors did not propose concrete characterization of the control law or computational methods. Subsequently, based on the tree structure, a deterministic quadratic minimization problem with linear equality constraints is formulated, and further employed in the derivation of necessary conditions for the optimal control law. This is useful for the development of Lloyd-Max type algorithms for the search of optimal quantization and controls.

II. The System Model

To make the analysis more tractable, we consider a scalar discrete-time system:

$$x_{t+1} = ax_t + bu_t, \qquad t \geq 0. \tag{1}$$

The initial condition x_0 is random with a density function $f(x)$ on $\mathbb{R}$ and $Ex_0^2 < \infty$. To avoid triviality we assume $a \neq 0$ and $b \neq 0$.

The structure of the control system is described as follows. The state space model specifies the dynamic behaviour of the

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plant at a certain site, and the controller is located at a remote site. The state variable x_t is measured by a sensor exactly and transmitted to an *encoder* Er which employs a quantizer to produce its output in a finite set. Then a *controller* Cr utilizes the information conveyed from the encoder Er via a noiseless digital channel with a fixed finite rate R. Here we simply use the controller to refer to the mechanism to produce the control input for the plant based on the output of the channel and it may have a composite structure including a *decoder* Dr corresponding to Er, and its own encoder at that remote site and a matching decoder located near the plant to generate the input.

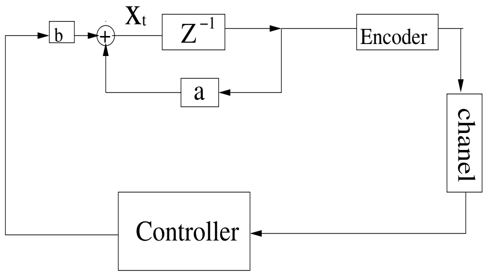

Fig. 1. Components of the system

For the LQR problem, the cost function to be minimized is of the form:

$$J = \sum_{t=0}^{N} E(x_{t+1}^2 + r u_t^2),$$

where $N+1$ is the time horizon and $r > 0$.

Suppose at each time instance, the encoder Er uses the same codebook $Q \triangleq \{0, 1, \cdots, M-1\}$, which may be realized by a set of $M = 2^R$ codewords of length R. We represent the output symbol s_t of the quantizer associated with the encoder Er at time t by a nonlinear function η mapping $(t, x_0, x_1, \cdots, x_t)$ to Q, i.e.,

$$s_t = \eta(t, x_0, x_1, \cdots, x_t), \qquad t \geq 0. \tag{2}$$

From now on, a sequence $\{y_k\}_{k=0}^t$ is denoted by $\tilde{y}_t$, and we may write $s_t = \eta(t, \tilde{x}_t)$. The function $\eta(t, \cdot)$ is Borel measurable on $\mathbb{R}^{t+1}$. The symbol s_t is transmitted by the noiseless channel and is available to the controller at the next time instant $t+1$.

We write the control u_t in the form:

$$u_t = \tau(t, \tilde{s}_{t-1}). \tag{3}$$

The control u_0 does not depend on the channel outputs, but is still formally written in the above form. For the case $u_t = \tau(t, s_{t-1})$, we call the controller memoryless. Let $\mathcal{U}_t = \{\tau(t, \tilde{s}_{t-1}) : \tilde{s}_{t-1} \in Q^t\}$. The set $\mathcal{U}_t$ is the image of the map $\tau(t, \cdot)$ and is called the set of reproduction alphabets. A major part of our optimal control problem is to determine $\mathcal{U}_t$ such that one can select the real time control input by matching the received symbol sequence $\tilde{s}_{t-1}$ to a value in $\mathcal{U}_t$. Since u_0 is determined using only the system's prior information, $\mathcal{U}_0$ is a singleton.

Following the technique in [10], by virtue of (1), (3) and induction, we may rewrite (2) in the form $s_t = \eta_t(x_0, \tilde{s}_{t-1})$.

III. The LQR Problem

We first express the state x_{k+1}, $k \geq 0$, in terms of the initial condition x_0 and the control inputs to get

$$x_{k+1} = a^{k+1} x_0 + \sum_{j=0}^{k} a^{k-j} b u_j. \tag{4}$$

Based on the cost J, we write $J_0 = \sum_{t=0}^{N} (x_{t+1}^2 + r u_t^2)$. By (4), it is obvious that

$$J_0 = \sum_{k=0}^{N} \Big[a^{k+1} x_0 + \sum_{j=0}^{k} a^{k-j} b u_j \Big]^2 + \sum_{k=0}^{N} r u_k^2.$$

