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Optimal Estimation with Limited Measurements

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Abstract— **We consider a sequential estimation problem with two decision makers, or agents, who work as members of a team. One of the agents sits at an observation post, and makes sequential observations about the state of an underlying stochastic process for a fixed period of time. The observer agent upon observing the process makes a decision as to whether to disclose some information about the process to the other agent who acts as an estimator. The estimator agent sequentially estimates the state of the process. The agents have the common objective of minimizing a performance criterion with the constraint that the observer agent may only act a limited number of times.**

I. INTRODUCTION

Recursive estimation of a linear stochastic process with full and partial state information has been extensively studied in the literature [1]. In this paper, we introduce the problem of recursive estimation with *limited* information. More specifically, we consider estimating a stochastic process over a decision horizon of length N using only $M \leq N$ measurements. Both the measurement and estimation of the process is carried out sequentially by two different decision makers, or agents, called the *observer* and the *estimator*,[1] respectively. Over the decision horizon of length N, the observer agent has exactly M opportunites to disclose some information about the process to the estimator. These information disclosures, or *transmissions*, are assumed to be error and noise free, and the problem is to jointly determine the best observation and estimation policies that minimize the average estimation error between the process and its estimate.

Estimation problems of this nature arise in many applications ranging from monitoring and control over wireless sensor networks [2], and scheduling of packet transmissions over time-allocation limited channels. For example, due to the power-limited nature of the wireless sensors, in most sensor net applications the wireless devices can only make a *limited* number of transmissions [3].

The rest of the paper is organized as follows. In Section II, we formally define the problem, and briefly discuss a potential application. Section III discusses estimating an i.i.d. random sequence with a limited number of mesaurements along with an extension to the case when the underlying process is Gauss-Markov. We present some illustrative examples in Section IV.

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[1]As we show next, in a communication-theoretic setting we may call them an *encoder* and a *decoder*, respectively.

II. PROBLEM STATEMENT

A. Problem Definition

The problem of optimal estimation with limited measurements can be treated in the more general framework of a communication system with limited channel uses. For this purpose, consider the generic communication system whose block diagram is given in Figure 1 [4]. The source outputs some data b_k for $0 \leq k \leq N-1$, that needs to be communicated to the user over a channel. The data b_k are generated according to some *a priori* known stochastic process, $\{b_k\}$, which may be i.i.d., or correlated as in a Markov process. An encoder (or an observer) and a decoder (or an estimator) is placed after the source output and the channel output, respectively, to communicate the data to the user efficiently. In the most general case, the encoder/observer may have access to a noise-corrupted version of the source output:

$$z_k = b_k + v_k, \; 0 \leq k \leq N-1$$

where $\{v_k\}$ is an independent[2] noise process.

The main constraint is that the encoder/observer can access the channel only a *limited*, $M \leq N$, number of times. The goal is to design an observer-estimator pair,[3] $(\mathscr{O},\mathscr{E})$, that will "causally" (or sequentially) observe (or encode) the data measurements, z_k, and estimate (or decode) the channel output, y_k, so as to minimize the *average distortion* or *error* between the observed data, b_k, and estimated data, $\hat{b}_k$.

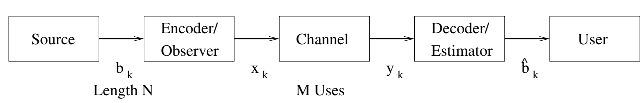

Fig. 1. Communication with limited channel use.

The channel is assumed to be memoryless, and is completely characterized by the conditional probability distribution $P_c(y|x)$ on $y \in \mathscr{Y}$ for each $x \in \mathscr{X}$, where $\mathscr{X}$ and $\mathscr{Y}$ are the set of allowable channel inputs, and the set of possible channel outputs, respectively.

The average distortion $D_{(M,N)}$ depends on the distortion measure and may vary depending on the underlying application. Some examples are the average mean-square error

$$D_{(M,N)} = E\left\{\frac{1}{N}\sum_{k=0}^{N-1}(b_k-\hat{b}_k)^2\right\} \tag{1}$$

[2]Independent across time and from the source output process b_k.

[3]Or depending on the application, an encoder-decoder pair $(\mathscr{E},\mathscr{D})$.

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or the Hamming (probability of error) distortion measure

$$D_{(M,N)} = E\left\{\frac{1}{N}\sum_{k=0}^{N-1}\mathscr{I}_{b_k\neq\hat{b}_k}\right\} \tag{2}$$

where $\mathscr{I}_S$ denotes the indicator function of the set S.

From a communication-theoretic standpoint, with the channel, source, and the distortion measure defined, we can formally state our main problem: Given a source and a memoryless channel, for a given decision-horizon N, and number of channel uses M, what is the minimum attainable value of the average distortion $D_{(M,N)}$? This minimization is carried out over the choice of possible encoder-decoder (observer-estimator) pairs which are *causal*.

