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Behavioral control in the presence of disturbances

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Abstract— **This paper deals with the issue of control with a constraint that in the controlled system certain variables remain free. By free, we mean that the variables are allowed to assume any trajectory. These variables can be interpreted as disturbances. The question of which controlled behaviors are a priori possible such that the controlled behaviors do not impose any constraints on the disturbance variables is addressed and solved. Certain generalities are imposed on the problem in the sense that a controller can act on only certain variables called control variables, while the performance specification is in terms of a different variable called the to-be-controlled variable. Another issue that we deal with is the notion of compatibility of such controllers. Further, the issue of controllability of the controller itself is dealt with, for the specific control problems of pole placement and stabilization. We use the behavioral approach to address these issues. In this setup control is considered as interconnection of two systems without distinguishing the variables of each system into inputs and outputs.**

Keywords: Interconnection, behavior, weak compatibility, regularity, disturbances

I. Introduction and preliminaries

Consider the following plant which contains three types of variables: disturbance variable e, to-be-controlled variable w and the control variable c. The reason for this distinction is as follows. The control variables are the ones that a controller can access. Viewing control as a restriction on the set of trajectories that the plant allows (this set we shall define as the *plant behavior*), the control variables are the only variables on which additional constraints specified by the controller can be specified. The disturbance variable e is a variable that should be allowed to assume any trajectory in the controlled system. In other words, in the *controlled behavior* the variable e is *free*. The variable w are variables that one actually wants to control. The performance specification is specified in terms of these variables. For example, in the stabilization problem one requires that these variables approach zero as time tends to infinity. Alternatively, in the pole placement problem, these variables are required to be linear combinations of only certain exponential trajectories. It is important to remember that we do not require that these sets of variables are disjoint.

In this paper we deal with the problem of finding conditions on a certain performance specification as to when a controller can achieve it without putting restrictions on the disturbance variables. The restriction on the plant behavior on the to-be-controlled variables is brought about by interconnecting the plant with the controller and we look into refinements on the interconnections that we allow. For example, we define directability of trajectories from the unconnected systems to the interconnected systems and this brings in the notion of compatibility of an interconnection. We finally address the issue of when a required controlled behavior can be obtained by a controller that is itself controllable, and moreover by a compatible interconnection.

In order to make the paper fairly self-contained we include the necessary definitions about the various interconnections in this paper. The rest of this section contains a few words about the notation that we use. Section 2 contains the problem formulation and the main results. Section 3 contains results related to compatibility of interconnections and disturbances being free. We next consider the problems of pole placement and stabilization, and we look into their relation with regular interconnection (with the plant) using a controller that is itself controllable (in section 4). We conclude this paper with some remarks in section 5.

The set of real numbers is denoted by $\mathbb{R}$, and the vector space of $\mathtt{v}-$tuples of real numbers is denoted by $\mathbb{R}^{\mathtt{v}}$. A behavior $\mathfrak{B}$ is a subset of $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{v}})$ the space of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$ that satisfy a set of linear constant coefficient differential equations of the kind $R(\frac{\mathrm{d}}{\mathrm{d}t})v=0$, i.e.

$$\mathfrak{B}=\{v\in\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{v}})\mid R(\tfrac{\mathrm{d}}{\mathrm{d}t})v=0\} \tag{1}$$

where $R(\xi)$ is a polynomial matrix with $\mathtt{v}$ columns and as many rows as the number of equations (say $\mathtt{g}$); $R\in\mathbb{R}^{\mathtt{g}\times\mathtt{v}}[\xi]$. Since we deal with several variables each with possibly

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different number of components, we use the same letter with a different font to denote the number of components, for example, $v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}})$, and $(e, w, c) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{e+w+c}})$. We shall be dealing with behaviors that can be described by linear constant coefficient ordinary differential equations, hence of the kind in equation (1). We denote such a behavior by $\mathfrak{B} \in \mathfrak{L}^{\mathtt{v}}$, $\mathtt{v}$ here denoting the number of components of a typical element in $\mathfrak{B}$. Further, a description of the behavior $\mathfrak{B}$ as in equation (1) is called a kernel representation.