Define $\theta_k = \sum_{j=0}^{k} a^{k-j} b u_j$, $k \geq 0$, and $\alpha_N = [\sum_{k=0}^{N} a^{2(k+1)}]^{\frac{1}{2}}$. It follows that

$$\begin{aligned} J_0 =& [\alpha_N x_0 + \alpha_N^{-1} \sum_{k=0}^{N} a^{k+1} \theta_k]^2 - \alpha_N^{-2} [\sum_{k=0}^{N} a^{k+1} \theta_k]^2 \\ & + \sum_{k=0}^{N-1} \theta_k^2 + \sum_{k=0}^{N} \frac{r}{b^2} (\theta_k - a\theta_{k-1})^2 \end{aligned}$$

where we have used the fact

$$u_k = \frac{1}{b} (\theta_k - a\theta_{k-1}),$$

with $\theta_{-1} \triangleq 0$. Let

$$\xi(\tilde{\theta}_N) \triangleq -\alpha_N^{-2} \sum_{k=0}^{N} a^{k+1} \theta_k, \tag{5}$$

$$\begin{aligned} H(\tilde{\theta}_N) \triangleq & -\alpha_N^{-2} [\sum_{k=0}^{N} a^{k+1} \theta_k]^2 + \sum_{k=0}^{N} \theta_k^2 \\ & + \sum_{k=0}^{N} \frac{r}{b^2} (\theta_k - a\theta_{k-1})^2. \end{aligned} \tag{6}$$

By use of (5) and (6), we may rewrite J_0 as

$$J_0 = \alpha_N^2 [x_0 - \xi(\tilde{\theta}_N)]^2 + H(\tilde{\theta}_N). \tag{7}$$

Now suppose a causal control law has been given in the form (3). Since $\xi(\tilde{\theta}_N)$ can take up to M^N different values dependent on the received symbol sequence, the first term at the right hand side of (7) may be regarded as a quantization error scaled by α_N^2. Hence the optimal control problem is essentially based on a tradeoff between minimizing the weighted mean square quantization error $\alpha_N^2 E[x_0 - \xi(\tilde{\theta}_N)]^2$ and minimizing $EH(\tilde{\theta}_N)$ which is called the *residual term*.

$H(\tilde{\theta}_N)$ may be treated as a quadratic form defined on $\mathbb{R}^{N+1}$ when no restriction is imposed on the range of $\tilde{\theta}_N$.

Proposition 1: The quadratic form $H(\tilde{\theta}_N)$ is strictly positive definite for $\tilde{\theta}_N \in \mathbb{R}^{N+1}$. □

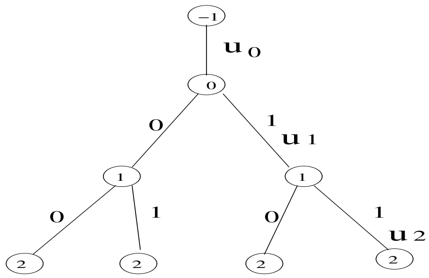

Fig. 2. A tree representation of the causal control. The structure may be used to represent any causal map $\tilde{\Phi}_N$.

IV. Representation of Causal Control Laws

In further analysis, we may denote by $\tilde{u}_N$ either a control law being a function from Q^N (the set of all possible symbol sequences up to time $N-1$) to $\mathbb{R}^{N+1}$, or just a vector in $\mathbb{R}^{N+1}$. There should be no risk of causing confusion.

Proposition 2: Let $-\infty < q_0 \leq q_1 \leq q_{M^N-1} < \infty$ be any given sequence. There exists a causal control law $\tilde{u}_N$ such that for each q_i, $0 \leq i \leq M^N - 1$, there exists $\tilde{s}_{N-1} \in Q^N$ satisfying $\sum_{k=0}^{N} a^{k+1}\theta_k = q_i$ where $\theta_k = \sum_{j=0}^{k} a^{k-j} b u_j(\tilde{s}_{j-1})$.

Proof: Suppose u_k, $0 \leq k \leq N-1$, have been selected for all possible symbol sequence $\tilde{s}_{N-2}$. This accordingly determines all possible values for $\sum_{k=0}^{N-1} a^{k+1}\theta_k$. Since u_N can be selected arbitrarily for a given $\tilde{s}_{N-1}$, we may adjust the range of θ_N to be a set of M^N arbitrarily chosen values, and the proposition follows. □

The above proposition indicates that if only causality is imposed for the controls, the system can adjust $\xi(\tilde{\theta}_N)$ to any form with a maximum of M^N levels.

To facilitate our analysis, we introduce the formal definition for causality for any mapping $\tilde{\Phi}_N$ from Q^N to $\mathbb{R}^{N+1}$.