In this paper, we present a solution to this problem when the source process is i.i.d. with a continuous or discrete probability density function, and the encoder/observer has access to the noiseless or a noisy version of the source output. We assume that the channel is noiseless, and hence, it is completely characterizied by the probability distribution $P_c(y|x) = \delta(y-x)$. We also present the solution to the case when the source process is Gauss-Markov.

B. An Example Application

In most industrial wireless sensing applications, wireless sensors communicate over an RF (radio frequency) channel, and a significant amount of power is consumed to transmit the sensor measurements [3]. In most sensor net applications, the wireless sensors are battery-powered [3], and therefore, it is important to use the wireless channel only when it is necessary to extend the life of the wireless device as long as possible. The desired length of time the wireless device will be in operation can be related to the decision horizon N in some appropriate time unit, and the size of the battery installed in the sensor can be related to the possible number of transmissions or channel uses M (see Figure 2).

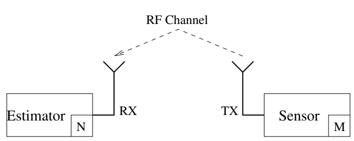

Fig. 2. Optimal transmission scheduling with limited channel access.

Hence, given an underlying performance criterion $D_{(M,N)}$, the problem is to design the best transmission schedule, and estimation policies for the wireless device and the remote monitoring station, respectively.

III. ESTIMATING AN I.I.D. RANDOM SEQUENCE

A. Problem Definition

Consider the special case of the general problem defined in Section II, where the source outputs a zero-mean[4] i.i.d. random sequence $b_k,\ 0 \le k \le N-1$. Let $\mathscr{B}$ denote the range of the random variable b_k. We assume that b_k's have a finite second moment, $\sigma_b^2 < \infty$, but their probabiliy distribution remains unspecified for now. At time k, the encoder/observer makes a sequential measurement of b_k, and determines whether to access the channel for transmission, which it can only do a limited, $M \le N$, number of times. The channel is noiseless and thus has a capacity to transmit the source output error-free when it is used to transmit. Note that, even when it decides not to use the channel for transmission, the observer/encoder may still convey a 1-bit information to the estimator/decoder. In view of this, the channel input x_k belongs to the set $\mathscr{X} := \mathscr{B}\cup\{\mathrm{NT}\}$, where NT stands for "no transmission."

More precisley, we let s_k denote the number of channel uses (or transmissions) left at time k. Now if $s_k \ge 1$, we have $y_k = x_k$ for $x_k \in \mathscr{B}\cup\{\mathrm{NT}\}$. If $s_k = 0$, on the other hand, the channel is useless, since we have exhausted the allocated number of channel uses. Note that, when the channel is noiseless, both the observer and the estimator can keep track of s_k by initializing $s_0 = M$ and decrementing it by 1 every time a transmission decision is taken.

We want to design an estimator/decoder

$$\hat{b}_k = \hat{\mu}_k(I_k^d) \text{ for } 0 \le k \le N-1$$

based on the available information I_k^d at time k. Clearly, the information available to the estimator is controlled by the observer. The distortion measure between the observed and estimated processes can be taken to be the average mean square error as given by (1), or the probability of error distortion measure which is given by (2).

The information I_k^d available to the estimator at time k is a result of an outcome of decisions taken by the observer up until time k. Let the observer's decision at time k be

$$x_k = \mu_k(I_k^e)$$

where I_k^e is the information available to the observer at time k. Assuming perfect recall, we have

$$\begin{aligned} I_0^e &= \{(s_0,t_0);b_0\} \\ I_k^e &= \{(s_k,t_k);b_0^k;x_0^{k-1}\},\ 1 \le k \le N-1 \end{aligned}$$

where t_k denotes the number of time or decision slots left at time k. We have $t_0 = N$ and $t_{k+1} = t_k - 1,\ 0 \le k \le N-2$. The range of $\mu_k(\cdot)$ is the space $\mathscr{X} = \mathscr{B}\cup\{\mathrm{NT}\}$. Let σ_k denote the decision whether the observer has decided to transmit or not. Assume $s_k \ge 1$, and let $\sigma_k = 1$ if a transmission takes place; i.e., $x_k \in \mathscr{B}$, and $\sigma_k = 0$ if no transmission takes place. We have $s_0 = M$ and $s_{k+1} = s_k - \sigma_k,\ 0 \le k \le N-2$. The observer's decision at time k is a function of its k past measurements, and $k-1$ past decisions, i.e.,

$$\mu_k(I_k^e) : \mathscr{B}^k \times \mathscr{X}^{k-1} \to \mathscr{X},\ 0 \le k \le N-1$$

Now, the information I_k^d available to the estimator at time k can be written as $I_k^d = \{(s_k,t_k);y_0^k\},\ 0 \le k \le N-1$. By definition, the channel output y_k satisfies $y_k = x_k$ if $s_k \ge 1$, and $y_k \in \emptyset$ (i.e., no information) if $s_k = 0$. Thus, for $s_k \ge 1$, having $y_k = x_k = \mathrm{NT}$ may still be considered as information.