II. Problem formulation and main results

Consider a plant behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e+w+c}}$ described by the kernel representation

$$F(\frac{d}{dt})e + R(\tfrac{\mathrm{d}}{\mathrm{d}t})w + M(\frac{d}{dt})c = 0. \qquad (2)$$

The signal e can be thought of as an external disturbance. A few words about the terminology. A given plant behavior is controlled by attaching a controller on the control variables. The behavior that results from this interconnection will be called *controlled behavior*, which is conventionally known as *closed loop behavior*. We denote this by $\mathfrak{B} \parallel \mathcal{C}$, and further, $\mathfrak{B} \parallel \mathcal{C} \in \mathfrak{L}^{\mathtt{e+w+c}}$. Formally,

$$\mathfrak{B} \parallel \mathcal{C} := \{(e, w, c) \in \mathfrak{B} \mid c \in \mathcal{C}\}.$$

When we say that a performance specification is given, it is the controlled behavior $\mathcal{K}$ of the $w-$trajectories that is given; $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Notice that in the full controlled behavior $\mathfrak{B} \parallel \mathcal{C}$, we are interested in just the to-be-controlled variables and this is the projection of the controlled behavior on the to-be-controlled variables. We call this *elimination* of the other variables from the full behavior. In this context we use the following notation. In addition to projection of a behavior, we often need to nullify certain variables (equate them to zero) in the behavior to obtain a new behavior as defined below.

Notation 1: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w+v}}$. We define $\mathfrak{B}_w \in \mathfrak{L}^{\mathtt{w}}$ as follows

$$\mathfrak{B}_w := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\ v \text{ such that } (w, v) \in \mathfrak{B}\}.$$

Further, we define $\mathcal{N}_w(\mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}$ as follows

$$\mathcal{N}_w(\mathfrak{B}) := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid (w, 0) \in \mathfrak{B}\}.$$

Thus $\mathcal{N}_w(\mathfrak{B})$ is the w-behavior obtained from $\mathfrak{B}$ by annulling all the variables in $\mathfrak{B}$ except the w-variable. Similarly, when $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e+w+c}}$, we often use $\mathcal{N}_e(\mathfrak{B})$ and $\mathfrak{B}_{ec}$ in the obvious way. In this notation, e being free in the controlled behavior is equivalent to $(\mathfrak{B} \parallel \mathcal{C})_e = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{e}})$.

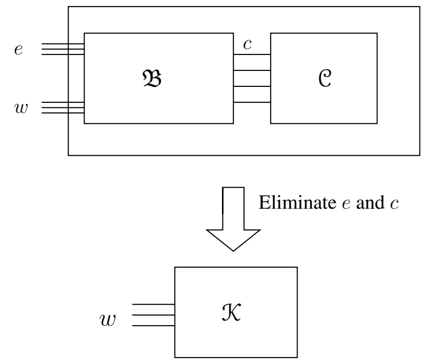

Fig. 1. The control interconnection diagram.

The second important issue in the main problems in this paper is that a controller is attached on just the control variables c while the controller attempts to meet a performance specification $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ on the w-trajectories. This brings in the notion of *implementability* of a given controlled behavior $\mathcal{K}$. A given controlled behavior $\mathcal{K}$ is called implementable if there exists a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that $(\mathfrak{B} \parallel \mathcal{C})_w = \mathcal{K}$. The above concepts have been dealt with in the literature and we refer to [10], [5], [1] for detailed expositions. We are now ready to formulate the main problem that is considered in this paper. Figure 1 shows the concerned behaviors and the interconnection of the plant and the controller.

Problem 2: Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e+w+c}}$. Find conditions on a given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that

1) $\mathcal{K}$ is implementable, i.e. there is a controller that implements this behavior by imposing additional laws on just the control variable c.
2) The controller puts no constraints on the disturbance e, i.e. e remains free after interconnection.

Before we state the main result, we provide some insight into the above problem using equations that the plant is described with. Equation (2) can be manipulated by row operations to obtain the following equivalent set of equations describing the full plant

$$\begin{bmatrix} F_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & M(\frac{\mathrm{d}}{\mathrm{d}t}) \\ F_2(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \\ 0 & R_3(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \end{bmatrix} \begin{bmatrix} e \\ w \\ c \end{bmatrix} = 0, \qquad (3)$$

where $R_3(\xi)$, $M(\xi)$ and $F_2(\xi)$ are polynomial matrices with full row rank.[1] The elimination theorem allows us to

[1]The full row rank assumption of these matrices does not limit generality because any system of equations (2) can be equivalently written to obtain equation (3) after elementary row operations and possibly after ignoring zero rows; this does not change the set of solutions to equation (2).

