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Output Feedback Stabilization of the Moore-Greitzer Compressor Model

Anton Shiriaev[1] Rolf Johansson[2] Anders Robertsson[2] Leonid Freidovich[1]

[1]Department of Applied Physics and Electronics
Umeå University, SE-901 87 Umeå, SWEDEN
{Anton.Shiriaev|Leonid.Freidovich}@tfe.umu.se

[2]Department of Automatic Control, Lund Institute of Technology
Lund University, P.O. Box 118, SE-221 00 Lund, SWEDEN
{Rolf.Johansson|Anders.Robertsson}@control.lth.se

Abstract. This paper presents a number of facts related to the output feedback stabilization of the Moore-Greitzer compressor model. We show that quadratic feedback stabilization of the surge subsystem of the three-state Moore-Greitzer compressor model, which ensures absence of additional equilibria and cycles in the closed-loop system augmented with stall dynamics, implies convergence of all solutions to the unique equilibrium at the origin. Then some steps in developing such output feedback controller for surge subsystem are discussed.
Keywords: Circle Criterion, Output Feedback Design

I. Introduction

This paper is devoted to design of output feedback controllers for a nonlinear control system—the so-called Moore-Greitzer model (MG model) [1]-[3]—that has been extensively used as approximation to model compressor dynamics [3], [1], [2], [5]

$$\dot{\phi} = -\psi + \frac{3}{2}\phi + \frac{1-(1+\phi)^3}{2} - 3R(1+\phi) \quad (1)$$

$$\dot{\psi} = (\phi - u)/\beta^2 \quad (2)$$

$$\dot{R} = -\sigma R^2 - \sigma R\left(2\phi + \phi^2\right), \quad R(0) \geq 0 \quad (3)$$

$$y = \psi \quad (4)$$

Here, u is the control variable to be defined and y is available to measurement; σ is a positive constant.

The main difficulty in designing feedback controllers for (1)–(3) is due to the presence of two nonlinearities in the equations (1) and (3) of the model. They make the search for an output feedback controller and an associated Lyapunov function for the closed-loop system a quite nontrivial mathematical problem that still appears to be open.

The common approach [8], [7], [9] is to consider an output feedback controller design only for the surge subsystem, *i.e.*, for the equations (1), (2); and then on the next step to analyze the behavior of the closed loop with the augmented stall dynamics (3). It turns out that this approach might not lead to a successful design of an output feedback controller for the system (1)–(3); even robust quadratic stabilization of the surge subsystem can result in unstable behavior of the closed loop when the stall dynamics are taken into consideration. An example of such problems is given below.

This paper suggests a test, which can be used for analysis of the closed-loop system with an output feedback controller designed to stabilize only the surge subsystem. It is shown that if such a controller ensures that the closed-loop system will dissipate so that all solutions enter a compact set of the phase space and remain there after some transition and that there are no additional equilibria and cycles for the closed-loop, then one can state that any solution of the closed-loop system converges to the origin. As for local stability based on a separation principle, results are reported in [11].

As a second contribution, this note suggests a series of steps for designing such an output feedback controller based on *Quadratic Constraints* (QC)—*i.e.*, the structural property of the nonlinearity in the surge dynamics (1)-(2).

The paper is organized as follows. Section II suggests a general test for global attractivity of the equilibrium of the MG model at the origin based on successful surge subsystem stabilization (Theorem 1). Section III collects design steps for developing stabilizing controllers for the surge subsystem based on the QC method to meet conditions of Theorem 1. Results of computer simulations are reported in Section IV, and some conclusions are drawn in Section V.

II. New Test for Global Attractivity of the Origin of the Moore-Greitzer Model

Theorem 1: Assume that:

1) (The case $R(t) \equiv 0$) The output feedback controller

$$\dot{z} = \mathcal{F}(z, y), \ u(t) = \mathcal{U}(z, y) \quad (5)$$

makes the surge subsystem (1)–(2), quadratically stable, *i.e.*, there are matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that the inequality

$$\frac{d}{dt}\left\{\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix}^T P \begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix}\right\} \leq -\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix}^T Q \begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} \quad (6)$$

holds along any solution of (1)–(2), (5);

2) The controller (5) satisfies the implication[1]

$$\text{If } y(t) = y_*, \ u(t) = u_* \ \forall\, t \ \Rightarrow \ z(t) = z_*, \quad (7)$$

[1]This is the formal restriction imposed on set of admissible controllers.

i.e. its internal state does not possess any dynamics that cannot be observed from the input y;

3) (The case $R(t) \not\equiv 0$) The closed-loop system (1)–(3), (5) has no cycles and only one stationary solution - the equilibrium at the origin.

Then any solution of the closed-loop system (1)–(3), (5) converges to the origin. ■

Proof of Theorem 1 consists of checking a number of statements, which are quoted and briefly commented below.