Consider the class of observer-estimator (encoder-decoder) policies consisting of a sequence of functions $\Pi =$

[4]This is not restrictive, as the known mean can be subtracted out by the estimator.

$\{\mu_0, \hat{\mu}_0, \dots, \mu_{N-1}, \hat{\mu}_{N-1}\}$, where each function μ_k maps I_k^e into $\mathscr{X}$, and $\hat{\mu}_k$ maps I_k^d into $\mathscr{B}$,[5] with the additional restriction that μ_k can map to $\mathscr{B}$ at most M times. Such policies are called *admissible*. We want to find an admissible policy $\pi^* \in \Pi$ that minimizes the average N-stage distortion, or estimation error:

$$e^{\pi}_{(M,N)} = E\left\{\sum_{k=0}^{N-1}(b_k - \hat{\mu}_k(I_k^d))^2\right\} \tag{3}$$

or for source processes with discrete probability densities:

$$e^{\pi}_{(M,N)} = E\left\{\sum_{k=0}^{N-1}\mathscr{I}_{b_k \neq \hat{\mu}_k(I_k^d)}\right\} \tag{4}$$

If $M \geq N$, this problem has the trivial solution where the observer writes the source output b_k directly into the channel at each time k (i.e., $\mu_k^*(b_k) = b_k$), and since the channel is noiseless, the estimator can use an identity mapping (i.e., $\hat{\mu}_k^*(I_k^d) = b_k$), resulting in zero distortion. Therefore, we only consider the case when $M < N$.

Before closing our account on this section, we would like to note the nonclassical nature of the information in this problem. Clearly, the observer's action affects the information available to the estimator, and there is no way in which the estimator can infer the information available to the observer. Also note the order of actions between the decision makers in the problem. At time k, first the random variable b_k becomes available, then the observer acts by transmitting some data or not, and finally, the estimator acts by estimating the state with $\hat{\mu}_k$, the cost is incurred, and we move to the next time $k+1$.

B. Structure of the Solution

We first consider the problem of finding the optimal estimator $\hat{\mu}_k^*$ at time k. Note that the estimator $\hat{\mu}_k$ appears only in a single term in the error expressions (3)-(4). Thus, for the mean-square error criterion, the optimal estimator is simply the solution of the quadratic minimization problem

$$\min_{\hat{\mu}_k(I_k^d)} E\left\{(b_k - \hat{\mu}_k(I_k^d))^2 | I_k^d\right\}$$

which is given by the conditional expectation of b_k given the available information at time k:

$$\hat{\mu}_k^*(I_k^d) = E\{b_k | I_k^d\} = E\{b_k | (s_k, t_k); y_0^k\} \tag{5}$$

Similarly, for the probability of error distortion criterion, the optimal estimator is the solution of the minimization problem

$$\min_{\hat{\mu}_k(I_k^d)} E\left\{\mathscr{I}_{b_k \neq \hat{\mu}_k(I_k^d)} | I_k^d\right\}$$

If at time k the channel can still be used ($s_k \geq 1$), the solution to this problem is given by the maximum *a posteriori* probability (MAP) estimate of the random variable b_k given the available information at time k:

$$\hat{\mu}_k^*(I_k^d) = \arg\max_{m_i \in \mathscr{B}_k(I_k^d)} \delta(y_k - i) p_i = \arg\max_{m_i \in \mathscr{B}_k((s_k,t_k);y_0^k)} p_i \tag{6}$$

where $\mathscr{B}_k(I_k^d) \subset \mathscr{B}$ is some subset of the range of the random variable b_k, which we assume is countable. Let m_i denote the values the random variable b_k takes. Then, p_i's denote the probability mass function of the random variable b_k, i.e., $p_i = P[b_k = m_i]$.

Note that, for the probability of errror distortion criterion, if the channel is useless at time k (i.e., $s_k = 0$), the best estimate of b_k is simply given by

$$\hat{\mu}_k^*(I_k^d) = \arg\max_{m_i \in \mathscr{B}} p_i \tag{7}$$

since the past channel outputs, y_0^{k-1}, are independent of b_k.

Similarly, for the mean-square error criterion, the channel output y_k has no information on b_k if $s_k = 0$. Thus, in this case, the conditional expectation in (5) equals

$$\hat{\mu}_k^*(I_k^d) = E\{b_k | (0, t_k); y_0^{k-1}, y_k\} = E\{b_k\} = 0 \tag{8}$$

since again the past channel outputs, y_0^{k-1}, are generated by the σ-algebra of random variables b_0^{k-1}, and hence are independent from b_k.