Definition 3: For any $\tilde{s} = (s_0, s_1, \cdots, s_{N-1}) \in Q^N$, $\tilde{s}' = (s'_0, s'_1, \cdots, s'_{N-1}) \in Q^N$ and $k \geq 1$, if (i) $s_j = s'_j$ for $0 \leq j \leq k-1$ implies $\Phi_k(\tilde{s}) = \Phi_k(\tilde{s}')$, and (ii) $\Phi_0(\tilde{s}) = \Phi_0(\tilde{s}')$ for all $\tilde{s}, \tilde{s}' \in Q^N$, then $\tilde{\Phi}_N$ is said to be causal. □

Notice that each entry Φ_k itself is treated as a map from Q^N to $\mathbb{R}$. In our control problem, the input u_t at time t only depends on $\tilde{s}_{t-1}$. However, we can always extend $\tilde{s}_{t-1}$ by adding an additional segment to form an entry in Q^N. Hence we may regard u_t as a function defined on Q^N.

Proposition 4: $\tilde{\theta}_N$ is causal if and only of $\tilde{u}_N$ is causal.

Proof: This follows from the nonsingular transform between $\tilde{\theta}_N$ and $\tilde{u}_N$. □

Proposition 5: If a map $\tilde{\Phi}_N = (\Phi_0, \cdots, \Phi_N)$ is causal, then each Φ_t, $0 \leq t \leq N$, can take from a set $\mathcal{U}_t$ with up to M^t values. On the other hand, for a collection of $\mathcal{U}_t$, $0 \leq t \leq N$, each with up to M^t values, we can always construct a causal map $\tilde{\Phi}_N = (\Phi_0, \cdots, \Phi_N)$ such that $\mathcal{U}_t$ is the image space of Φ_t.

Proof: The first assertion is evident. We prove the second one. Without loss of generality we assume each $\mathcal{U}_k$ contains M^k possibly repeated values. Thus, by a slight abuse of the terminology for a set, $\mathcal{U}_k$ may contain more than one copies of the same number. We compose the product set $\mathcal{U}_0 \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_N$.

Now we construct a tree with $N+1$ layer of nodes, labelled by $-1, 0, \cdots, N-1$, where the kth layer, $-1 \leq k \leq N-1$ has $M^{\max\{k,0\}}$ nodes. Starting from the 0th layer, each (parent) node at an upper layer is connected by an edge with M (child) nodes at the next layer. We label by $0, 1, \cdots, M-1$ these M edges sharing a parent node. Obviously, the M integers may appear more than once between two neighboring layers of nodes since they can be used by multiple parent nodes at the same layer. We do not label the edge between nodes -1 and 0 as illustrated in Fig. 2.

We attach each entry in $\mathcal{U}_k$, $k = 0, 1, 2\cdots$, to an edge between the $(k-1)$th and kth layers of nodes. In the end, each edge is associated with a number in $\mathcal{U}_k$.

Now for any symbol sequence $\tilde{s} \stackrel{\triangle}{=} s_0 s_1 \cdots s_{N-1}$ we can identify a unique path (also represented by $s_0 s_1 \cdots s_{N-1}$ in the tree starting from the single node at level 0, going along edge s_0 to reach the next node, continuing there with edge s_1, and so on, until reaching a bottom node. For the given $\tilde{s}$, we define Φ_k, $k \geq 1$, to be the value (in $\mathcal{U}_k$) which is previously attached to the corresponding edge along the path $s_0 s_1 \cdots s_{N-1}$. Let Φ_0 be the single entry in $\mathcal{U}_0$. We set $\tilde{\Phi}(\tilde{s}) = (\Phi_0, \cdots, \Phi_N)$. Then the map $\tilde{\Phi}$ is causal and the image of Φ_t is equal to the whole set $\mathcal{U}_t$, $0 \leq t \leq N$. □

By Proposition 5, it is clearly seen that each causal control law can be parametrized by use of a tree structure where each edge is attached with a real number. Since at $t = 0$ there is no symbol available at the controller, one can only set u_0 and hence θ_0 as a single value to be optimized.

V. Necessary Conditions for Optimal Controls

In this section we show that the data rate constrained LQR problem can be re-expressed as a combined quantization and quadratic minimization problem subject to linear constraints.

A. Specification of linear equality constraints

Recalling Proposition 4 and the nonsingular linear transform between two $N+1$ dimensional values of $\tilde{\theta}_N$ and $\tilde{u}_N$, it is clear that finding an optimal control law $\tilde{u}_N$ is equivalent to finding a causal map $\tilde{\theta}_N$ (with an associated quantization of x_0) such that $EJ_0 = E\{\alpha_N^2[x_0 - \xi(\tilde{\theta}_N)]^2 + H(\tilde{\theta}_N)\}$ is minimized. Recall that we have defined $\xi(\tilde{\theta}_N) = \alpha_N^{-2}\sum_{k=0}^{N} a^{k+1}\theta_k$. For the following analysis we regard $\tilde{\theta}_N$ as a function from Q^N to $\mathbb{R}^{N+1}$, and hence $\xi(\tilde{\theta}_N)$ may take up to M^N distinct values. We list two possible scenarios:

(a) Under the optimal $\tilde{\theta}_N$, $\xi(\tilde{\theta}_N)$, as a random variable since each symbol sequence $\tilde{s}_{N-1}$ is associated with a certain region of x_0, takes M^N different values each with a positive probability.