Step 1: "The quadratic stability of the surge subsystem (1)–(2) and the controller (5) implies that any solution of the closed-loop system (1)–(3), (5), that is the system with nontrivial stall dynamics, is bounded. Furthermore, the stall variable $R(t)$ resides within the interval $[0, 1]$ after transition."

This technical statement is proven based on the inequality (6). The details are excluded here due to the lack of space.

Step 2: If for a solution $[\phi(t), \psi(t), R(t), z(t)]$ of the closed-loop the stall variable $R(t)$ tends to zero as $t \to +\infty$, then the other components of the solution tend to zero as well

$$\phi(t) \to 0, \quad \psi(t) \to 0, \quad z(t) \to 0 \quad \text{as} \quad t \to +\infty.$$

This is proven following common arguments applied for quadratically stable systems with vanishing perturbations.

Step 3: If the closed-loop system (1)–(3), (5) has a solution that does not converge to the origin, then there exists a solution of (1)–(3), (5), which is periodic in time and is not an equilibrium.

Consider first the case when along a solution the function $\dot{R}(t)$ preserves its sign, say, $\dot{R} > 0$, *i.e.*, $R(t)$ is monotonically growing approaching some constant value $R_* > 0$ remaining within the interval $[0, 1]$.

This solution has an ω-limit set γ_0, which is non-empty, compact and consists of solutions of the closed-loop system. Then one can easily deduce from Eq. (3) that along any closed loop system solution belonging to γ_0 the variable $\phi(t)$ remains constant. In the same fashion, it follows from Eq. (1) that the variable $\psi(t)$ along any solution on γ_0 is some constant. Hence, Eq. (2) shows that the control action $u(t)$ becomes equal to constant value. Based on Assumption 2, one concludes then that $z(t)$ is also a constant on γ_0. Summarizing, the set γ_0 is an equilibrium of the closed loop system, but by Assumption 3 it can only be the origin. This contradicts the hypothesis that $\dot{R}$ always is positive.

In the same way, one can prove that if $\dot{R}$ preserves its sign remaining negative along some solution, then by necessity this solution will converge to the origin.

Consider now a solution of the closed-loop system, if it exists, that does not converge to the origin. Then by necessity its time derivative should change its sign infinitely many times. Therefore, such a solution should pass through the hypersurface

$$\Gamma = \{[\phi, \psi, R, z] : \dot{R} = 0\} \tag{8}$$

infinitely many times. In turn, the intersection of Γ with a ball of sufficiently large radius centered at the origin, gives a compact subset Γ_c of Γ, where all the passages of this solution through Γ take place. Following standard arguments, one concludes that there is at least one accumulation point of these passages on Γ_c and this accumulation point corresponds to the presence of a cycle in the dynamics for the closed-loop system. By assumption, there are no cycles in the closed-loop system. Therefore, we conclude that any solution of (1)–(3), (5) converges to the origin in positive time. ■

III. Design of Output Feedback Controllers (5) Based the Quadratic Constraints method

Following Theorem 1 one can separate the output controller design problem for the MG model into two steps: Firstly, to elaborate output feedback controller(s) for the surge subsystem; and secondly, to check that the closed-loop system with the stall dynamics (3) taken into consideration does not have equilibria other than the one at the origin and does not have nontrivial cycles. Here we present a series of static and dynamical feedback controllers for the surge subsystem.

A. Quadratic Constraints for Nonlinearity in the Surge Subsystem

The nonlinearity of the surge subsystem $w(\phi)$ and a linear output

$$\begin{aligned} w(\phi) &= 1 - (1+\phi)^3 && (9) \\ v &= -\phi && (10) \end{aligned}$$

satisfy the inequality

$$\begin{aligned} w \cdot v &= \left[1 - (1+\phi)^3\right](-\phi) \\ &= \phi^2\big(1 + (1+\phi) + (1+\phi)^2\big) = \phi^2\big(3 + 3\phi + \phi^2\big) \\ &\geq 3\,\phi^2/4 = 3\,v^2/4 && (11) \end{aligned}$$

along a solution of the MG model with any feedback.

Another QC of an incremental nature can be mentioned for (9). Indeed, $\forall\,\phi_1$ and $\forall\,\phi_2$ the inequality below holds

$$\begin{aligned} &(\phi_1 - \phi_2)\left[w(\phi_2) - w(\phi_1)\right] = && (12) \\ &= (\phi_1 - \phi_2)^2\left[(\phi_1+1)^2 + (\phi_2+1)^2 + (\phi_1+1)(\phi_2+1)\right] \geq 0 \end{aligned}$$

B. State Feedback Law Design for the Surge Subsystem

Suppose that both components ϕ and ψ of the surge system (1)–(2) are available from measurements. Introduce a family of feedback controllers of the form

$$u = \phi - \beta^2\left\{\lambda_1\phi + \lambda_2\psi + \alpha(1 - (1+\phi)^3)\right\} \tag{13}$$

where λ_1, λ_2 and α are constant to be defined. With such a choice of feedback, the surge subsystem looks as

$$\frac{d}{dt}\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} 3/2 & -1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \phi \\ \psi \end{bmatrix} + \begin{bmatrix} 1/2 \\ \alpha \end{bmatrix} w(\phi) \tag{14}$$

where $w(\phi)$ is defined in (9). Applying conditions of the Circle criterion to the closed-loop system (14) with the QC (11), results in parameters λ_1, λ_2 and α of the controller (13), which makes the closed-loop quadratically stable.