If $s_k \geq 1$, the channel output $y_k = x_k$, but since $y_0^{k-1} = x_0^{k-1}$ is the outcome of a Borel-measurable function defined on the σ-algebra generated by b_0^{k-1}, the conditional expectation in (5) is equivalent to

$$\hat{\mu}_k^*(I_k^d) = E\{b_k | (s_k, t_k); x_k\} \tag{9}$$

By a similar argument we can write (6) as

$$\hat{\mu}_k^*(I_k^d) = \arg\max_{m_i \in \mathscr{B}_k((s_k,t_k);x_k)} p_i \tag{10}$$

Now, substituting the optimal estimators (9)-(10) back into the estimation error expressions (3)-(4) yields

$$e^{\pi}_{(M,N)} = E\left\{\sum_{k=0}^{N-1}(b_k - E\{b_k | (s_k, t_k); x_k\})^2\right\} \tag{11}$$

$$e^{\pi}_{(M,N)} = E\left\{\sum_{k=0}^{N-1}\mathscr{I}_{b_k \neq \arg\max_{m_i \in \mathscr{B}_k((s_k,t_k);x_k)} p_i}\right\} \tag{12}$$

which we seek to minimize over the observer/encoder policies $\mu_k(I_k^e), 0 \leq k \leq N-1$. Since $x_k = \mu_k(I_k^e)$, we see that the choice of an observer policy affects the cost only through the information made available to the estimator.

In general, the observer's decision μ_k at time k depends on (s_k, t_k), all past measurements b_0^{k-1}, the present measurement b_k, and its past actions x_0^{k-1}. However, as we show next, there is nothing the observer can gain by having access to its past measurements b_0^{k-1} and its past actions x_0^{k-1} as far as the optimization of the criteria (11)-(12) are concerned. Thus, a *sufficient statistics* for the observer are the current measurement b_k and the remaning number of channel uses (transmission opportunities) and decision instances, i.e. (s_k, t_k). We have the following result whose proof is in [2].

Proposition 1: The set $S_k^e = \{(s_k, t_k); b_k\}$ constitutes sufficient statistics $S_k^e(I_k^e)$ for the optimal policy μ_k^* of the observer. In other words,

$$\mu_k^*(I_k^e) = \bar{\mu}(S_k^e(I_k^e))$$

[5]Note that we do not distinguish between the source and user sets.

for some function $\bar{\mu}$.

A consequence of Proposition 1 is that the observer's decision to use the channel to transmit a source measurement or not is based purely on the current observation b_k and its past actions only through (s_k, t_k).

Since μ_k depends explicitly only on the current source output b_k, the search for an optimal observer policy can be narrowed down to the class of policies of the form[6]

$$\mu_k(I_k^e) = \bar{\mu}((s_k,t_k);b_k) = \begin{cases} b_k & \text{if } b_k \in \mathscr{T}_{(s_k,t_k)} \\ \text{NT} & \text{if } b_k \in \mathscr{T}^c_{(s_k,t_k)} \end{cases} \tag{13}$$

where $\mathscr{T}_{(s_k,t_k)}$ is a measurable set on $\mathscr{B}$ and is a function of (s_k,t_k). The complement of the set $\mathscr{T}_{(s_k,t_k)}$ is taken with respect to $\mathscr{B}$, i.e., $\mathscr{T}^c_{(s_k,t_k)} = \mathscr{B} \backslash \mathscr{T}_{(s_k,t_k)}$. When probability of error distortion criterion is used, Proposition 1 implies that $\mathscr{B}_k((s_k,t_k);\text{NT}) = \mathscr{T}^c_{(s_k,t_k)}$, and $\mathscr{B}_k((s_k,t_k);m_i) = m_i$.

Note that the optimal estimators (9) and (10) have access to (s_k,t_k) as well. Thus, even when the observer chooses not to transmit b_k, it can still pass a 1-bit information about b_k to the estimator provided that $s_k \geq 1$. If k is such that all M transmissions are concluded prior to time k (i.e., $s_k = 0$), the estimators are given by (7)-(8), irrespective of b_k.

C. The Solution with Mean-Square Error Criterion

Let $(s_k,t_k) = (s,t)$, and $e^*_{(s,t)}$ denote the optimal value of the estimation error (or distortion) (11) when the decision horizon is of length t, and the observer is limited to s channel uses, where $s \leq t$. From (13), we know that at time k, the optimal observation policy will be of the form (13). Now, at time $k+1$, depending on the realization of the random variable b_k, the remaining $(t-1)$-stage estimation error is either $e^*_{(s-1,t-1)}$, or $e^*_{(s,t-1)}$. Thus, inductively by the DP equation [5], we can write[7]

$$\begin{aligned} e^*_{(s,t)} = \min_{\mathscr{T}_{(s,t)}} \Big\{ & e^*_{(s-1,t-1)} \int_{b \in \mathscr{T}_{(s,t)}} f(b)db \\ & + e^*_{(s,t-1)} \int_{b \in \mathscr{T}^c_{(s,t)}} f(b)db \\ & + \int_{b \in \mathscr{T}^c_{(s,t)}} \left[b - E\{b | b \in \mathscr{T}^c_{(s,t)}\} \right]^2 f(b)db \Big\} \end{aligned} \tag{14}$$

where $f(b)$ is the pdf of the random variable b_k. To solve for $e^*_{(s,t)}$, we first note the boundary conditions $e^*_{(t,t)} = 0$, and $e^*_{(0,t)} = t\sigma_b^2$, $\forall t \geq 0$. The optimal sets satisfy the boundary conditions $\mathscr{T}^*_{(t,t)} = \mathscr{B}$, and $\mathscr{T}^*_{(0,t)} = \emptyset$, $\forall t \geq 0$. The recursion of (14) needs to be solved offline and the optimal sets $\mathscr{T}^*_{(s,t)}$ must be tabulated starting with smaller values of (s,t).[8] The solution to the original problem can then be determined as follows:

