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A Recursive Algorithm for Reducing the Order of Controller while Guaranteeing Performance

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***Abstract*— In this paper, a procedure for the recursive reduction of the order of the stabilizing controller is introduced for SISO systems. Since all achievable closed loop maps are affine in the $\mathcal{Q}$ (Youla) parameter, we devise a sufficient condition for order reduction: Suppose there exists a $\mathcal{Q}$ parameter to induce a pole zero cancellation in the closed loop map to decrease the order of the closed loop system by m, then the corresponding controller is reduced in order by m. By appropriately choosing $\mathcal{Q}$, we formulate a procedure for the recursive reduction of the order of the stabilizing controller and guarantee a performance describable through a complex stabilization technique.**

I. INTRODUCTION

In the last two decades, there have been numerous results in the field of $\mathcal{H}_\infty$ control theory ([1], [2]). These provide a precise formulation and solution of the problem of synthesizing a controller which minimizes the $\mathcal{H}_\infty$-norm of a given transfer function. Many robust stability and performance problems can be cast as similar problems of optimization.

The order of the $\mathcal{H}_\infty$ optimal controller obtained through these traditional technique is almost always very high, being equal to that of the generalized plant. The difficulty involved in implementing a high order controller for practical applications has been a deterrent for the use of these controllers. The need for low order controllers arises when simplicity, hardware limitations or reliability in the implementation of a controller dictates low order of stabilization [3]–[5].

There are in general three basic approaches to obtain a low order controller [6]. The first method is to directly generate a low order controller from the given plant data. The second method is to find a simpler lower order representation of the plant which captures the plant dynamics, and then generate a possible lower order controller using the lower order representation of the plant. The third approach is to compute a high order controller directly from the higher order plant. Controller order reduction schemes are then applied to synthesize a lower order controller.

The direct synthesis of low order controllers involves the problem of fixed-order stabilization. A good survey of the attempts to solve the fixed order control problem and the related Static Output Feedback (SOF) problem is given in [7], [8]. The set of all fixed order/structure stabilizing controllers is *non-convex* and in general, *disconnected* in the space of controller parameters [9]. This is a major source of difficulty in its computation.

The parametrization of all stabilizing controllers of fixed order via Quadratic Lyapunov Functions is presented in [10]. It is accomplished through the use of two coupled Riccati equations. In [11], the synthesis of a low order stabilizing controller is posed as the feasibility of a pair of LMIs with a coupling rank constraint.

Methods for synthesizing $\mathcal{H}_\infty$ controllers with a constraint on the controller order and/or structure are available in [12], [13], but these approaches suffer from computational intractability or conservatism.

Another approach to achieve a low order controller is to approximate the original system, and design a controller based on the approximated plant. The model of the system is approximated by various existing methods (see [14]) which are all based on minimization of some error. A method based on truncating the balanced realization was proposed by Moore [15]. In many applications, the interest is in approximating the full order plant only in a specific frequency interval. The use of weighted-frequency improves the model reduction by trying to reduce the error only over a specified frequency range [16]. Comparison of different model reduction techniques is given in [17]. The main drawback of this method is that the errors due to model approximation will cause problems in subsequent controller design synthesis.

The procedure of direct controller order reduction can be categorized in two parts, the open-loop and closed-loop methods. In open loop methods, it is required that the reduced controller, $C_r(s)$ is a good approximation of the original controller $C(s)$. Requiring $C_r(s)$ to be a good approximation to $C(s)$ may not provide the desired closed-loop performance. The controller reduction requires taking the plant dynamics into account and hence closed-loop methods are used. This is generally achieved through frequency weighting (see [6], [14]). In frequency-weighted controller reduction, the aim is to find a lower order controller C_r that minimizes the weighted error $||W_o(C - C_r)W_i||_\infty$, where W_i and W_o are appropriate frequency weighting. These weights can be chosen to satisfy the closed loop stability and performance.

This paper provides a procedure to recursively reduce the order of the high order controller and can be applied to high order controllers obtained through classical control synthesis techniques. The procedure uses the fact that a controller of order higher than the minimal order is unbounded in the

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controller parameter space. Since all achievable closed loop maps are affine in the $\mathcal{Q}$ (Youla) parameter, the procedure finds a $\mathcal{Q}$ parameter to induce a pole zero cancellation in the closed loop map and obtain a lower order controller. Preliminary results regarding stabilization were provided in [18]. An algorithm is developed which uses the ability of specifying certain performance criteria as the condition for a family of complex polynomials to be Hurwitz. The proposed procedure provides a *sufficient* recursive reduction scheme, which can be used to reduce the controller to the minimal order possible while satisfying a given performance specification.

II. Main Results

In this section, we provide a recursive order reduction procedure based on pole-zero cancellation, which guarantees a specified performance specification. This procedure is *sufficient* for order reduction and may not be necessary.