Theorem 2: Those constant parameters λ_1, λ_2 and α that satisfy the following two conditions:

1) the inequality

$$\mathrm{Re}\{G(j\omega)\} - 3|G(j\omega)|^2/4 < 0 \tag{15}$$

holds[2] for any $\omega \in R^1$, where

$$G(s) = \frac{-s + \alpha + \lambda_2/2}{s^2 - s(\lambda_2 + 3/2) + \lambda_1 + 3\lambda_2/2}; \tag{16}$$

2) the following matrix is strictly Hurwitz

$$\begin{bmatrix} 3/2 & -1 \\ \lambda_1 & \lambda_2 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 0.5 \\ \alpha \end{bmatrix}[-1,\, 0]$$

characterize the feedback controllers (13), which make the closed-loop system (14) quadratically stable. ■

The conditions mentioned in Theorem 2 describe a non-empty set of parameters. For example, the values

$$\lambda_1 = 5, \quad \lambda_2 = -3, \quad \alpha = 1 \tag{17}$$

belong to this set, the transfer function (16) is then

$$G(s) = \frac{-s - 1/2}{s^2 + 3s/2 + 1/2}.$$

It is seen that $-G(s)$ satisfies the SPR condition [4]. As known, the conditions of Theorem 2 are equivalent to the fact that there exists a 2×2 matrix $P = P^T > 0$ such that the following matrix relations hold

$$\begin{bmatrix} 3/2 & -1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^T P + P\begin{bmatrix} 3/2 & -1 \\ \lambda_1 & \lambda_2 \end{bmatrix} < \begin{bmatrix} 3/4 & 0 \\ 0 & 0 \end{bmatrix} \tag{18}$$

$$[1/2,\, \alpha]\, P = [1,\, 0] \tag{19}$$

C. State Feedback with Integrator for the Surge Subsystem

One of the critical conditions of Theorem 1 is the absence of equilibria in the closed-loop system different from the equilibrium at the origin. From this perspective, modify the state feedback controller (13) by adding an integrator of ϕ in the loop

$$\begin{aligned} u &= \phi - \beta^2\left\{\lambda_1\phi + \lambda_2\psi + \alpha(1-(1+\phi)^3) + \varepsilon\, q\right\} \\ \dot{q} &= -\phi \end{aligned} \tag{20}$$

where ε is a constant. The surge subsystem with the controller (20) takes on the form

$$\frac{d}{dt}\begin{bmatrix} \phi \\ \psi \\ q \end{bmatrix} = \underbrace{\begin{bmatrix} 3/2 & -1 & 0 \\ \lambda_1 & \lambda_2 & \varepsilon \\ -1 & 0 & 0 \end{bmatrix}}_{A}\begin{bmatrix} \phi \\ \psi \\ q \end{bmatrix} + \underbrace{\begin{bmatrix} 1/2 \\ \alpha \\ 0 \end{bmatrix}}_{B} w(\phi) \tag{21}$$

and the linear output (10) is now

$$v = -\phi = \underbrace{[-1,\, 0,\, 0]}_{C}\begin{bmatrix} \phi \\ \psi \\ q \end{bmatrix} \tag{22}$$

[2]The inequalities (15), (23) degenerate at $\omega = \pm\infty$ because $G(s)$ is a strictly proper transfer function, but (15), (23) are assumed to remain valid when ω approaching $\pm\infty$ provided that the left hand side of (15) and (23), respectively, is premultiplied by ω^2.

The stability conditions of the closed-loop system (21) could be obtained from the Circle criterion applied to the QC (11), in the same way as it was done in Theorem 2.