Initialize $s_0 = M$, $t_0 = N$. For each k in $0 \leq k \leq N-1$ do:

1) Look up the optimal set $\mathscr{T}^*_{(s_k,t_k)}$ from the table that was determined offline.
2) Observe b_k and apply the observation policy
$$\bar{\mu}^*((s_k,t_k);b_k) = \begin{cases} b_k & \text{if } b_k \in \mathscr{T}^*_{(s_k,t_k)} \\ \text{NT} & \text{if } b_k \in \mathscr{T}^{*c}_{(s_k,t_k)} \end{cases}$$
3) Apply the estimation policy
$$\hat{\mu}_k^*(\mathscr{T}^*_{(s_k,t_k)}) = E\{b_k | b_k \in \mathscr{T}^{*c}_{(s_k,t_k)}\} = \frac{\int_{b \in \mathscr{T}^{*c}_{(s_k,t_k)}} b f(b)db}{\int_{b \in \mathscr{T}^{*c}_{(s_k,t_k)}} f(b)db}$$
4) Update
$$s_{k+1} = s_k - \sigma_k,\ t_{k+1} = t_k - 1$$

In tabulating $\mathscr{T}^*_{(s,t)}$ one should start with solving for $\mathscr{T}^*_{(1,2)}$, and the corresponding estimation error $e^*_{(1,2)}$. To determine the optimal set at (s,t), we need to know the optimal costs at $(s,t-1)$, and $(s-1,t-1)$. Hence, we can propagate our calculations as shown in Figure 3.

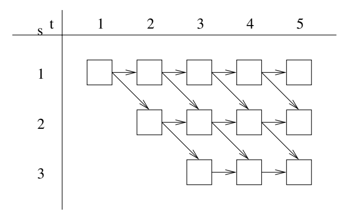

Fig. 3. Recursive calculation of $e^*_{(s,t)}$.

Now, we return to the problem of minimizing (14) over $\mathscr{T}_{(s,t)}$. This is an optimization problem over measurable sets $\mathscr{T}_{(s,t)}$ on the real line, and since these sets are not countable, there is no known method for carrying out this minimization in a systematic manner. Therefore, we restrict our search to the sets that are in the form of simple symmetric intervals, i.e., $\mathscr{T}^c_{(s,t)} = [-\beta_{(s,t)}, \beta_{(s,t)}]$, where $0 \leq \beta_{(s,t)} \leq \infty$. The optimum choice for $\beta_{(s,t)}$ is $\beta^*_{(s,t)} = \sqrt{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}$. Note that, we always have $e^*_{(s,t-1)} \leq e^*_{(s-1,t-1)}$, since for the same decision horizon, $t-1$, the minimum average distortion achieved by s channel uses, is always less than that achieved by $s-1$ channel uses. So, $\beta^*_{(s,t)}$ always exists. Hence, in the class of symmetric intervals, the best set $\mathscr{T}^c_{(s,t)}$ is given by the interval

$$\mathscr{T}^{*c}_{(s,t)} = [-\sqrt{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}, \sqrt{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}] \tag{15}$$

D. The Solution with Probability of Error Criterion

As in Section III-C, let $(s_k,t_k) = (s,t)$, and let $e^*_{(s,t)}$ denote the optimal value of the estimation error. At time $k+1$, depending on the realization of the random variable b_k, the remaining $(t-1)$-stage estimation error is either $e^*_{(s-1,t-1)}$, or $e^*_{(s,t-1)}$. Thus, assuming that $s \geq 1$, inductively by the DP equation, we can write

$$\begin{aligned} e^*_{(s,t)} = \min_{\mathscr{T}_{(s,t)}} \Big\{ & P[b_k \in \mathscr{T}_{(s,t)}] e^*_{(s-1,t-1)} + P[b_k \in \mathscr{T}^c_{(s,t)}] e^*_{(s,t-1)} \\ & + P[b_k \in \mathscr{T}^c_{(s,t)}] - \max_{m_j \in \mathscr{T}^c_{(s,t)}} p_j \Big\} \end{aligned}$$

[6] As long as k is such that all M measurements are not exhausted.

[7] Assuming that the random variables $\{b_k\}$ are continuous with a well-defined probability density function (pdf) $f(b)$.

[8] Note that $(1,2)$ is the smallest possible nontrivial value.