It is shown that the set of rational, strictly proper stabilizing controllers will form a bounded (can even be empty) set in the controller parameter space if and only if the order of the stabilizing controller can not be reduced any further; if the set of proper stabilizing controllers of order r is not empty and the set of strictly proper controllers of order r is bounded, then r is the minimal order of stabilization.

A. Some Properties of the Set of Stabilizing Controllers

It is a known fact that an n^{th} order plant can be stabilized by a $(n-1)^{st}$ order stabilizing controller.

The following lemmas are simple observations which provide key basis for the proposed algorithm on stabilizing controller order reduction.

Lemma 1. *If $C_r(s) = \frac{N_r}{D_r}(s)$ is a r^{th} order rational, proper controller that stabilizes $P(s) = \frac{N_p}{D_p}(s)$, then given any $\tilde{N}_r(s)$ and $\tilde{D}_r(s)$ of degree r, there is a $\tau^* > 0$ such that the $(r+1)^{st}$ order strictly proper, rational controller $\frac{N_r(s)+\tau\tilde{N}_r(s)}{D_r(s)+\tau(s^{d+1}+\tilde{D}_r(s))}$ also stabilizes $P(s)$ for every $0 < \tau \leq \tau^*$.*

Proof. Let $\Delta(s) = N_p(s)N_r(s) + D_p(s)D_r(s)$. The characteristic polynomial, $\Delta_{pert}(s,\tau)$, associated with the perturbed controller is $\Delta(s) + \tau(s^{r+1}D_p(s) + (\tilde{N}_r(s)N_p(s) + \tilde{D}_r(s)D_p(s)))$. If τ is treated as a variable in the following root locus problem,

$$1 + \frac{1}{\tau}\frac{\Delta(s)}{s^{r+1}D_p(s) + \tilde{N}_r(s)N_p(s) + \tilde{D}_r(s)D_p(s)} = 0$$

and noticing that the relative degree of the rational proper transfer function in the above equation is one, it follows that there is a $\tau^* > 0$ such that for all $0 < \tau \leq \tau^*$, the polynomial, $\Delta_{pert}(s,\tau)$, is Hurwitz. □

The following are the consequences of Lemma 1:

1) If there is a r^{th} order stabilizing controller, then there is a (strictly proper) stabilizing controller of order $r+1$. Therefore, there is no gap in the order of stabilization. Hence, minimal order compensators can be synthesized by recursively reducing the order of stabilizing controller by one.
2) Let us associate a vector $K = (k_0, k_1, \ldots, k_r, k_{r+1}, \ldots, k_{2r})$ with a rational, proper controller, $C_r(s)$, where

$$C_r(s) = \frac{k_0 + k_1 s + \ldots + k_r s^r}{k_{r+1} + k_{r+2}s + \ldots + k_{2r}s^{r-1} + s^r}$$

Clearly, there is a one-to-one correspondence with $K \in \Re^{2r+1}$ and a rational, proper r^{th} order controller $C_r(s)$. Without any loss of generality, we will use K and $C_r(s)$ interchangeably.
Let $\tilde{N}_r(s) = \tilde{k}_0 + \tilde{k}_1 s + \ldots + \tilde{k}_r s^r$, and $\tilde{D}_r(s) = \tilde{k}_{r+1} + \tilde{k}_{r+2}s + \ldots + \tilde{k}_{2r}s^{r-1} + \tilde{k}_{2r+1}s^r$, so that, by Lemma 1, there is a τ^* such that for all $0 < \tau \leq \tau^*$, the following $(r+1)^{st}$ order controller, $\tilde{C}_{r+1}(s)$, is also stabilizing:

$$\tilde{C}_{r+1}(s) = \frac{\tilde{N}_{r+1}(s)}{\tilde{D}_{r+1}(s)}$$

$$\tilde{N}_{r+1}(s) = (k_0+\tau\tilde{k}_0)+(k_1+\tau\tilde{k}_1)s+\ldots+(k_r+\tau\tilde{k}_r)s^r$$
$$\tilde{D}_{r+1}(s) = (k_{r+1} + \tau\tilde{k}_{r+1}) + \ldots$$
$$\ldots + (k_{2r} + \tau\tilde{k}_{2r})s^{r-1} + (1 + \tau\tilde{k}_{2r+1})s^r + \tau s^{r+1}$$