Theorem 3: Those constant parameters λ_1, λ_2, α and ε that satisfy the two conditions:

1) the inequality

$$\mathrm{Re}\{G(j\omega)\} - 3|G(j\omega)|^2/4 < 0 \tag{23}$$

holds[2] for any $\omega \in R^1$, where

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \frac{-\frac{1}{2}s^2 + \left(\alpha + \frac{1}{2}\lambda_2\right)s}{s^3 - \left(\lambda_2 + \frac{3}{2}\right)s^2 + \left(\lambda_1 + \frac{3}{2}\lambda_2\right)s - \varepsilon}; \end{aligned} \tag{24}$$

2) the matrix $\left(A + \frac{3}{4}BC\right)$ is strictly Hurwitz,

describe the feedback controllers (20), which make the closed-loop system (21) quadratically stable. Here the matrices A, B and C are defined in (21) and (22). ■

Theorem 3 describes a non-empty set of parameters, *e.g.*

$$\lambda_1 = 5, \quad \lambda_2 = -3, \quad \alpha = 1, \quad \varepsilon = -0.8 \tag{25}$$

belong to this set. Furthermore, as in Theorem 2 the conditions of Theorem 3 are equivalent to the the fact the there exists a 3×3 positive definite matrix $P = P^T$ such that the following matrix relations hold (*cf.* Eqs. (18-19), [4])

$$A^T P + PA - 3C^T C/4 < 0 \tag{26}$$

$$PB = -C \tag{27}$$

D. Output Feedback Control for the Surge Subsystem

Unfortunately, the controllers suggested in Theorems 2–3 cannot be directly used for output feedback stabilization of the surge subsystem (1)-(2). Indeed both controllers (13) and (20) use the ϕ component of the surge state, while it is not available to measurement, see Eq. (4). Here we present modifications and dynamical extension of the controllers (13), (20) that result in a new family of output feedback stabilizing controllers for the surge subsystem.

Consider a dynamical output feedback controller

$$u = \lambda_1\psi + \lambda_2 z + \alpha_u\left(1 - (1 + c_\psi\psi + c_z z)^3\right) \tag{28}$$

$$\dot{z} = \lambda_3\psi + \lambda_4 z + \alpha_z\left(1 - (1 + c_\psi\psi + c_z z)^3\right) \tag{29}$$

where λ_1-λ_4, α_u, α_z, c_ψ and c_z are constant parameters.

Augmenting the surge subsystem (1)-(2) with such feedback results in the dynamical system

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} &= \underbrace{\begin{bmatrix} 1.5 & -1 & 0 \\ 1/\beta^2 & -\lambda_1/\beta^2 & -\lambda_2/\beta^2 \\ 0 & \lambda_3 & \lambda_4 \end{bmatrix}}_{A}\begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} + \\ &+ \underbrace{\begin{bmatrix} 0 \\ -\frac{\alpha_u}{\beta^2} \\ \alpha_z \end{bmatrix}}_{B_2} w_2 + \underbrace{\begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}}_{B_1} w_1 \end{aligned} \tag{30}$$

Here w_1 and w_2 are the nonlinearities with the first one defined in (9), $w_1(\phi) = w(\phi)$, while the second one is introduced into the system (30) by the controller (28)-(29)

$$w_2 = \left(1 - (1 + c_\psi \psi + c_z z)^3\right) \quad (31)$$

The new nonlinearity (31) is of particular form and added to the controller so that both the quadratic constraints (11) and (12) can be re-used for developing stability conditions for the closed-loop system (30). Indeed, based on the relations (11) and (12) at least 3 QCs could be mentioned

$$\begin{aligned} -\phi\, w_1 &\geq 3\phi^2/4 & (32)\\ (-c_\psi \psi - c_z z)\, w_2 &\geq 3\left(-c_\psi \psi - c_z z\right)^2/4 & (33)\\ (c_\psi \psi + c_z z - \phi)(w_1 - w_2) &\geq 0 & (34) \end{aligned}$$

which hold along a solution of the closed-loop system (30).

The Circle criterion applied to the closed-loop system (30) with the QCs (32)–(34) states that (30) is quadratically stable if

1) the 'frequency condition' holds, *i.e.*, there exist non-negative τ_1, τ_2, τ_3 such that $\tau_1 + \tau_2 + \tau_3 > 0$ and the inequality

$$\mathrm{Re}\left\{\tau_1\left(v_1^*\xi_1 - \frac{3}{4}|\xi_1|^2\right) + \tau_2\left(v_2^*\xi_2 - \frac{3}{4}|\xi_2|^2\right) + \right. \left. +\tau_3\left(v_1 - v_2\right)^*\left(\xi_1 - \xi_2\right)\right\} < 0 \quad (35)$$

is valid for any ξ_1, $\xi_2 \in C$ and any $\omega \in R$, where

$$v_1 = \underbrace{[-1, 0, 0]}_{C_1}(j\omega I_3 - A)^{-1}\left(B_2\xi_2 + B_1\xi_1\right) \quad (36)$$

$$v_2 = \underbrace{[0, -c_\psi, -c_z]}_{C_2}(j\omega I_3 - A)^{-1}\left(B_2\xi_2 + B_1\xi_1\right) \quad (37)$$

2) the matrix

$$A + 3\left(B_2 C_2 + B_1 C_1\right)/4 \quad (38)$$

is strictly Hurwitz. Here the matrices A, B_1, B_2, C_1 and C_2 are defined in (30), (36) and (37).