conclude that

$$\begin{bmatrix} F_2(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & R_3(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix} = 0, \tag{4}$$

is a kernel representation of $\mathfrak{B}_{ew}$. Suppose $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is described by the set of equations $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. Then, we are looking for a condition on K (and F_i, R_i and M) such that the behavior $\mathcal{K}$ can be implemented by some controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ and e is free in $\mathfrak{B} \parallel \mathcal{C}$. These conditions are stated in the theorem below. We need a behavior called the hidden behavior, which is defined as $\mathcal{N} := \mathcal{N}_{ew}(\mathfrak{B})$. Further, this is projected onto the w-trajectories, and we call this $\mathcal{G}$. Written explicitly, $\mathcal{G} := (\mathcal{N}_{ew}(\mathfrak{B}))_w$.

Theorem 3: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e+w+c}}$ and let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Suppose that $\mathfrak{B}$ is given by the kernel representation as in equation (3) (with F_2, R_3 and M of full row rank), and let $\mathcal{K}$ be described by $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. Then, $\mathcal{K}$ is implementable (i.e. there exists a controller, say, $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that results in this controlled behavior) and e is free in $\mathfrak{B} \parallel \mathcal{C}$, if and only if

(c1) $\mathcal{G} \subseteq \mathcal{K} \subseteq \mathfrak{B}_w$, and

(c2) $\operatorname{rank} \begin{bmatrix} R_2 \\ K \end{bmatrix} = \operatorname{rank} R_2 + \operatorname{rank} K.$

The above theorem provides a concrete way of checking whether or not a given performance specification meets our criteria (of implementability and of allowing disturbance variables to remain free). The explicit rank condition turns out to lead the way to another result which we state below. In this context we define a notion of regularity of an interconnection as introduced in [9]. This notion is closely related to that of *weak compatibility* of interconnections as studied in [6], [7], and more results on such interconnections follow in this paper.

Given a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{v}}$ we described above how certain variables are free. The maximum number of free variables in a behavior turns out to be independent of the particular choice of the variables. We call this maximum number the *input cardinality* of the behavior $\mathfrak{B}$, and denote it by $\mathtt{m}(\mathfrak{B})$. The number of remaining variables is defined as *output cardinality* of $\mathfrak{B}$ and is denoted by $\mathtt{p}(\mathfrak{B})$. Consider the kernel representation of $\mathfrak{B}$ in equation (1). Here $\mathtt{p}(\mathfrak{B})$ equals $\operatorname{rank}(R)$, the rank of the polynomial matrix $R(\xi)$. Regularity of an interconnection depends on whether the output cardinalities add up to the output cardinality of the interconnected system. For the behavior $\mathfrak{B} \parallel \mathcal{C} \in \mathfrak{L}^{\mathtt{e+w+c}}$ obtained by interconnection of the plant $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e+w+c}}$ and the controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, the interconnection of $\mathfrak{B}$ and the controller $\mathcal{C}$ is said to be regular if $\mathtt{p}(\mathfrak{B} \parallel \mathcal{C}) = \mathtt{p}(\mathfrak{B}) + \mathtt{p}(\mathcal{C})$. Regular interconnection was introduced in [9] and was studied further in [2], [1], [3] amongst others.

We now unravel the implications of e being free in $\mathfrak{B} \parallel \mathcal{C}$. A necessary condition for e to be free in $\mathfrak{B} \parallel \mathcal{C}$ is that e is free in $\mathfrak{B}$ and hence $\mathfrak{B}_{ew}$. Consider the kernel representation of $\mathfrak{B}_{ew}$ as in equation (4). Then e is free in $\mathfrak{B}_{ew}$ if and only if R_2 has full row rank. Notice that $\mathcal{G}$ defined before theorem 3 is contained in $\mathfrak{B}_w$. Further, $\mathcal{G}$ can be obtained from $\mathfrak{B}_w$ by regular interconnection of $\mathfrak{B}_w$ and a 'controller' that acts on the $w-$ variables, say, $\mathcal{C}^w$. Then, condition **c2** of theorem 3 is equivalent to the regularity of the interconnection of $\mathcal{C}^w$ and $\mathcal{K}$.

We have thus outlined the proof of the following reformulation of theorem 3.