Dividing the numerator and denominator by τ, the controller $\tilde{C}_{r+1}(s)$ can be expressed in terms of $\tilde{K}(\tau)$. By defining $K_0 := \tilde{K}(\tau^*), \lambda := \frac{1}{\tau} - \frac{1}{\tau^*}$, one can express $\tilde{K}$ as $K_0 + \lambda K_1$ and is stabilizing for every $\lambda \geq 0$, by Lemma 1. where K_1 is:

$$K_1 := (k_0, k_1, \ldots, k_r, 0, k_{r+1}, k_{r+1}, \ldots, k_{2r}, 1)$$

Thus, $\tilde{K}$ is a ray originating at K_0 and is in the direction of K_1 in the space of parameters corresponding to a $(r+1)^{st}$ strictly proper stabilizing compensator. Two things can be inferred from above:
   a) If an r^{th} order stabilizing compensator exists, the set of $(r+1)^{st}$ order strictly proper stabilizing controller parameters is unbounded. In particular, the set of $(r+1)^{st}$ order strictly proper stabilizing controllers contains a ray of the form $K_0 + \lambda K_1$ in $\Re^{2r+2}$ that is stabilizing for every $\lambda \geq 0$.
   b) If, by some means, one were to find a ray, $\{K_0 + \lambda K_1,\ \lambda \geq 0\}$, of strictly proper $(r+1)^{st}$ order stabilizing controllers, with K_1 having the $(r+2)^{nd}$ entry to be zero and the last entry to be unity, then it seems likely to recover a lower order controller from K_1 considering the correspondence between K_1 and $C(s)$.

Lemma 2. *If $C_r(s) = \frac{N_r}{D_r}$ is a strictly proper stabilizing controller of order r for the plant $P(s) = \frac{N_p}{D_p}$ of order n, then there also exists a proper, but not strictly proper, stabilizing controller of order "r" for $P(s)$.*

Proof. We provide a sketch of the proof. Let $\Delta(s) = N_p(s)N_r(s)+D_p(s)D_r(s)$. If we set a proper, but not strictly proper, controller to be $(\varepsilon s+1)C_r(s)$, then the corresponding closed loop characteristic polynomial is $\Delta(s) + \varepsilon s N_p N_r$. In the standard form for the root locus, since the characteristic equation for the perturbed case can be expressed as

$$1+\varepsilon\frac{sN_pN_r}{\Delta(s)}=0$$

it follows that the closed loop system with the perturbed controller becomes stable by a standard root locus argument if ε is sufficiently small. □

Lemma 2 indicates that if ever order reduction of stabilizing controller of order from r to $r-1$ is possible there always exists a proper, but not strictly proper, stabilizing controller of order $r-1$.

Lemma 3. *If a stabilizing controller of order r exists for the given plant of order $n(>r)$, then stabilizing controllers of order between r and n exist.*

Proof. It is easily proved by applying Lemma 1 and Lemma 2 recursively. □

In the rest of this section we provide a necessary and sufficient condition for the recursive reduction of the higher order controller. Let $P_0(s)$ denote a proper transfer function of an LTI system and let $P_0(s)$ be given by:

$$P_0(s)=\frac{\bar{a}_0+\bar{a}_1s+\ldots+\bar{a}_ms^m}{s^n+\bar{b}_{n-1}s^{n-1}+\ldots+\bar{b}_0}.$$

Let $\mathcal{P}_\epsilon$ be a ball of interval plants around P_0 and be given by:

$$\mathcal{P}_\epsilon:=\{P(s):P(s)=\frac{a_0+a_1s+\ldots+a_ms^m}{s^n+b_{n-1}s^{n-1}+\ldots+b_0},$$
$$\max\{\max_{k=0,1,..,m}|a_k-\bar{a}_k|,\max_{l=0,1,..,n-1}|b_k-\bar{b}_k|\}<\epsilon.\}$$

Let $C(s)$ be a rational proper stabilizing controller of order r for $P_0(s)$. Specifically, $C(s)$ is of the form:

$$C(s)=\frac{c_0+c_1s+\ldots+c_rs^r}{s^r+d_{r-1}s^{r-1}+\ldots+d_0}.$$

The following theorem provides the conditions for the existence of a lower order controller from the boundedness of the set of higher order controllers:

Theorem 1. *A proper controller of order $r-1$ stabilizing $P_0(s)$ exists iff there exists a ray of strictly proper controllers of order r, namely $\{K_0+\lambda K_1,\ \lambda>0\}$, that stabilize an interval of plants $\mathcal{P}_\epsilon$ for some $\epsilon>0$.*

Proof. A controller of order $n-1$ always exists for a SISO plant of order n. Hence, we will assume that $r\le n-1$. (Necessity) Suppose an $r-1^{st}$ order proper controller, $C(s)$ stabilizes $P_0(s)$; then, clearly, there exists an $\epsilon>0$ such that $C(s)$ stabilizes every $P(s)\in\mathcal{P}_\epsilon$. Consider a controller of the form $\frac{1}{\tau s+1}C(s)$. Let τ^* be the smallest positive real number such that the characteristic polynomial of the closed loop system has a purely imaginary root. Clearly, τ^* is a continuous function of $a_0,\ \ldots,\ a_m,\ b_0,\ b_1,\ldots,b_{n-1}$, all of which belong to a compact set. Therefore, there exists a minimum value $\bar{\tau}>0$ such that for all $\tau\in(0,\bar{\tau})$, the r^{th} order strictly proper controller $\frac{C(s)}{\tau s+1}$ stabilizes all the plants in $\mathcal{P}_\epsilon$.