As known [4], the conditions of the Circle criterion applied to the QCs (32)–(34) are equivalent to the existence of a 3×3 matrix $P = P^T > 0$ satisfying the matrix relations

$$A^T P + PA - 3\left(C_1^T C_1 + C_2^T C_2\right)/4 < 0 \quad (39)$$

$$2PB_1 = \tau_1 C_1^T + \tau_3\left(C_1^T - C_2^T\right) \quad (40)$$

$$2PB_2 = \tau_2 C_2^T + \tau_3\left(C_2^T - C_1^T\right) \quad (41)$$

These relations are bilinear matrix inequalities (BMIs), and there are no efficient computational methods for solving these equations to identify even one set of the feedback controller parameters, see (28)-(29).

To show that the matrix equations (39)–(41) are feasible, choose the parameters of the controller (28)-(29) as follows $\lambda_2 = \gamma_1$, $\alpha_u = \gamma_3$, $c_\psi = \gamma_4$, $c_z = 1$ and

$$\begin{aligned} \lambda_1 &= \gamma_2 + \gamma_1\gamma_4, \quad \alpha_z = 0.5 + \gamma_3\gamma_4/\beta^2 \\ \lambda_3 &= \gamma_4^2\left(\gamma_1 - 1\right)/\beta^2 + \gamma_4\left(3/2 + \gamma_2/\beta^2\right) - 1 \\ \lambda_4 &= 3/2 + \gamma_4\left(\gamma_1 - 1\right)/\beta^2 \end{aligned} \quad (42)$$

where γ_1–γ_4 are some constants. Introduce new coordinates for the closed-loop system (30)

$$\psi = \psi, \quad \hat\phi = z + \gamma_4\psi, \quad e = \phi - z - \gamma_4\psi \quad (43)$$

Direct calculations show that in the new coordinates the closed-loop system is of the form

$$\frac{d}{dt}\begin{bmatrix}\hat\phi\\ \psi\\ e\end{bmatrix} = \begin{bmatrix}3/2 & -1 & \gamma_4/\beta^2\\ 1/\beta^2 & 0 & 1/\beta^2\\ 0 & 0 & \left(3/2 - \gamma_4/\beta^2\right)\end{bmatrix}\begin{bmatrix}\hat\phi\\ \psi\\ e\end{bmatrix} + \begin{bmatrix}0\\ -1/\beta^2\\ 0\end{bmatrix}u^* + \begin{bmatrix}0.5\\ 0\\ 0\end{bmatrix}w_2 + \begin{bmatrix}0\\ 0\\ 0.5\end{bmatrix}(w_1 - w_2) \quad (44)$$

$$u^* = \gamma_1\,\hat\phi + \gamma_2\,\psi + \gamma_3\,w_2 \quad (45)$$

Let us rewrite (44) as two coupled systems

$$\frac{d}{dt}\begin{bmatrix}\hat\phi\\ \psi\end{bmatrix} = \begin{bmatrix}3/2 & -1\\ 1/\beta^2 & 0\end{bmatrix}\begin{bmatrix}\hat\phi\\ \psi\end{bmatrix} + \begin{bmatrix}0\\ -1/\beta^2\end{bmatrix}u^* + \begin{bmatrix}0.5\\ 0\end{bmatrix}w_2 + \begin{bmatrix}\gamma_4\\ 1\end{bmatrix}\frac{e}{\beta^2} \quad (46)$$

$$\dot e = \left(3/2 - \gamma_4/\beta^2\right)e + 0.5(w_1 - w_2) \quad (47)$$

The system (47) is stable with the Lyapunov function

$$V_e(e) = p_2 e^2 \quad (48)$$

with $p_2 > 0$ provided that the linear part is negative[3], i.e.

$$3/2 - \gamma_4/\beta^2 < 0 \quad (49)$$

Another observation is that the subsystem (46) coincides with the surge subsystem provided that $e = 0$, but for this subsystem (46) the signal $\hat\phi$ is available from measurements; it is a linear combination of ψ and z. Therefore, when $e = 0$, (46) can be stabilized by the state feedback laws elaborated in Theorem 2 and the closed loop has the Lyapunov function

$$V_1(\hat\phi, \psi) = \begin{bmatrix}\hat\phi\\ \psi\end{bmatrix}^T P_1 \begin{bmatrix}\hat\phi\\ \psi\end{bmatrix}. \quad (50)$$

To prove that such controller is stabilizing when $e \neq 0$, consider the next Lyapunov function candidate for (46), (47)

$$V(\hat\phi, \psi, e) = V_1(\hat\phi, \psi) + \rho V_2(e), \quad \rho > 0. \quad (51)$$

Its time derivative along a solution satisfies the inequality

$$\frac{dV}{dt} < -\begin{bmatrix}\hat\phi\\ \psi\end{bmatrix}^T Q_1 \begin{bmatrix}\hat\phi\\ \psi\end{bmatrix} + 2\begin{bmatrix}\hat\phi\\ \psi\end{bmatrix}^T P_1 \begin{bmatrix}\gamma_4\\ 1\end{bmatrix}\frac{e}{\beta^2} - \rho q_2 e^2 \quad (52)$$

where Q_1 is a 2×2 positive definite matrix and $q_2 > 0$. It is then clear that there are some positive values for the parameter ρ so that the right hand side of (52) is a negative definite quadratic form, and the quadratic stability is shown. To summarize, the next statement has just been proven.