(b) Under the optimal $\tilde{\theta}_N$, $\xi(\tilde{\theta}_N)$ takes less than M^N different values.

If we are only required to minimize $E|x_0 - \xi(\tilde{\theta}_N)|^2$, $\xi(\tilde{\theta}_N)$ will belong to scenario (a) since x_0 has a density function. However, in the optimal LQR problem, this cannot be immediately claimed due to the additional term $H(\tilde{\theta}_N)$. In general, for an optimal control law falling into scenario (b), the parametrization of the control is more difficult since it is implied that at certain time stages some symbols in the codebook are never used regardless of the value of x_0. Thus one faces the issue of optimally deleting some branches in the original tree structure (with M^N bottom nodes) such that the number of bottom nodes is equal to the number of values of $\xi(\tilde{\theta}_N)$. In the present work, we do not intend to characterize optimal controls for this scenario[1]. Our intuition is that scenario (b), if existing at all, rarely occurs.

In the following, we analyze optimality conditions for scenario (a) which means that all symbols in the codebook are exhaustively used to give the finest possible partition of the range space of the initial state x_0. Now assume the causal map $\tilde{\theta}_N$ minimizes EJ_0. We compose the tree representation of $\tilde{\theta}_N$ and label from left to right the M^N paths connect the top and bottom nodes by $\tilde{s}_{N-1}(0), \tilde{s}_{N-1}(1), \cdots, \tilde{s}_{N-1}(M^N-1)$. Since each vector value of $\tilde{\theta}_N$ is associated with a path, we write the vector as $\tilde{\theta}_N(\tilde{s}_{N-1}(i))$, $0 \le i \le M^N-1$, which is associated with a constituent region in the partition of x_0.

For scenario (a), suppose for the optimal $\tilde{\theta}_N$ we have the distinct numbers $q(0), \cdots, q(M^N-1)$ which are matched to the paths by the relation

$$\xi[\tilde{\theta}_N(\tilde{s}_{N-1}(i))] = q(i), \qquad 0 \le i \le M^N-1. \tag{8}$$

B. Structure of optimal partition

To compute the optimal control law, we partition the range space of x_0 for constructing an optimal causal map $\tilde{\theta}_N$ such that $q(i)$ in (8) is matched to a certain region for x_0.

Before proceeding, we first analyze the following optimal matching problem, which is essentially quantization subject to probability constraints and frozen centers in contrast to the usual Voronoi partition [6].

Suppose $g(x)$ is a probability density function on $\mathbb{R}$. Let the real numbers $y_0 < y_1 < \cdots < y_k$ be given, and $\pi = (\pi_0, \pi_1, \cdots, \pi_k)$ a probability vector with $\pi_i > 0$ for all $0 \le i \le k$. Let $\{I_i, 0 \le i \le k\}$ be a disjoint partition of $\mathbb{R}$ such that $\cup_{i=0}^k I_i = \mathbb{R}$ and $P(I_i) = \int_{x\in I_i} g(x)dx = \pi_i$. Define the error term $\epsilon(I_0, \cdots, I_k) = \sum_{i=0}^k \int_{x\in I_i} |x-y_i|^2 g(x)dx$. We now select the smallest $\tilde{x}_0$ such that $\int_{-\infty}^{\tilde{x}_0} g(x)dx = \pi_0$. Then the smallest $\tilde{x}_1$ is selected such that $\int_{\tilde{x}_0}^{\tilde{x}_1} g(x)dx = \pi_1$. This process is repeated until $\tilde{x}_{k-1}$ is obtained with $\int_{\tilde{x}_{k-1}}^{\infty} g(x)dx = \pi_k$.

Set $\hat{I}_i = (\tilde{x}_{i-1}, \tilde{x}_i]$ for $i = 0, \cdots, k-1$, and $\hat{I}_k = (\tilde{x}_{k-1}, \infty)$. We denote $\tilde{x}_{-1} = -\infty$ and $\tilde{x}_k = \infty$. We will call $\{\hat{I}_i, 0 \le i \le k\}$ a nominal partition with respect to $\pi = (\pi_0, \cdots, \pi_k)$, $\pi_i > 0$ for all $i \ge 0$.