If $C(s)$ is of the form

$$C(s)=\frac{c_0+c_1s+\ldots+c_{r-1}s^{r-1}}{s^{r-1}+d_{r-2}s^{r-2}+\ldots+d_0},$$

then $\frac{C(s)}{\tau s+1}$ is of the form,

$$\frac{\frac{1}{\tau}(c_0+c_1s+\ldots+c_{r-1}s^{r-1})}{s^r+d_{r-2}s^{r-1}+..+d_0s+\frac{1}{\tau}(s^{r-1}+d_{r-2}s^{r-2}+..+d_0)}$$

In parameter vector form, it is of the form, $K_0+\lambda K_1$, where

$$\lambda:=\frac{1}{\tau}-\frac{1}{\bar{\tau}},$$
$$K_1=(c_0,\ c_1,\ \ldots,c_{r-1},0,\ d_0,\ d_1,\ \ldots,d_{r-2},\ 1),$$
$$K_0=\frac{1}{\bar{\tau}}K_1+(\underbrace{0,\ldots,0}_{r+2\ zeros}\ d_0\ d_1\ \ldots,d_{r-2}),$$

and this ray of controllers, $\{K_0+\lambda K_1,\ \lambda>0\}$ stabilize the family of plants $\mathcal{P}_\epsilon$.

(Sufficiency) Consider a ray of strictly proper controllers of order r, $C(s,\lambda)=$

$$\frac{\overbrace{c_0+c_1s+\ldots+c_{r-1}s^{r-1}}^{N_c(s)}+\lambda\overbrace{e_0+e_1s+\ldots+e_{r-1}s^{r-1}}^{N_c^*(s)}}{\underbrace{s^r+d_0+d_1s+\ldots+d_{r-1}s^{r-1}}_{D_c(s)}+\lambda\underbrace{f_0+..+f_{r-1}s^{r-1}}_{D_c^*(s)}}$$

Suppose there exists a ray of strictly proper controllers of order r that stabilize a family of plants $\mathcal{P}_\epsilon$ for some $\epsilon>0$. If $P(s)=\frac{N_p}{D_p}(s)$, then, the closed loop characteristic polynomial for the plant $P(s)$ in the family with a controller from the ray (identified by λ) may be written as:

$$\Delta(P(s),\lambda)=\Delta_0(P(s))+\lambda\Delta_1(P(s)),$$

where $\Delta_0=N_p(s)N_c(s)+D_p(s)D_c(s)$ and $\Delta_1=N_p(s)N_c^*(s)+D_p(s)D_c^*(s)$. Notice that the degree of $\Delta_1(s)$ is less than that of $\Delta_0(s)$, since we are considering a ray of strictly proper controllers. Since $\Delta(P(s),\lambda)$ is Hurwitz for all $\lambda>0$, from a root locus argument, it must be true that the roots of Δ_1 must lie in the closed left half plane for every $P(s)\in\mathcal{P}_\epsilon$.

We prove, by contradiction, that $\Delta_1(P_0(s))$ is Hurwitz in the following steps:

1) If N_c^* and D_c^* are co-prime, then $\Delta_1(P_0(s))$ is Hurwitz and $\frac{N_c^*}{D_c^*}$ is a stabilizing controller.
2) If N_c^* and D_c^* are not co-prime, then $\frac{N_c^*}{D_c^*}$ in its reduced form stabilizes $P_0(s)$.

Claim 1: If N_c^* and D_c^* are co-prime, then $\Delta_1(P_0(s))$ is Hurwitz.

Proof of Claim: Suppose not. Then, $\Delta_1(P_0(s))$ has a root on the imaginary axis. Given any $\delta>0$, there is a polynomial, $\tilde{\Delta}_1$, whose coefficients differ from those of $\Delta_1(P_0(s))$ by no more than δ, and has a root with positive real part. For a sufficiently small δ, $\tilde{\Delta}_1$ corresponds to a characteristic polynomial of a plant $P(s)\in\mathcal{P}_\epsilon$ with a controller $\frac{N_c^*}{D_c^*}$. This can be concluded by viewing $\frac{N_c^*}{D_c^*}$ as the plant and $\frac{N_p}{D_p} \in \mathcal{P}_\epsilon$ as a controller. Since N_c^* and D_c^* are co-prime (and hence, their resultant is non-singular), one can always find a solution to $N_c^*\tilde{N}_p + D_c^*\tilde{D}_p = \tilde{\Delta}_1(s) - \Delta_1(P_0(s))$, where the coefficients of $\tilde{N}_p$ and $\tilde{D}_p$, where the degrees of $\tilde{N}_p$ and $\tilde{D}_p$ are greater than that of r. Let $P_0(s) = \frac{N_p^0}{D_p^0}(s)$. In particular, if δ is sufficiently small, then $P(s) = \frac{N_p^0+\tilde{N}_p}{D_p^0+\tilde{D}_p}$ belongs to $\mathcal{P}_\epsilon$. This is a contradiction, because, the roots of $\Delta_1(P(s))$ must lie in the closed left half plane.