Plugging in $P[b_k \in \mathcal{T}^c_{(s,t)}] = \sum_{m_i \in \mathcal{T}^c_{(s,t)}} p_i$, and rearranging the terms, we obtain the following error recursion:

$$\begin{aligned} e^*_{(s,t)} &= e^*_{(s-1,t-1)} \\ &\quad + \min_{\mathcal{T}^c_{(s,t)}} \left\{ -(e^*_{(s-1,t-1)} - e^*_{(s,t-1)}) \sum_{m_i \in \mathcal{T}^c_{(s,t)}} p_i \right. \\ &\quad \left. + \sum_{m_i \in \mathcal{T}^c_{(s,t)}} p_i - \max_{m_j \in \mathcal{T}^c_{(s,t)}} p_j \right\} \end{aligned} \tag{16}$$

We have the following property whose proof is in [2].

Proposition 2: Suppose $1 \le s \le t$. Then, the error difference $e^*_{(s-1,t-1)} - e^*_{(s,t-1)}$ satisfies:

$$0 \le e^*_{(s-1,t-1)} - e^*_{(s,t-1)} \le 1$$

Using Propostion 2, in [2] we show that the optimum choice for the sets $\mathcal{T}^c_{(s,t)}$ is the singleton $\mathcal{T}^{c*}_{(s,t)} = \{m_{i^*}\}$, where $i^* = \arg\max_{m_i \in \mathcal{B}} p_i$. In other words, the optimal solution is not to transmit the *most likely* outcome, and transmit all the other outcomes. Moreover, this policy is independent of the number of decision instances left, and the number of transmission opportunities left, provided that $s_k \ge 1$.

E. Gaussian Case

Suppose b_k's are zero-mean, i.i.d. Gaussian with variance σ_b^2. If we generalize the search for an optimum in (14) to intervals of the form $\mathcal{T}^c_{(s,t)} = [\alpha_{(s,t)}, \beta_{(s,t)}]$, where $-\infty \le \alpha_{(s,t)} \le \beta_{(s,t)} \; +\infty$, it can be shown that the solution is still a symmetric interval around zero in the Gaussian case [2]. To evaluate the optimum estimation error $e^*_{(s,t)}$ in terms of $e^*_{(s-1,t-1)}$ and $e^*_{(s,t-1)}$, we substitute the optimum interval solution (15) into the right-hand side of (14) and obtain

$$\begin{aligned} e^*_{(s,t)} &= e^*_{(s-1,t-1)} - \left[e^*_{(s-1,t-1)} - e^*_{(s,t-1)} - \sigma_b^2 \right] \\ &\quad \times \left[2\Phi\left(\sqrt{\frac{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}{\sigma_b^2}} \right) - 1 \right] \\ &\quad - \frac{2\sigma_b^2}{\sqrt{2\pi\sigma_b^2}} \sqrt{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}} e^{-\frac{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}{2\sigma_b^2}} \end{aligned} \tag{17}$$

We can normalize the optimal estimation error by letting $\varepsilon_{(s,t)} = \frac{e^*_{(s,t)}}{\sigma_b^2}$, and rewrite the recursion (17) in a simpler form:

$$\begin{aligned} \varepsilon_{(s,t)} &= \varepsilon_{(s-1,t-1)} - \left[\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)} - 1 \right] \\ &\quad \times \left[2\Phi\left(\sqrt{\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)}} \right) - 1 \right] \\ &\quad - \frac{2}{\sqrt{2\pi}} \sqrt{\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)}} e^{-\frac{\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)}}{2}} \end{aligned} \tag{18}$$

with the initial conditions (IC) $\varepsilon(t,t) = 0,\ \varepsilon(0,t) = t, \forall t \ge 0$.

F. Limiting Solution for the Gaussian Case

The normalized recursion of the estimation error in the Gaussian case is given by (18). Note that, this is a Riccati-like equation, which needs to be solved offline to determine the optimal observation and estimation policies. However, unlike the RE, (18) is a two-dimensional recursion with given boundary conditions. In order to see how this recursion behaves when (s,t) are large, we proceed as follows. Let $t = s + k$, where $k \in \mathcal{N}$ is an arbitrary integer. We substitute $s + k$ for t in the recursion (18) starting with $k = 1$, and obtain

$$\begin{aligned} \varepsilon_{(s,s+1)} &= \varepsilon_{(s-1,s)} - [\varepsilon_{(s-1,s)} - 1] \times [2\Phi(\sqrt{\varepsilon_{(s-1,s)}}) - 1] \\ &\quad - \frac{2}{\sqrt{2\pi}} \sqrt{\varepsilon_{(s-1,s)}} e^{-\frac{\varepsilon_{(s-1,s)}}{2}} \end{aligned} \tag{19}$$

since $e(s,s) = 0$. The sequence (19), can be shown to converge from the initial condition $\varepsilon(0,1) = 1$ to the limit $l_\varepsilon \approx 0.3345$, which is the solution of the nonlinear equation:

$$(1 - l_\varepsilon)(2\Phi(\sqrt{l_\varepsilon}) - 1) = \frac{2}{\sqrt{2\pi}} \sqrt{l_\varepsilon} e^{-\frac{l_\varepsilon}{2}}$$