For two subsets D_i, $i = 1, 2$, of $\mathbb{R}$, we introduce the symmetric difference $D_1 \Delta D_2 = (D_1 \backslash D_2) \cup (D_2 \backslash D_1)$, where $D_1 \backslash D_2 = D_1 \cap D_2^c$, etc.

Lemma 6: Let $y_0 < y_1 < \cdots < y_k$ and the probability density function $g(x)$ on $\mathbb{R}$ be given. If a disjoint partition $\{\bar{I}_i, 0 \le i \le k\}$ for $\mathbb{R}$ attains the minimum of $\epsilon(I_0, \cdots, I_k)$ subject to the constraints $P(\bar{I}_i) = \pi_i > 0$, $0 \le i \le k$, then $\sum_{i=0}^k P(\bar{I}_i \Delta \hat{I}_i) = 0$, where $P(\bar{I}_i \Delta \hat{I}_i) = \int_{x\in \bar{I}_i \Delta \hat{I}_i} g(x)dx$.

This lemma characterizes to what extend an optimal partition of $\mathbb{R}$ for minimization of $\epsilon(I_0, \cdots, I_k)$ coincides with the nominal partition which has connectedness for each subset. We give the proof by induction.

Proof: Step 1. We prove the case $k = 1$ by contradiction. Let $A = \hat{I}_0 \backslash \bar{I}_0$ and $A' = \bar{I}_0 \backslash \hat{I}_0$. Obviously $P(A) = P(A')$. We assume $P(A) = P(A') > 0$. It follows that

$$\begin{aligned}&\epsilon(\bar{I}_0, \bar{I}_1) - \epsilon(\hat{I}_0, \hat{I}_1)\\ =&2(y_1-y_0)\int_{x\in A'} xg(x)dx - 2(y_1-y_0)\int_{x\in A} xg(x)dx.\end{aligned}$$

Suppose the boundary of $\hat{I}_0$ is $x = \tilde{x}_0$, then the positive probability sets $A' \subset (x_0, \infty)$ and $A \subset (-\infty, x_0]$. Hence

$$\begin{aligned}&\epsilon(\bar{I}_0, \bar{I}_1) - \epsilon(\hat{I}_0, \hat{I}_1)\\ >&2(y_1-y_0)\int_{x\in A'} \tilde{x}_0 g(x)dx - 2(y_1-y_0)\int_{x\in A} \tilde{x}_0 g(x)dx\\ =&2(y_1-y_0)\tilde{x}_0[P(A')-P(A)] = 0.\end{aligned}$$

Then it easily follows that the lemma holds for $k = 1$.

Step 2. We show the lemma is true for $k = l+1$ if it is true for $k = l \ge 1$. Assume $\{\bar{I}_i, 0 \le i \le l+1\}$ attains the minimum for $\epsilon(I_0, \cdots, I_{l+1})$.

We first establish $P(\bar{I}_0 \Delta \hat{I}_0) = 0$. Assume $P(\bar{I}_0 \Delta \hat{I}_0) > 0$. Then we can necessarily find two sets A and B such that (i) $A \subset \hat{I}_0 \cap \bar{I}_i$, $A \cap \bar{I}_0 = \varnothing$, (ii) $B \subset \hat{I}_j \cap \bar{I}_0$, and (iii) $P(A) = P(B) > 0$ for some $i, j > 0$. For introducing a contradiction, below we need to show that $\epsilon(\bar{I}_0 \cdots, \bar{I}_{l+1})$ cannot attain a minimum.

Let $I_i' = (\bar{I}_i \backslash A) \cup B$ and $I_0' = (\bar{I}_0 \backslash B) \cup A$, and set $I_m' = \bar{I}_m$ for $m \ne 0, i$. We get the new partition $(I_0', \cdots, I_{l+1}')$.

Now we have

$$\begin{aligned}&\epsilon(\bar{I}_0, \cdots, \bar{I}_{l+1}) - \epsilon(I_0', \cdots, I_{l+1}')\\ =&2(y_i-y_0)[\int_{x\in B} xg(x)dx - \int_{x\in A} xg(x)dx] > 0\end{aligned} \tag{9}$$

since $A \subset \hat{I}_0$ and $B \subset \hat{I}_j$, $j > 0$. This is a contradiction. Hence we have $P(\bar{I}_0 \Delta \hat{I}_0) = 0$.