Claim 2: If N_c^* and D_c^* are not co-prime, then $\frac{N_c^*}{D_c^*} = \frac{N_1}{D_1}$, where N_1 and D_1 are co-prime, and N_1 is of lower degree than N_c^* and D_1 is of lower degree than D_c^*. Consider $N_p^0 N_1 + D_p^0 D_1$; this must be Hurwitz, by an argument similar to that in claim 1, since N_1 and D_1 are co-prime. Therefore, $\frac{N_1}{D_1}$ is stabilizing.

There is a possibility that $C(s) = \frac{N_c^*}{D_c^*}$ may not be proper; since $C(s)$ stabilizes $P_0(s)$, and since $\frac{C(s)}{\tau s+1}$ stabilizes $P_0(s)$ for a sufficiently small $\tau > 0$, and since the degree of N_c^* is no more than $r-1$, one can always synthesize a proper controller of order $r-1$ that stabilizes $P_0(s)$. $\square$

III. Formulation of the Problem of Recursive Order Reduction

A. A Variant Form of Youla Parametrization of All Stabilizing Controllers

Consider a rational, proper transfer function, $P(s) = \frac{N_p(s)}{D_p(s)}$ of order n, where $N_p(s)$ and $D_p(s)$ are co-prime polynomials and a rational, proper but not strictly proper, stable transfer function, $C_1(s) = \frac{N_{c1}(s)}{D_{c1}(s)}$ of order $r (< n)$, where $N_c(s)$ and $D_c(s)$ are co-prime polynomials. The problem is to find a low order controller, $C_2(s)$ which stabilizes the plant, $P(s)$ and meets some specified $\mathcal{H}_\infty$-norm performance specification. Let us assume that the closed loop characteristic polynomial, $\Delta(s)$ corresponding to $P(s)$ and $C_1(s)$ be factorized as $\Delta(s) = \delta_1(s)\cdot\delta_2(s)$ so that the following proper rational transfer functions $P_n(s)$, $P_d(s)$, $C_{1n}(s)$ and $C_{1d}(s)$ are stable and $P_n(s)C_{1n}(s) + P_d(s)C_{1d}(s) = 1$; where,

$$P(s) = \frac{P_n(s)}{P_d(s)}, \qquad C_1(s) = \frac{C_{1n}(s)}{C_{1d}(s)}$$
$$P_n(s) = \frac{N_p(s)}{\delta_1(s)}, \qquad P_d(s) = \frac{D_p(s)}{\delta_1(s)}$$
$$C_{1n}(s) = \frac{N_{c1}(s)}{\delta_2(s)}, \qquad C_{1d}(s) = \frac{D_{c1}(s)}{\delta_2(s)}$$

With this factorization, by Youla Parametrization, all stabilizing controllers, $C_s(s)$, are characterized as

$$C_s(s) = \frac{C_{1n}(s) + Q\cdot P_d(s)}{C_{1d}(s) - Q\cdot P_n(s)}$$

where Q is in the set of stable, proper real rational transfer functions.

In order to obtain a controller order reduction, it is sufficient to to consider Youla parameters, which are proper but not strictly proper, and are of the form

$$Q = \frac{\delta_1(s)}{\delta_2(s)}\cdot\bar{Q} = \frac{\delta_1(s)}{\delta_2(s)}\cdot\frac{k_m s^m + \ldots + k_0}{q_{n-r+m}}$$

where the order of q_{n-r+m} is $n-r+m$. Then, all stabilizing controllers associated with Q result in

$$\begin{aligned} C_s(s) &= \frac{C_{1n}(s) + Q\cdot P_d(s)}{C_{1d}(s) - Q\cdot P_n(s)} \\ &= \frac{N_{c1}(s)\cdot q_{n-r+m} + (k_m s^m + \ldots + k_1 s + k_0)\cdot D_p(s)}{D_{c1}(s)\cdot q_{n-r+m} - (k_m s^m + \ldots + k_1 s + k_0)\cdot N_p(s)} \end{aligned} \quad (1)$$