Theorem 4: Consider a plant behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e+w+c}}$, and let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be a required performance specification. Then, $\mathcal{K}$ can be implemented by a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that e is free in $\mathfrak{B} \parallel \mathcal{C}$ if and only if

(c1) $\mathcal{G} \subseteq \mathcal{K} \subseteq \mathfrak{B}_w$,

(c2) there exists a controller $\mathcal{C}^w \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{G} = \mathfrak{B}_w \parallel \mathcal{C}^w$ and this interconnection is regular,

(c3) the interconnection of any controller $\mathcal{C}^w$ of **(c2)** and $\mathcal{K}$ is regular.

Notice the close relation between a controller leaving disturbances free and the regularity of the interconnection between this controller and a suitable behavior obtained from the plant. This is further explored in the following section within the concept of compatibility of interconnection.

III. Compatibility and directability

As remarked above, regularity of interconnections is related to *weak compatibility* of an interconnection. The exact relation is brought out in proposition 10. Since regularity and compatibility of an interconnection form an important part of this paper, we give a quick review of these concepts and their close relation to controllability of a behavior. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{v}}$ is called controllable if for any two trajectories $v_1, v_2 \in \mathfrak{B}$, there exist a $v \in \mathfrak{B}$ and a $T \in \mathbb{R}$ such that $v(t) = v_1(t)$ for $t \leqslant 0$ and $v(t) = v_2(t)$ for $t \geqslant T$. This motivates us to define that v_1 is *weakly directable* to v_2. In other words, a behavior $\mathfrak{B}$ is controllable if and only if any two trajectories in $\mathfrak{B}$ are weakly directable to each other. (The weakness is about the allowance of the time delay T to be different from 0.) We now come to the notion of weak compatibility of two behaviors. $\mathfrak{B}_1$ and $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{v}}$ are said to be weakly compatible if for every $v_i \in \mathfrak{B}_i$ ($i = 1, 2$), there exists a $v \in \mathfrak{B}_1 \cap \mathfrak{B}_2$ such that v_1 and v_2 are weakly directable to v, with possibly different required time delays T_i's (see [3] for a thorough analysis).

We now extend the definition of directability under the presence of disturbances. The presence of disturbances gives rise to *partial* directability as follows.

Definition 5 (Directability under disturbance): Consider a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e}+\mathtt{x}}$. The signal e is interpreted as a disturbance. For any $\tilde{e} \in \mathfrak{B}_e$, define

$$\mathfrak{B}^{\tilde{e}} := \{x \mid (\tilde{e}, x) \in \mathfrak{B}\}.$$

A trajectory $x_1 \in \mathfrak{B}_x$ is called *(weakly) directable* to $x_2 \in \mathfrak{B}_x$ in the presence of the disturbance e if for all $\tilde{e}$ such that $(\tilde{e}, x_1),\ (\tilde{e}, x_2) \in \mathfrak{B}$, we have that $(\tilde{e}, x_1)$ is (weakly) directable to $(\tilde{e}, x_2)$ in $\mathfrak{B}^{\tilde{e}}$, in the usual sense. We denote this fact as $x_1 D_{\mathfrak{B},e} x_2$ for strong compatibility and $x_1 D^*_{\mathfrak{B},e} x_2$ for weak compatibility.

Note that the above definition accommodates the usual sense of directability as a special case with $e = 0$. In several results below, certain properties are stated in terms of the behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{x}}$ that is obtained by setting the disturbance variable equal to zero in $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e}+\mathtt{x}}$. For $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e}+\mathtt{x}}$, recall that $\mathcal{N}_x(\mathfrak{B}) \in \mathfrak{L}^{\mathtt{x}}$ denotes the behavior of the x-trajectories when $e = 0$. We shall also use $\mathcal{M} \in \mathfrak{L}^{\mathtt{x}}$ to denote $\mathcal{N}_x(\mathfrak{B})$.

Lemma 6: Consider a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e}+\mathtt{x}}$. We have

$$\left(x_1 D^*_{\mathfrak{B},e} x_2\right) \Leftrightarrow (x_1 - x_2) \in (\mathcal{N}_x(\mathfrak{B}))^{\mathrm{ctr}},$$

where $(\mathcal{M})^{\mathrm{ctr}}$ is the controllable part of $\mathcal{M}$.

(The property of directability is linked to the controllable part of a behavior. The controllable part of $\mathfrak{B}$ is defined as the largest controllable behavior contained in $\mathfrak{B}$. See [1] for more properties of the controllable part of a behavior.) Loosely speaking, according to the above lemma, only the difference in two given trajectories needs to be directed (to zero), and hence their difference needs to be contained in the controllable part of the concerned behavior.