Theorem 4: Consider the surge subsystem (1)–(2) augmented with the nonlinear dynamical feedback controller (28)–(29) and rewritten as (30). Then the Circle criterion

[3] thanks to the QC (34)

applied to that closed-loop system (30) with the QC (32)–(34) results in the BMIs, which are feasible and some solutions can be found as follows:

1) Choose any state feedback (13) developed for the surge subsystem, this gives values for λ_1, λ_2 and α;
2) Choose γ_4 so that the inequality (49) is valid, and[4]

$$\gamma_1 = (1-\beta^2\lambda_1), \quad \gamma_2 = -\beta^2\lambda_2, \quad \gamma_3 = -\beta^2\alpha; \quad (53)$$

3) Choose parameters of the controller (28)–(29) based on the found γ_1–γ_4 as it is done in (42). ■

Remark 1: Theorem 4 gives a constructive procedure for solving the associated BMI, but it does not describe all stabilizing controllers that could be found with QCs (32)-(34). For instance, the parameter c_z can be different from 1, and we have not used the constraint (32) in the analysis.

The output feedback law found might not stabilize the MG model with nontrivial stall dynamics. In fact, additional, off-origin equilibria can be introduced for the closed-loop system. ■

To determine controllers that ensure the presence of only one equilibrium in the closed loop, consider further modification of the feedback controller (28)–(29)

$$u = \lambda_1\psi + \lambda_2 z + \alpha_u\left(1-(1+c_\psi\psi+c_z z)^3\right) + \varepsilon_u q \quad (54)$$
$$\dot z = \lambda_3\psi + \lambda_4 z + \alpha_z\left(1-(1+c_\psi\psi+c_z z)^3\right) + \varepsilon_z q \quad (55)$$
$$\dot q = -(c_\psi\psi + c_z z) \quad (56)$$

where λ_1–λ_4, α_u, α_z, c_ψ, c_z, ε_u and ε_z are constant parameters. Here we have added an additional state to the controller that will later be linked to the integrator in the state feedback controller (20). The surge subsystem and the controller (54)–(56) result in the dynamical system

$$\frac{d}{dt}\begin{bmatrix}\phi\\ \psi\\ z\\ q\end{bmatrix} = \begin{bmatrix}1.5 & -1 & 0 & 0\\ \frac{1}{\beta^2} & -\frac{\lambda_1}{\beta^2} & -\frac{\lambda_2}{\beta^2} & -\frac{\varepsilon_u}{\beta^2}\\ 0 & \lambda_3 & \lambda_4 & \varepsilon_z\\ 0 & -c_\psi & -c_z & 0\end{bmatrix}\begin{bmatrix}\phi\\ \psi\\ z\\ q\end{bmatrix} + \quad (57)$$
$$+\begin{bmatrix}0\\ -\alpha_u/\beta^2\\ \alpha_z\\ 0\end{bmatrix} w_2 + \begin{bmatrix}0.5\\ 0\\ 0\\ 0\end{bmatrix} w_1$$

whose stability can be approached via the Circle criterion with the QCs (32)-(34) in the same way as it has been done for the closed-loop system (30).

Following the discussion above, we know that the circle criterion conditions are equivalent to feasibility of an associated BMI and a search for quadratic Lyapunov function. We will skip repeating such a discussion here, and come to search of solutions. In doing so it is useful to repeat the assignments (42) for the parameters of the controller and make the change of coordinates (43) with $q = q$. Then, the closed-loop system is

$$\frac{d}{dt}\begin{bmatrix}\hat\phi\\ \psi\\ q\\ e\end{bmatrix} = \begin{bmatrix}1.5 & -1 & (\varepsilon_z - \frac{\varepsilon_u\gamma_4}{\beta^2}) & \frac{\gamma_4}{\beta^2}\\ \frac{1}{\beta^2} & 0 & \frac{-\varepsilon_u}{\beta^2} & \frac{1}{\beta^2}\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & (\frac{3}{2}-\frac{\gamma_4}{\beta^2})\end{bmatrix}\begin{bmatrix}\hat\phi\\ \psi\\ q\\ e\end{bmatrix}$$
$$= \begin{bmatrix}0\\ -1/\beta^2\\ 0\\ 0\end{bmatrix}u^* + \begin{bmatrix}0.5\\ 0\\ 0\\ 0\end{bmatrix}w_2 + \begin{bmatrix}0\\ 0\\ 0\\ 0.5\end{bmatrix}(w_1 - w_2)$$
$$u^* = \gamma_1\hat\phi + \gamma_2\psi + \gamma_3 w_2 \quad (58)$$