The modified closed loop characteristic polynomial is given by

$$\begin{aligned} &N_p(s)[N_{c1}(s)\cdot q_{n-r+m} + (k_m s^m + \ldots + k_0)\cdot D_p(s)] \\ &\quad + D_p(s)[D_{c1}(s)\cdot q_{n-r+m} - (k_m s^m + \ldots + k_0)\cdot N_p(s)] \\ &\qquad = \Delta\cdot q_{n-r+m} \end{aligned}$$

It is shown that by using the parameter Q, we add $n-r+m$ poles to the closed loop system. For an order reduction of the controller, the polynomials $N_{c1}(s)\cdot q_{n-r+m} + (k_m s^m + k_{m-1}s^{m-1} + \ldots + k_1 s + k_0)\cdot D_p(s)$ and $D_{c1}(s)\cdot q_{n-r+m} - (k_m s^m + k_{m-1}s^{m-1} + \ldots + k_1 s + k_0)\cdot N_p(s)$ must have at least $n-r+m+1$ factors in common; otherwise, the resulting controller will not be of reduced order. If they have a polynomial factor, $\bar{q}_{n-r+m+1}$ of order $n-r+m+1$ in common, this factor must divide $\Delta(s)\cdot q_{n-r+m}$. This indicates that $n-r+m+1$ poles of the closed loop system corresponding to $C_1(s)$ have been taken out to obtain one reduced order controller. That is, at λ_i, $i = 1, ..., n-r+m+1$ with $\Delta(\lambda_i) = 0$, $i = 1, ..., n+r$, we must have

$$N_{c1}(s)\cdot q_{n-r+m} + (k_m s^m + \ldots + k_1 s + k_0)\cdot D_p(s)|_{s=\lambda_i} = 0, \quad i = 1, ..., n-r+m+1 \quad (2)$$

and therefore, have the following dependent set of equations:

$$D_{c1}(s)\cdot q_{n-r+m} - (k_m s^m + \ldots + k_1 s + k_0)\cdot N_p(s)|_{s=\lambda_i} = 0, \quad i = 1, ..., n-r+m+1 \quad (3)$$

The construction of q_{n-r+m} does not care which of the $(n-r+m+1)$ roots of $\triangle_1$ are picked, as long as complex conjugates are chosen together. Hence, to obtain a controller order reduction by one with a proper controller, it is sufficient to solve the (2) for q_{n-r+m} and $(k_m s^m + k_{m-1}s^{m-1} + \ldots + k_1 s + k_0)$. Without any loss of generality, q_{n-r+m} may be chosen to be a monic polynomial. Therefore, there are $(n-r+2m+1)$ unknowns and $(n-r+m+1)$ linear equations. The formulation of the problem and the procedure for the solution in terms of these variables is provided in the following sections. If a Hurwitz polynomial q_{n-r+m} satisfying (2) is found, then a stabilizing controller, whose order is reduced by one, is obtained by (3).

B. Problem Formulation

This section presents the formulation of the Youla parametrization, introduced in the above section, in a compact form and presents the problem in a similar representation.

Equation (2) can be expressed as follows:

$$\mathbb{A}[\ \tilde{q}_0 \quad \tilde{q}_1 \quad \cdots \quad \tilde{q}_{n-r+m} \quad k_0 \quad \cdots \quad k_m\ ]^T = \mathbb{B}$$

where $\mathbb{A}$ is a numeric matrix of size $(n-r+m+1) \times (n-r+2m+1)$, $\mathbb{B}$ is a numeric matrix of size $(n-r+m+1) \times (1)$. The vector $\alpha = [\ \tilde{q}_0 \quad \tilde{q}_1 \quad \cdots \quad \tilde{q}_{n-r+m} \quad k_0 \quad \cdots \quad k_m\ ]^T$ are the variable parameters in this procedure. An appropriate solution to α, which makes q_{n-r+m} Hurwitz, will yield the desired low order controller.

Let the desired solution be $\alpha = \alpha^{\dagger} + \lambda\alpha_i^{\mathcal{N}}$, where $\alpha^{\dagger}$ is the minimum norm solution and $\alpha^{\mathcal{N}}$ is the null space of the above system of equations. $\alpha_i^{\mathcal{N}}$ represents i^{th} vector in a basis of the null space. Hence, the solution can be represented in terms of one parameter, λ. The span of the null space can be controlled by choosing m, i.e. it depends on the form of the $\mathcal{Q}$ parametrization.

The closed loop controller (one order lower) can be expressed as $C_s(s) = \frac{\tilde{N}_c}{\tilde{D}_c} = \dfrac{N_{c0} + \lambda N_{c1}}{D_{c0} + \lambda D_{c1}}$. This equation is obtained by substituting α into (1) and removing the $(n-r+m+1)$ roots of $\triangle_1$, which we picked earlier, from both the numerator and the denominator.