We proceed to define (weak) compatibility for the configuration shown in figure 1.

Definition 7 (Compatibility of the controller): Consider the configuration shown in the figure. The plant is subjected to the disturbance e. The interconnection is (weakly) compatible if for any $(w_b, c_b) \in \mathfrak{B}_{wc}$ and $c_c \in \mathcal{C}$ there exists a $c \in \mathfrak{B}_c \cap \mathcal{C}$ such that

- There exists a $w \in \mathfrak{B}_w$ such that $(w_b, c_b) D_{\mathfrak{B},e} (w, c)$ (respectively $(w_b, c_b) D^*_{\mathfrak{B},e} (w, c)$), and
- $c_c D_{\mathcal{C}} c$ (respectively $c_c D^*_{\mathcal{C}} c$).

Notice that for $\mathcal{C}$ we use the usual kind of directability (i.e. without the presence of disturbance). The fact that we assume $\mathcal{C}$ can be directed without the presence of any disturbance does not exclude the possibility of including some disturbances in the control variable of the plant (i.e. assuming them to be measurable). Rather, we assume that prior to the formation of the interconnection, the controller can be isolated from the disturbance. We do not make the same assumption on the plant, since the disturbance is assumed to be an inherent part of the plant. Hence, although the disturbance can appear in the control variable, there is an asymmetry in the role it plays in the plant and in the controller.

In the previous discussion we highlight the issue of strong and weak compatibility in the presence of disturbances. In the subsequent discussion, we will use only weak compatibility. Recall that for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e}+\mathtt{w}+\mathtt{c}}$, we have $\mathcal{M} := \mathcal{N}_e(\mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ obtained by setting $e = 0$.

Theorem 8: **(Geometric condition of weakly compatible controller)** Let a plant behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e}+\mathtt{w}+\mathtt{c}}$. A controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is weakly compatible with the plant if and only if

$$\mathfrak{B}_c \subseteq (\mathcal{M}_c)^{\mathrm{ctr}} + \mathcal{C}^{\mathrm{ctr}}, \tag{5}$$

$$\mathcal{C} \subseteq (\mathcal{M}_c)^{\mathrm{ctr}} + \mathcal{C}^{\mathrm{ctr}}. \tag{6}$$

As mentioned in Notation 1, the behavior $\mathcal{M}_c$ is obtained by eliminating w from $\mathcal{M}$, and $(\mathcal{M}_c)^{\mathrm{ctr}}$ is its controllable part.

Notice that (5) and (6) can be (equivalently) written as

$$\mathfrak{B}_c + \mathcal{C} \subseteq (\mathcal{M}_c)^{\mathrm{ctr}} + \mathcal{C}^{\mathrm{ctr}}.$$

We now come to another main result of this paper that essentially uses the previous results. This result characterizes the conditions on a given performance specification under which a weakly compatible controller can implement it. Since implementability of $\mathcal{K}$ is necessary we assume this and look for further conditions that are essential.

Theorem 9 (Weakly compatible achievability): Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{e}+\mathtt{w}+\mathtt{c}}$ be given. An implementable controlled behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is implementable by a weakly compatible controller, if and only if

$$\mathcal{K} + \mathcal{M}_w^{\mathrm{ctr}} = \mathfrak{B}_w. \tag{7}$$

The behavior $\mathcal{M}_w$ is obtained by eliminating c from $\mathcal{M}$, and $\mathcal{M}_w^{\mathrm{ctr}}$ is its controllable part.

IV. Pole placement and stabilization

In order to provide connections between weak compatibility, regular implementability and more familiar control problems like stabilization and pole placement, we recapitulate a few results about pole placement and stabilization. We then move on to the question as to which controlled behaviors can be obtained by using a *controllable* controller.

For simplicity we deal with the case that the control variable c is exactly the same as the to-be-controlled variable w and there are no disturbances. This makes our plant have just one kind of variable w, and the controller acts on w directly. The following proposition establishes the close link between weak compatibility and regular implementability.

Proposition 10: **:** Consider a plant $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be a required controlled behavior. Then the following statements are equivalent.