Proceeding in the same manner, we can calculate the limits of the sequences $\varepsilon_{(s,s+k)}$ for an arbitrary but otherwise fixed k, and it turns out that $\lim_{s \to \infty} \varepsilon_{(s,s+k)} = l_\varepsilon k \approx 0.3345k$. Since k is arbitrary, we conclude that for $t \ge s$,

$$\lim_{s \to \infty, t \to \infty} \varepsilon_{(s,t)} = l_\varepsilon (t - s)$$

As a result, from (15), the optimum interval for the observer not to transmit is given by $[-\sqrt{l_\varepsilon}, \sqrt{l_\varepsilon}] \approx [-0.5784, 05784]$. Note that this interval is for the normalized Gaussian density. For an arbitrary Gaussian distribution with variance σ_b^2, the interval must be scaled by the standard deviation σ_b, i.e., $[-\sqrt{l_\varepsilon}\sigma_b, \sqrt{l_\varepsilon}\sigma_b]$. In the limit as (M,N) get large, the N-stage optimal average distortion, $D^*_{(M,N)} = \frac{1}{N} e^*_{(M,N)}$, for a Gaussian i.i.d. source with variance σ_b^2 is given by

$$\lim_{M \to \infty, N \to \infty} D^*_{(M,N)} = l_\varepsilon \frac{1}{N}(N - M)\sigma_b^2 = l_\varepsilon \left(1 - \frac{M}{N}\right)\sigma_b^2$$

One can identify the term $\left(1 - \frac{M}{N}\right)\sigma_b^2$ as the average distortion when no observer is used. More precisely, say there is no observer agent, and thus the source is allowed to make M transmissions at some arbitrary times. When the estimator receives the source data, clearly the estimation error for that stage is zero. When no measurement is received on the other hand, the estimator simply estimates the process by its *a priori* density, and incurs an error of size σ_b^2. This process of estimation leads to an average estimation error of size $e_{no-observer} = \left(1 - \frac{M}{N}\right)\sigma_b^2$. Comparing this error with the asymptotic optimum estimation error obtained by using an optimum observer-estimator pair, we see that they differ by a factor of l_ε. Hence, $(1 - l_\varepsilon)$ can be thought of as the encoder-decoder gain of the communication system in Figure 1 for a Gaussian i.i.d. source and a noiseless channel.

G. Gaussian Case with Noisy Measurements

Let the source process b_k be i.i.d. Gaussian. If the observer has access to a noisy version of the source output, i.e., $z_k = b_k + v_k$, where v_k is an independent zero-mean i.i.d. Gaussian process with variance σ_v^2, the solution is similar to the noiseless case [2]. The optimal estimator is given by

$$\hat{\mu}_k((s_k,t_k);x_k) = \begin{cases} \frac{\sigma_b^2}{\sigma_b^2 + \sigma_v^2} z_k & \text{if } z_k \in \mathcal{T}_{(s_k,t_k)} \\ E\left\{ b_k | z_k \in \mathcal{T}^c_{(s_k,t_k)} \right\} & \text{if } z_k \in \mathcal{T}^c_{(s_k,t_k)} \end{cases}$$

and the optimum interval where the observer chooses not to transmit is given by $\mathscr{T}^c_{(s,t)} = [\alpha_{(s,t)}, \beta_{(s,t)}]$, where $\beta^*_{(s,t)} = \frac{\sigma_b^2+\sigma_v^2}{\sigma_b^2}\sqrt{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}$, and $\alpha^*_{(s,t)} = -\beta^*_{(s,t)}$. Substituting these values into the error recursion and normalizing, $\varepsilon_{(s,t)} := \frac{\sigma_b^2+\sigma_v^2}{(\sigma_b^2)^2} e^*_{(s,t)}$, we obtain the two-dimensional recursion:

$$\begin{aligned} \varepsilon_{(s,t)} \quad = \quad & \varepsilon_{(s-1,t-1)} - \left[\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)} - 1\right] \\ & \times \left[2\Phi\left(\sqrt{\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)}}\right) - 1\right] \\ & + \frac{\sigma_v^2}{\sigma_b^2} - \frac{2}{\sqrt{2\pi}}\sqrt{\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)}}\, e^{-\frac{\varepsilon_{(s-1,t-1)}-\varepsilon_{(s,t-1)}}{2}} \end{aligned} \tag{20}$$

with the ICs $\varepsilon(t,t) = \frac{\sigma_v^2}{\sigma_b^2}t,\ \varepsilon(0,t) = \left(1 + \frac{\sigma_v^2}{\sigma_b^2}\right)t, \forall t \geq 0$.

H. Gauss-Markov Case

Suppose the source process is Markov $b_{k+1} = Ab_k + w_k$, driven by an i.i.d. Gaussian process $\{w_k\}$ with zero-mean. If the observer has access to the source output, b_k, without noise, the solution is similar to the i.i.d. case. The only difference is that, now the observer-estimator pair has to keep track of three variables (r_k, s_k, t_k), where r_k keeps track of the number of time units passed since the last use of the channel for transmission. A similar DP recursion, now in three dimensions, can be obtained; see [2] for details.