Our main objective is to ensure that this lower order controller *stabilizes* the system. It should also satisfy some prescribed *performance specification* which can be expressed as a complex stabilization problem. A large class of performance problems such as, desired phase margin, desired upper bound on the $\mathcal{H}_\infty$ norm of a weighted sensitivity transfer function, or a requirement that a certain closed loop transfer function be SPR etc., can be reduced to the problem of determining a set of stabilizing controllers that render a set of complex polynomials Hurwitz [9].

1) Lower order stabilizing controller.: The closed loop characteristic equation, for the reduced order controller, is given by:

$$\tilde{\Delta}(s,\lambda) = \tilde{N}_c N_p + \tilde{D}_c D_p = \Delta_0 + \lambda\Delta_1 \tag{4}$$

Problem 1. Find λ such that $\tilde{\Delta}(s,\lambda)$ is Hurwitz.

Procedure. The controller can be reduced to a one order lower stabilizing controller if there exists a Hurwitz q_{n-r+m}. Hence, we need to find λ such that the monic polynomial q_{n-r+m} is Hurwitz. This problem reduces to a real root locus problem for the range of λ such that the following polynomial is Hurwitz.

$$s^{n-r+m} + \tilde{q}^{\dagger}_{n-r+m-1}s^{n-r+m-1} + \ldots + \tilde{q}^{\dagger}_0 + \lambda\left[\tilde{q}^{\mathcal{N}}_{n-r+m-1}s^{n-r+m-1} + \ldots + \tilde{q}^{\mathcal{N}}_0\right]$$

2) Lower order controller satisfying given performance specification.: In this paper we will consider a performance specification which can be expressed as a complex stabilization problem. Consider the performance specification to be a desired upper bound on the $\mathcal{H}_\infty$ norm of a weighted sensitivity transfer function. The given performance ($\mathcal{H}_\infty$) specification is expressed as:

$$\left\| \frac{N_w}{D_w} \frac{N_p(N_{c0} + \lambda N_{c1})}{(D_p(D_{c0} + \lambda D_{c1}) + N_p(N_{c0} + \lambda N_{c1}))} \right\|_\infty \leq \gamma$$

This can be expressed as, $\left\| \dfrac{N_0 + \lambda N_1}{D_0 + \lambda D_1} \right\|_\infty \leq \gamma.$

The above $\mathcal{H}_\infty$ specification is expressed as a complex stabilization problem, i.e. $\gamma(D_0 + \lambda D_1) + e^{j\theta}(N_0 + \lambda N_1)$ should be Hurwitz $\forall\theta \in [0, 2\pi]$. Using Euler's formula ($e^{j\theta} = cos\theta + jsin\theta$), this can be converted into a problem of simultaneous stabilization of family of complex polynomial given by,

$$\begin{aligned} P(jw,\lambda,\theta) &= P_r(w,\lambda,\theta) + jP_i(w,\lambda,\theta) \\ &= P_1(w,\theta) + \lambda P_2(w,\theta) \end{aligned} \tag{5}$$

Problem 2. Find λ such that the family of complex polynomials $P(jw,\lambda,\theta)$ is Hurwitz $\forall\theta \in [0, 2\pi]$.

Procedure. Equation (5) represents a family of polynomial. $P_r(w,\theta)$ and $P_i(w,\theta)$ are real polynomial. Then the range of λ which stabilizes this family of polynomial can be found by discretizing θ and solving the complex root locus for each θ. The the range of λ is found by finding the intersection of $\lambda's$ stabilizing each complex polynomial in the family of complex polynomials. For each θ_j, the range of λ_j is found by first solving for the λ_j, where a root lies on the imaginary axis. If $P_j(w,\lambda)$ has a root on the imaginary axis, then the real polynomials P_{rj} and P_{ij} have a common real root. The values of λ_j for which a common root exists, is found by solving the resultant of P_{rj} and P_{ij}. The number of roots in the left half plane and right half plane changes only at the $\lambda_j's$ found earlier. This provides a range of λ_j for which the j^{th} complex polynomial is Hurwitz. The intersection of stabilizing ranges for all polynomials in the family of polynomial provides a range of value for λ, which is used to obtain a one order lower controller which satisfies a pre-specified performance criterion.

IV. Example

Consider a fourth order plant,

$$P(s) = \frac{s^2 + 3s + 2}{s^4 - 10s^3 + 35s^2 - 50s + 24}.$$

The initial controller is

$$C(s) = \frac{1000s^3 + 13000s^2 + 54000s + 72000}{s^3 + 42s^2 + 395s + 1050}.$$

The weighting function considered is $W = 1$. The $\mathcal{H}_\infty$ norm of the complementary sensitivity function is 3.2704. The aim is to recursively find reduced order controllers with $\mathcal{H}_\infty \leq 2$. It is interesting to note that initial system can have an H_∞ norm greater than the desired value.