We form the new partition $\bar{I}_0 \cup \bar{I}_1, \bar{I}_2, \cdots, \bar{I}_{l+1}$ with $y_1 < y_2 < \cdots < y_{l+1}$ and define the new density function

$$g^*(x) = \begin{cases} \frac{1}{\sum_{i=1}^{l+1}\pi_i} g(x), & \text{for } x \in \cup_{i=1}^{l+1} \bar{I}_i \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

We see that $\epsilon(\bar{I}_0 \cup \bar{I}_1, \bar{I}_2 \cdots, \bar{I}_{l+1})$ (with $(y_1, \cdots, y_{l+1})$) attains the minimum subject to the constraints $\int_{x\in \bar{I}_0\cup\bar{I}_1} g^*(x)dx = \pi_1\delta_{1,l+1}, \cdots, \int_{x\in\bar{I}_{l+1}} g^*(x)dx = \pi_{l+1}\delta_{1,l+1}$, where $\delta_{1,l+1} = \frac{1}{\sum_{i=1}^{l+1}\pi_i}$. Using the induction hypothesis (w.r.t. the modified density function g^* and its nominal partition $\{\hat{I}_0 \cup \hat{I}_1, \hat{I}_i, 2 \le i \le l+1\}$), we have

$$\int_{x\in(\bar{I}_0\cup\bar{I}_1)\Delta(\hat{I}_0\cup\hat{I}_1)} g^*(x)dx = 0 \tag{11}$$

$$\int_{x\in\bar{I}_i\Delta\hat{I}_i} g^*(x)dx = 0, \quad 2 \le i \le l+1, \tag{12}$$

[1]However, we note that the constrained quadratic minimization method developed subsequently for control computation is still applicable for any candidate tree structure (for scenario (b)) under examination.

which yields $P(\bar{I}_i\Delta\hat{I}_i) = \int_{x\in\bar{I}_i\Delta\hat{I}_i} g(x)dx = 0,\ 1 \le i \le l+1$. So we have proven the case $k = l+1$.

Step 3. By induction, the lemma holds for all $k \ge 1$. □

Lemma 6 is useful for the parametrization of the quantizer for the optimal control law.

C. Necessary conditions for optimal control

For the given set of $q(i)$, $0 \le i \le M^N - 1$, we perform the following quadratic minimization:

$$\text{(P1):} \quad \min \ EH(\tilde{\theta}_N) = \sum_{i=0}^{M^N-1} p_i H[\tilde{\theta}_N(\tilde{s}_{N-1}(i))] \quad (13)$$

subject to M^N linear constraints given by (8). Here p_i is the probability for the subset within the partition of x_0 which is matched to $q_i \triangleq q(i)$ in (8). Having determined p_i in this manner, each entry in the vector $\tilde{\theta}_N(\tilde{s}_{N-1}(i))$ may be looked at as a usual deterministic variable.

Notice that in the tree representation of the causal map $\tilde{\theta}_N$ we have a total of $K = 1 + M + \cdots + M^N$ parameters or variables attached to all edges. Using a prescribed order, we denote these parameters by K variables y_j, $0 \le j \le K$. Then the above constrained minimization problem may be equivalently stated as

$$\text{(P2):} \quad \min \quad EH(\tilde{\theta}_N) = H_0(y_0, y_1, \cdots, y_{K-1})$$

subject to $h_i(y_0, y_1, \cdots, y_{K-1}) = q_i \triangleq q(i)$, $0 \le i \le M^N - 1$, where H_0 and h_i are easily determined from (8) and (13). In H_0 we do not explicitly indicate the M^N constants p_i. Denote the minimum obtained in Problem (P2) by

$$\hat{H}(q_0, q_1, \cdots, q_{M^N-1}),$$

which is to be called the optimal residual term.

Proposition 7: Let q_i, $0 \le i \le M^N - 1$, be M^N distinct values, and $p_i > 0$ for all $0 \le i \le M^N - 1$ be the associated probabilities specified in (13). Then Problem (P2) admits a unique set of $(\hat{y}_0, \hat{y}_1, \cdots, \hat{y}_{K-1})$ which attains the minimum $\hat{H}(q_0, q_1, \cdots, q_{M^N-1})$.

Proof: (P2) is a standard quadratic minimization problem subject to linear equality constraints. In addition, we can verify that H_0 is strictly convex w.r.t. the argument $(y_0, y_1, \cdots, y_{K-1})$. Hence there is a unique minima. □

Remark 8: By the method of elimination of arguments we can convert (P2) into an equivalent unconstrained quadratic minimization problem. Using the method of completion of squares, we can show that $\hat{H}$ is quadratic in $(q_0, q_1, \cdots, q_{M^N-1})$ with all p_i treated as fixed constants. □

Now we have the necessary condition for the optimal values for q_i in the control problem. In contrast to the Voronoi diagram, we call q_i the shifted centers for the quantization associated with the optimal control law.