IV. ILLUSTRATIVE EXAMPLES

A. Example 1

The first example is just solving the problem of Section III-E for $(s,t) = (1,2)$. So, the observer can use the channel for tranmission only once, at time $k = 0$ or 1, and the observer and the estimator are jointly trying to minimize the average distortion (or estimation error):

$$e = E\left\{(b_0 - \hat{b}_0)^2 + (b_1 - \hat{b}_1)^2\right\}$$

where b_0, b_1 are i.i.d. Gaussian with zero mean, and variance σ_b^2. If we arbitrarily choose to transmit the first source output, or the second one, the estimation error would be $e^*_{no-observer} = \sigma_b^2$, which is the best error that can be achieved without a decision maker that observes the source output. Now, suppose the observer is aware of the fact that the estimator knows the *a priori* distribution of b_0. So, it makes sense for the observer not to transmit the realized value of b_0 if this value happens to be close to the *a priori* estimate of it, which in this case is the mean value of b_0, i.e., zero.

Motivated by this intuition, the observer adopts a policy in which it will not use the channel to transmit b_0 if it lies in an interval $[\alpha, \beta]$ around zero. The decision for the second stage would already have been made once α and β are determined, because, if $b_0 \in [\alpha, \beta]$, then the observer cannot use the channel to transmit at time 1, and if $b_0 \notin [\alpha, \beta]$, there is no reason why it should not transmit at time 1.

Now, the optimization problem faced by the observer is to choose α and β such that the following error is minimized:

$$\begin{aligned} e_{(\alpha,\beta)} \quad = \quad & \int_\alpha^\beta (b - E\{b | b \in [\alpha,\beta]\})^2 f(b) db \\ & + \sigma_b^2 P\{b_0 \notin [\alpha,\beta]\} \end{aligned}$$

The solution can be easily obtained by checking the first, and second order optimality conditions, and is given by $(\alpha^*, \beta^*) = (-\sigma_b, \sigma_b)$. Thus, the observer should not use the channel to transmit the source output b_0 if it falls within one standard deviation of its mean. For these values of α and β, the optimal value of the estimation error can be calculated as

$$e_{(\alpha^*,\beta^*)} = \sigma_b^2\left[1 - \sqrt{\frac{2}{\pi e}}\right]$$

Comparing this to the no-observer policy, $e^*_{no-observer} = \sigma_b^2$, we see that there is a $\sqrt{\frac{2}{\pi e}} \approx 48\%$ improvement in the estimation error.

B. Example 2

The second example we will discuss considers the following design problem. We are given a time-horizon of a fixed length, say 100. For this $N = 100$ time units, we would like to sequentially estimate the state of a zero-mean, i.i.d. Gaussian process with unit variance. We have a design criterion which says that the *aggregate* estimation error should not exceed 20. The solution to this problem without an observer agent is to reveal 80 arbitrary observations to the estimator and achieve an aggregate estimation error of 20. Suppose now we use the optimal observer-estimator pair. In Figure 4, we plot the optimal value of the 100-stage estimation error for different values of M.

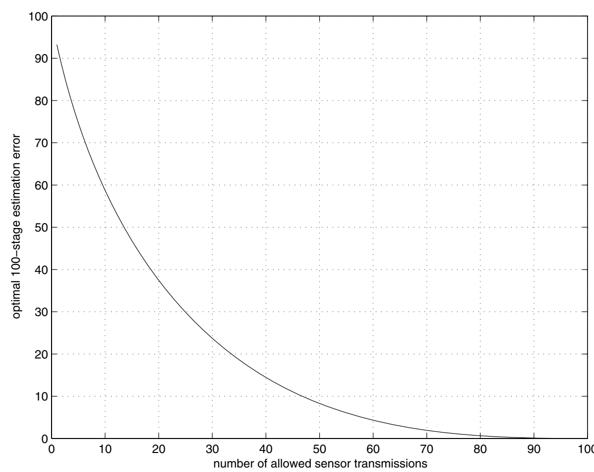

Fig. 4. Optimal 100-stage estimation error vs the number of channel uses.

It is striking that a cumulative estimation error of 20 can be achieved with only 34 transmissions. This is approximately a $\frac{80-34}{80} \times 100 \approx 58\%$ improvement over the no-observer policy.

REFERENCES

[1] B. D. Anderson and J. B. Moore *Optimal Filtering*. Englewood Cliffs, NJ: Prentice-Hall, Inc., 1979.

[2] O. C. Imer, "Optimal estimation and control under communication network constraints," *Ph.D. Thesis*, University of Illinois, 2005.

[3] "Industrial wireless technology for the 21st century," based on views of the industrial wireless community, US DOE, 2002.

[4] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York, NY: John Wiley & Sons, Inc., 1991.

[5] D. P. Bertsekas, *Dynamic Programming and Optimal Control*. Belmont, MA: Athena Scientific, 1995.