Comparing the matrices of the linear part in (21) and the corresponding 3×3 sub-matrix of the linear part of (58), gives the relation for ε_u and ε_z

$$\varepsilon_z = \varepsilon_u\gamma_4/\beta^2 \quad (59)$$

With such a choice, we can interpret the closed-loop system (58) as two coupled quadratically stable subsystem. Its stability follows from the previous arguments elaborated in Theorem 4 and proves the next

Theorem 5: Consider the surge subsystem (1)–(2) augmented with the nonlinear dynamical feedback controller (54)–(56) and rewritten as (57). Then the Circle criterion applied to the closed-loop system (57) with the QCs (32)–(34) resulting in feasible BMIs. Some solutions can be found as follows:

1) Choose a state feedback (20) developed for the surge subsystem, it gives values for λ_1, λ_2, α and ε;
2) Choose γ_4 so that the inequality (49) is valid, and

$$\gamma_1 = (1-\beta^2\lambda_1), \quad \gamma_2 = -\beta^2\lambda_2, \quad \gamma_3 = -\beta^2\alpha; \quad (60)$$

3) Choose parameters of the controller (54)–(56) based on the found γ_1–γ_4 as it is done in the assignments (42) with $\varepsilon_u = -\beta^2\varepsilon$ and ε_z defined in (59). ■

Theorem 6: Consider the MG model with nontrivial stall dynamics, (1)–(3). Suppose that the output feedback controller is chosen as in Theorem 5. Then the closed-loop system has the unique equilibrium at the origin. ■

Proof: The equilibria of the closed-loop system are the points, where the right hand sides of the differential equations become zero. The first and third lines of (58) are then

$$0 = \dot q = -\hat\phi = -(c_\psi + c_z z)$$
$$0 = \dot{\hat\phi} = 1.5\hat\phi - \psi + 0.5(1-(1+\hat\phi)^3) + \gamma_4(\phi-\hat\phi)/\beta^2$$

and one can conclude that at any equilibrium satisfies $\psi = \gamma_4\phi/\beta^2$. Furthermore, from equation (3)

$$0 = \dot R = -\sigma R(R + \phi^2 + 2\phi)$$

then either $R = 0$ or $R = -\phi^2 - 2\phi$. The first case cannot lead to any other equilibrium than at the origin, because this reduces the closed-loop dynamics to the already stabilized surge subsystem.

[4]These relations come from equating the coefficients of the state feedback controller (13) with coefficients of the controller (58) under the assumption that $\phi = \hat\phi$.

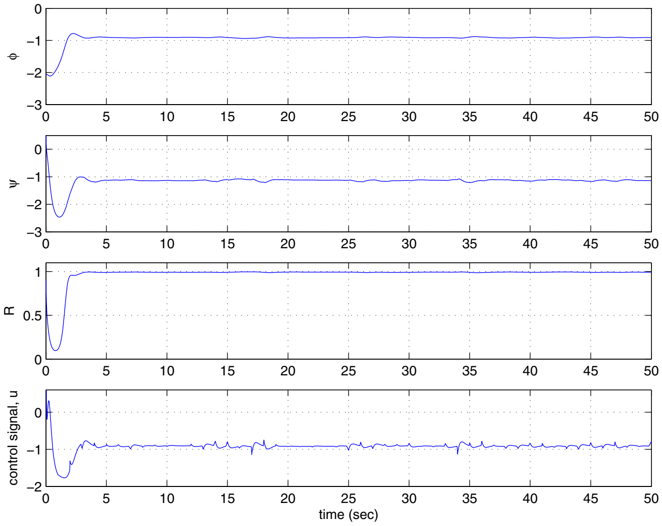

Fig. 1. The solution of the closed-loop system with the dynamical controller from Theorem 4 that is built from the state feedback with the parameters (17) when $\gamma_4 = 3.1$

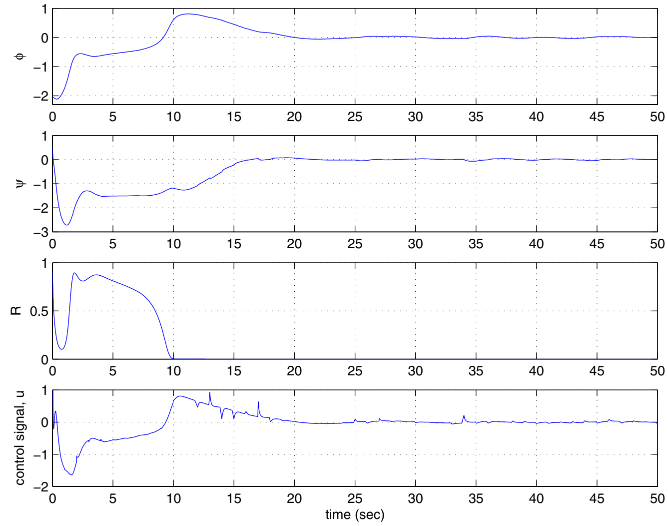

Fig. 2. The solution of the closed-loop system with the dynamical controller from Theorem 4 that is built from the state feedback with the parameters (25) with $\gamma_4 = 3.1$