Recursive order reduction results using the two procedure in the above section is presented. Order reduction without any performance performance specification takes around 1-2 seconds on a desktop computer with Pentium $2Ghz$ processor. The order reduction with performance criterion takes about 10-15 seconds.

Without performance criterion: The form of the Youla Parameter is $Q(s) = \dfrac{k_1 s + k_0}{s^2 + q_1 s + q_0}$. The closed loop poles are $[-6.123 \pm j24.195,\ -6.691 \pm j2.834,\ -0.592 \pm$

$j0.800481,\ -5.187]$. We choose to remove $-6.123 \pm j24.195,\ -5.187$. The range of λ is found to be $(-\infty, 2365.9523)$. Choosing $\lambda = 2000$, the reduced controller obtained is :

$$C_1(s) = \frac{668.1s^2 + 5682s + 1.037e04}{s^2 + 27.6s + 152.7}.$$

The $\mathcal{H}_\infty$ norm of reduced system is 9.928.

The poles of the closed loop of the reduced system is $[-1.516 \pm j21.533,\ -6.691 \pm j2.833,\ -0.592 \pm j0.801]$ Choosing to remove $-1.516 \pm j21.533, -6.691 \pm j2.833$, the range of λ is $(-\infty, -2229.335)$. Picking $\lambda = -2500$, the reduced order controller obtained is,

$$C_2(s) = \frac{640.7s + 1584}{s + 23.53}.$$

The $\mathcal{H}_\infty$ norm of reduced system is 10.2588 .

No further reduction of the controller occurs. Failure to calculate does not guarantee that we have achieved the minimal order controller. however in this example, the minimal order of stabilization is indeed first order.

With performance criterion: The same form of the Youla Parameter is considered. θ is discretized at intervals of $45\,\text{deg}$. The roots which are sought to be removed are $-6.123 \pm j24.195,\ -5.187$. For $\theta = 0\,\text{deg}$, using Sturm sequences the resultant is found to have 1 real root. The value of this root is 7340.372. This is a value of λ where one root of the polynomial $P(jw)$ is on the imaginary axis. The number of roots are checked for two intervals of λ, $[-\infty,\ 7340.372]$ and $[7340.372,\ \infty]$. It is found that for $\lambda \in (-\infty,\ 7340.372)$, all roots of $P(jw, \lambda)$ are in the Left Half Plane. For other values of θ, the ranges of $\lambda's$ is:

$$\begin{array}{ll} 45\,\text{deg} \Rightarrow (-\infty, 954.659) & 90\,\text{deg} \Rightarrow (-\infty, -1432.454) \\ 135\,\text{deg} \Rightarrow (-\infty, -2842.117) & 180\,\text{deg} \Rightarrow (-\infty, 1339.214) \\ 225\,\text{deg} \Rightarrow (-\infty, -2842.117) & 270\,\text{deg}(-\infty, -1432.454) \\ 315\,\text{deg} \Rightarrow (-\infty, 954.659) & \end{array}$$

Choosing $\lambda = -10000$, a reduced order controller which provides a system with $\mathcal{H}_\infty$ norm of 1.6365 is obtained. The reduced order controller is given by,

$$C_1(s) = \frac{3013s^2 + 1.95e04s + 3.375e04}{s^2 + 94.03s + 485.2}.$$

The same procedure is repeated to recursively obtain a lower order controller $C_2(s)$ which yields a system with $\mathcal{H}_\infty$ norm of 1.6049. The controller is given by

$$C_2(s) = \frac{3054s + 3013}{s + 89.23}.$$

V. Conclusions

In this paper, we have shown the structure of the set of minimal order stabilizing controllers for continuous time LTI plants in the controller parameter space by proving the result: The set of proper stabilizing controllers of order r is not empty and the set of strictly proper stabilizing controllers of order r is bounded *iff* r is the minimal order of stabilization for the plant. The basis for the above result is the following: if there is a stabilizing controller of order r, then there is a stabilizing, strictly proper controller of order $r + 1$ and hence, strictly proper controllers of all orders higher than r. Therefore, one can recursively reduce the order of the controllers to arrive at a minimal order of stabilization for a plant. Since all achievable closed loop maps are affine in the Q parameter, we devise a sufficient condition for order reduction: Suppose there exists a Q parameter to induce a pole zero cancellation in the closed loop map to decrease the order of the closed loop system by one, then the corresponding controller is reduced in order by one. Using the ability of specifying certain performance criteria as the condition for a family of complex polynomial to be Hurwitz, an algorithm was devised in which a lower order controller can be obtained with a pre-specified performance criterion. It is not necessary for the initial controller to satisfy the performance criteria. Finally, we provide numerical examples to illustrate the procedure developed in this paper.

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