Theorem 9: Suppose the optimal control is specified by scenario (a) with $q_0, q_1, \cdots, q_{M^N-1}$ being M^N distinct values for $\xi(\tilde{\theta}_N)$ in (8), each with probability $p_i > 0$, $\sum_{i=0}^{M^N-1} p_i = 1$. Let q_i be associated with $(\underline{x}_i, \overline{x}_i]$, the interval in the nominal partition with respect to the probability vector $(p_i, 0 \le i \le M^N - 1)$. The optimal values for q_i's satisfy the condition $\alpha_2^2 \int_{\underline{x}_i}^{\overline{x}_i} 2(c-x)f(x)dx + \frac{\partial \hat{H}}{\partial c} = 0$, where f is the density function of the initial condition x_0, and $c(= q_i)$ is any one of the shifted centers.

Proof: The equality is obtained by differentiating the cost J^* (obtained from minimizing EJ_0 w.r.t. $\tilde{\theta}_N$ for a given partition of x_0) w.r.t. $c = q_i$. From Remark 8 we see that $\frac{\partial \hat{H}}{\partial c}$ is a linear function in c. □

The necessary condition in Theorem 9 is essentially a generalization of the well known necessary condition for optimal quantizers due to Lloyd and Max [8], [5]. Based on our previous structural results in Lemma 6 for the optimal partition, here we can specify the region of q_i by two numbers $\underline{x}_i$ and $\overline{x}_i$ (possibly including $-\infty$ and ∞).

If the density $f(x)$ is continuous at the boundary points such as $\overline{x}_i$, we can further consider taking partial differential of the cost J^* with respect to $\overline{x}_i$ to get a second set of necessary conditions. It should be noted that this also involves the partial differential of $\hat{H}$ w.r.t. $\overline{x}_i$ which affects the coefficient p_i, and also p_{i+1} if $i < M^N - 1$. To avoid more complicated notation, we will not state this set of necessary conditions in details. Instead, we illustrate them in the computation of the next section.

VI. A Computational Example

For illustration, we consider a two stage optimization problem. The cost to be minimized is given as $J = E(x_1^2 + ru_0^2 + x_2^2 + ru_1^2)$. We set the rate R as one (producing binary controls) and parametrize the causal map $\tilde{\theta}_1$ on its tree representation by three parameters θ_0, θ_1 (for $s_0 = 0$) and θ_1' (for $s_0 = 1$). Suppose $q_0 < q_1$ are two values for $\xi(\tilde{\theta}_1)$. The constraints corresponding to (8) reduce to:

$$a\theta_0 + a^2\theta_1 = -\alpha_2^2 q_0, \quad a\theta_0 + a^2\theta_1' = -\alpha_2^2 q_1.$$

After taking expectation, we write the residual term

$$\begin{aligned} EH(\tilde{\theta}_1) =& [-\alpha_2^2 q_0^2 + \theta_0^2 + \theta_1^2 + \hat{r}\theta_0^2 + \hat{r}(\theta_1 - a\theta_0)^2]p_0 \\ &+ [-\alpha_2^2 q_1^2 + \theta_0^2 + \theta_1^{'2} + \hat{r}\theta_0^2 + \hat{r}(\theta_1' - a\theta_0)^2]p_1 \end{aligned}$$

where $\hat{r} = r/b^2$. Making use of the linear constraints to eliminate θ_1 and θ_1', we have

$$\begin{aligned} EH =& -\alpha_2^2(q_0^2 p_0 + q_1^2 p_1) + (1 + \hat{r} + \hat{r}a^2)\theta_0^2 \\ &+ (1+\hat{r})p_0[\frac{\alpha_2^2 q_0}{a^2} + \frac{\theta_0}{a}]^2 + 2\hat{r}p_0 a\theta_0[\frac{\alpha_2^2 q_0}{a^2} + \frac{\theta_0}{a}] \\ &+ (1+\hat{r})p_1[\frac{\alpha_2^2 q_1}{a^2} + \frac{\theta_0}{a}]^2 + 2\hat{r}p_1 a\theta_0[\frac{\alpha_2^2 q_1}{a^2} + \frac{\theta_0}{a}] \\ \triangleq& \beta_2\theta_0^2 + 2\beta_1(p_0 q_0 + p_1 q_1)\theta_0 + \beta_0(p_0 q_0^2 + p_1 q_1^2) \\ \triangleq& H_1(\theta_0), \end{aligned} \quad (14)$$

where $\beta_2 = 1 + 3\hat{r} + \hat{r}a^2 + \frac{1+\hat{r}}{a^2}$, $\beta_1 = \frac{\alpha_2^2}{a}[\hat{r} + \frac{1+\hat{r}}{a^2}]$, $\beta_0 = \frac{(1+\hat{r})\alpha_2^4}{a^4} - \alpha_2^2$. It follows that

$$\begin{aligned} \hat{H} &= \inf H_1(\theta_0) \\ &= -\frac{\beta_1^2}{\beta_2}(p_0 q_0 + p_1 q_1)^2 + \beta_0(p_0 q_0^2 + p_1 q_1^2). \end{aligned} \quad (15)$$