Consider the second case: Equation (1) then becomes

$$\begin{aligned} 0 &= \dot{\phi} = 1.5\phi - \psi + 0.5(1 - (1+\phi)^3) - 3R(1+\phi) \\ &= 1.5\phi - \frac{\gamma_4}{\beta^2}\phi + \frac{1}{2}(1 - (1+\phi)^3) + 3(\phi^2 + 2\phi)(1+\phi) \\ &= 2.5\phi\left[\phi^2 + 3\phi + 2\left(6 - \gamma_4/\beta^2\right)/5\right] \end{aligned}$$

The case $\phi = 0$ is disregarded because then $R = 0$. Two other solutions of the last equation are given by

$$\phi_{1,2} = 0.5\left(-9 \pm \sqrt{8\gamma_4/(5\beta^2) - 3/5}\right) \tag{61}$$

Having in mind the inequality (49), it is seen that these roots are always real and it can be verified that $R = -\phi_i^2 - 2\phi_i < 0$, $i = 1,\ 2$, if γ_4 is chosen to satisfy (49). However, negative values for R are not possible. ■

IV. Computer Simulations

In this Section some simulations show the performance of the dynamical output feedback controllers presented above. The system has two parameters to be chosen; β and σ. Their values were chosen $\beta = 1/\sqrt{2}$, $\sigma = 7$. The initial conditions for the simulations were chosen as

$$\phi_0 = -2.07,\ \ \psi_0 = 0.5,\ \ R_0 = 0.9,\ \ q_0 = z_0 = 0$$

Figure 1 shows the solution of the closed-loop system with the controller from Theorem 4 that is made from the state feedback with the parameters mentioned in Eq. (17) when $\gamma_4 = 3.1$. As seen, it robustly stabilizes the surge subsystem, but it does not stabilize the whole system when the stall dynamics are added.

Figure 2 shows the solution of the closed-loop system with the controller from Theorem 5 that is built from the state feedback with the parameters (25) when $\gamma_4 = 3.1$. As seen, it robustly stabilizes the surge subsystem, and it does stabilize the system when the stall dynamics are added.

V. Acknowledgements

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VI. Conclusions

This note has two contributions: Firstly, it is shown when stabilization of the surge subsystem leads to convergence of solutions to the origin. Secondly, a family of controllers that possess such properties are presented.

References

[1] E. M. Greitzer and F. K. Moore, "A Theory of Post-stall Transients in Axial Compressor Systems: Part II-Application.", *J. Eng. Gas Turbines and Power*, 108, pp 231-239, 1986.

[2] F. K. Moore and E. M. Greitzer, "A Theory of Post-stall Transients in Axial Compressor Systems: Part I-Development of Equations," *J. Eng. Gas Turbines and Power*, 108, pp 68-76, 1986.

[3] F. K. Moore, "A Theory of rotating stall of Multistage Axial Compressors: Parts I-III", *J. Eng. Gas Turbines and Power*, 106, pp 313-336, 1984

[4] A. K. Gelig, G. A. Leonov, and V. A. Yakubovich, *Stability of Nonlinear Systems with Non-Unique State of Equilibrium.* Moskva Nauka, Moscow, 1978.

[5] J. T. Gravdahl and O. Egeland, "Compressor surge and rotating stall: Modeling and control ", Springer-Verlag, London, 1999

[6] H. Khalil, *Nonlinear Systems (3rd ed.)*, Prentice Hall, Upper Saddle River, NJ, 2002

[7] M. Krstic, D. Fontaine, P. V. Kokotovic, and J. D. Paduano, "Useful nonlinearities and global stabilization of bifurcations in a model of jet engine surge and stall", *IEEE Trans. Automatic Control*, vol. 43, pp. 1739–1745, 1998.

[8] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, "Nonlinear and Adaptive Control Design". New York: Wiley, 1995.

[9] J. D. Paduano, L. Valavani, A. Epstein, E. Greitzer, and G. R. Guenette, "Modeling for control of rotating stall", *Automatica*, vol. 30(9), pp. 1357–1373, 1994.

[10] Shiriaev A., R. Johansson and A. Robertsson. "Some comments on output feedback stabilization of Moore-Greitzer compressor model," *Proc. 43rd IEEE Conf. Decision and Control*, pp.4465-4466, 2004.

[11] M. Maggiore and K. M. Passino "A separation principle for a class of non-UCO systems", *IEEE Trans. Automatic Control*, vol. 48(7), pp. 1122–1133, 2003.