1) there exists a controller $\mathcal{C}' \in \mathfrak{L}^{\mathtt{w}}$ that implements $\mathcal{K}$ by a weakly compatible interconnection
2) there exists a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ that implements $\mathcal{K}$ by regular interconnection
3) $\mathcal{K} + \mathfrak{B}^{\text{ctr}} = \mathfrak{B}$.

Notice how three seemingly different concepts get interlinked:

- the directability properties of $\mathfrak{B}$ (with respect to a suitable controller) into a given behavior $\mathcal{K}$;
- the output cardinalities of $\mathcal{K}$ and $\mathfrak{B}$ and the existence of a controller (possibly different than the one above) with the right output cardinality;
- the sets $\mathcal{K}$ and $\mathfrak{B}$ 'differing' by at most the controllable part of $\mathfrak{B}$.

We now proceed to state the first of the main results of this section: one which characterizes conditions on a plant $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ under which given any set of 'poles', there exists a controllable controller that implements a controlled autonomous behavior with precisely this set of poles. A behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is said to be autonomous if its input cardinality is zero. Such a behavior is a finite dimensional subspace of $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ and any element of $\mathcal{K}$ is a linear combination of (vectors of) exponential trajectories. The exponents of the exponential trajectories is defined as the poles of $\mathcal{K}$. (If the poles are complex, they ought to occur in conjugate pairs.) If all the poles of an autonomous behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ are in the open left half complex plane, then the behavior is stable, i.e. all trajectories $w(t) \in \mathcal{K}$ satisfy $w(t) \rightarrow 0$ as $t \rightarrow \infty$. The monic polynomial whose roots are precisely the poles of an autonomous behavior $\mathcal{K}$ (counted with multiplicity) is called the characteristic polynomial of $\mathcal{K}$. Hence to specify the required set of poles for an autonomous behavior, one only needs to specify the corresponding monic polynomial $r \in \mathbb{R}[\xi]$ which has this set of poles as its roots. A detailed exposition of this can be found in [5]. The following theorem gives conditions on the plant under which the pole placement problem is solvable using a controllable controller. We will call a plant behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ *nontrivial* if it is neither the behavior with just the zero element nor the complete space $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

Theorem 11: **:** Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. For each monic $r \in \mathbb{R}[\xi]$, there exists a controllable controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ that implements a $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ by regular interconnection with $\mathfrak{B}$ such that $\mathcal{K}$ has r as its characteristic polynomial if and only if $\mathfrak{B}$ is controllable and nontrivial.

We now proceed to consider the stabilization problem. In this context we need the definition of stabilizability of a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be stabilizable if for any $w_1 \in \mathfrak{B}$, there exists a $w \in \mathfrak{B}$ such that $w_1(t) = w(t)$ for $t \leqslant 0$ and w satisfies $w(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence, any trajectory in a stabilizable behavior $\mathfrak{B}$ can be patched to some stable trajectory $w \in \mathfrak{B}$. Using this concept we address the problem of stabilizing a plant using a controllable controller which, moreover, achieves stability by regular interconnection.

Theorem 12: **:** Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. There exists a controllable controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ that implements a stable $\mathcal{K}$ by regular interconnection with $\mathfrak{B}$ if and only if $\mathfrak{B}$ is stabilizable.

We briefly remark the main advantage of imposing the controllability property on the controller. The controller's controllability assures that every trajectory in the controller behavior is directable to any trajectory that is allowed after interconnection (with the plant). A second aspect is that the requirement of controllability on the controller does not limit the conditions on the plant under which pole placement and/or stabilization problems are solvable (see [2]).

V. Concluding remarks

We showed various results pertaining to interconnection of a plant with a controller to obtain a given performance specification under the important constraint that the controller should not impose restrictions on the disturbances affecting the plant. After exploring the property in a performance specification (given controlled behavior $\mathcal{K}$) that allows us to obtain without restricting the disturbances, we related this property to regularity of interconnection of certain behaviors intrinsic to the plant and the controlled behavior $\mathcal{K}$.

The concept of weak compatibility of an interconnection was extended for the case that disturbances affect the plant. Finally, we also related the solvability of two important problems, namely, pole placement and stabilization of a plant with the controllability and stabilizability of the plant, respectively. This equivalence holds when the control has to be achieved using a weakly compatible or regular interconnection. (That controllability is not essential for pole placement if the interconnection did not need to be regular was established in [9].) Imposing the controller to be controllable turns out to not put any further restrictions on the plant in the case of either pole placement or stabilization.

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