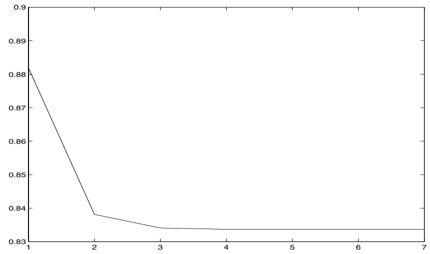

Fig. 3. The monotonic decrease of the cost following the alternative improvement of the boundaries and shifted centers of the quantizer.

Now the necessary condition in Theorem 9 reduces to:

$$0 = \alpha_2^2 \int_{-\infty}^{\bar{x}} (q_0 - x) f(x) dx - \frac{\beta_1^2}{\beta_2}(p_0^2 q_0 + p_0 p_1 q_1) + \beta_0 p_0 q_0$$
$$0 = \alpha_2^2 \int_{\bar{x}}^{\infty} (q_1 - x) f(x) dx - \frac{\beta_1^2}{\beta_2}(p_0 p_1 q_0 + p_1^2 q_1) + \beta_0 p_1 q_1$$

where p_0, p_1 are the probability on the intervals $(-\infty, \bar{x}]$ and $(\bar{x}, \infty)$, respectively.

If f is continuous at $\bar{x}$, differentiating the cost J^* w.r.t. $\bar{x}$ yields the additional necessary condition

$$\begin{aligned} 0 =& f(\bar{x})[\alpha_2^2(\bar{x} - q_0)^2 - \alpha_2^2(\bar{x} - q_1)^2 \\ & - \frac{2\beta_1^2}{\beta_2}(q_0 p_0 + q_1 p_1)(q_0 - q_1) + \beta_0(q_0^2 - q_1^2)]. \end{aligned}$$

which is equivalent to

$$0 = f(\bar{x})[2\alpha_2^2 \bar{x} + \frac{2\beta_1^2}{\beta_2}(q_0 p_0 + q_1 p_1) - (\beta_0 + \alpha_2^2)(q_0 + q_1)],$$

where p_0, p_1 involved in $\hat{H}$, are treated as functions of $\bar{x}$.

A. A simple numerical example

In the two stage control problem we select the parameters as: $a = 1.5$, $b = r = 1$. x_0 has a uniform distribution on $[0, 1]$. We compute the optimal quantizer and controller by a Lloyd-Max type algorithm — (i) Choose an initial value for q_0, q_1 and $\bar{x}$; (ii) fix $\bar{x}$ and update q_0, q_1 using the equation in the necessary condition, which will lead to a decrease of the associated cost; (iii) fix q_0, q_1, minimize the cost by updating $\bar{x}$ using the optimality condition. The process is repeated by alternatively carrying out (ii) and (iii).

After $\tilde{\theta}_1$ is computed, we can easily retrieve the value for u_0 and u_1. It turns out x_0 should be quantized as: $s_0 = 0$ for $x_0 \in [0, 0.5]$, and $s_0 = (0.5, 1]$. The control law is given as $u_0 = -0.51$,

$$u_1 = \begin{cases} 0.10125 & \text{for} \quad s_0 = 0 \\ -0.46125 & \text{for} \quad s_0 = 1. \end{cases}$$

The associated optimal cost is $J = 0.83367$.

B. Localized computation for long horizon

For the case of control with a long horizon, it is difficult to compute $\hat{H}$ directly. However, we note that numerical optimization may be useful in the context of quadratic minimization subject to linear constraints.

For reducing complexity, a localized optimization algorithm may be employed. Notice that H is strictly convex with respect to its arguments. For a given set of linear constraints, one can fix the values attached to all edges in the tree except $M + 1$ edges connected to a given node. Then for a given set of p_i, q_i, $0 \le i \le M^N - 1$, one can minimize H with respect to $\tilde{\theta}_N$ by tuning these $M + 1$ values (as restricted by M equalities) such that the original set of M^N equality constraints (8) still holds. Such a basic step can be analytically performed when M is small. One can exhaust all the nodes in the tree in a prescribed order and then repeat. This constitutes a basic step for further adjustment of the quantizer's boundaries and shifted centers in the Lloyd-Max type algorithm.

There is a very clear geometric interpretation for the above method. Indeed, such a procedure amounts to gradually descending along lower dimensional subsets (as the intersection of hyperplanes) of the set (also as the intersection of hyperplanes; see (8)) specified by the linear equality constraints. The convergence analysis for the above minimization procedure will be investigated in future work.

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