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Dissipative Systems Synthesis: a Linear Algebraic Approach

Madhu N. Belur, Harish K. Pillai and H.L. Trentelman

Abstract— In this paper we consider the problem of synthesis of dissipative systems for the case that first and higher order derivatives of the concerned variables also appear in the weighting function. The problem is formulated and solved using the behavioral approach to systems and control. It turns out that this problem can be reformulated analogously as a problem of finding a non-negative subspace (non-negative with respect to a given indefinite constant symmetric matrix) within a finite dimensional vector space satisfying certain inclusion and dimensionality constraints.

Keywords: Dissipative systems, behaviors, indefinite weighting functional, non-negative subspace

I. INTRODUCTION AND PRELIMINARIES

In this paper we consider the problem of synthesis of a dissipative dynamical system (henceforth called Dissipativity Synthesis Problem (DSP)) using a behavioral approach along the lines of papers [5], [7]. Further, we reformulate this into an analogous problem as that of finding a non-negative subspace within a given subspace of a finite dimensional vector space and under certain dimensionality constraints. The behavioral approach of formulating the DSP plays a crucial role in this linear algebraic analogy, and we outline a proof of the above problem of finding a non-negative subspace with respect to a given indefinite symmetric constant matrix. (We restrict ourselves to only an outline of the proof together with some auxiliary intermediate results due to space constraints.)

The paper is structured as follows. The notation and other basic definitions form the remainder of this section. The next section (section II) contains the definition of dissipativity and other concepts that are essential for the formulation of the DSP. Section III contains the main result and the concepts necessary to state this main result. We then move on to section IV to study the analogous problem concerning subspaces within a finite dimensional vector space. As mentioned above, the DSP has a parallel problem in this context and this problem can be of interest in its own right.

M. N. Belur and H.K. Pillai are with the Department of Electrical Engineering, Indian Institute of Technology Bombay, Powai, Mumbai 400 076, India. Email: {Belur,Hp}@ee.iitb.ac.in

H.L. Trentelman is with the Institute for Mathematics and Computing Science, University of Groningen, P.O.Box 800, 9700 AV, Groningen, The Netherlands. Email: H.L.Trentelman@math.rug.nl

An outline of the proof of the linear algebraic formulation of the DSP is covered in explained in section V. A few remarks about this paper are finally summarized in section VI.

The notation we use is standard. $\mathbb{R}$ stands for the field of real numbers and $\mathbb{R}^n$ for the n-dimensional real vector space. $\mathbb{R}[\xi]$ is the ring of polynomials in one indeterminate, ξ, with real coefficients. We also consider polynomial matrices in one and two indeterminates: $\mathbb{R}^{n \times m}[\xi]$ and $\mathbb{R}^{n \times m}[\zeta, \eta]$ are the sets of polynomial matrices in the corresponding indeterminates, each matrix having n rows and m columns. We use $\bullet$ when it is unnecessary to specify the number of rows, for example, $\mathbb{R}^{\bullet \times m}[\xi]$. $\mathbb{Z}_+$ stands for the set of non-negative integers.

In order to keep track of the number of components in a vector v, we use the same variable $\mathtt{v}$ (in a different font) to indicate the dimension. Let $v \in \mathbb{R}^{\mathtt{v}}$, and let $\Sigma \in \mathbb{R}^{\mathtt{v} \times \mathtt{v}}$ be a symmetric matrix. Then $|v|_\Sigma^2$ denotes $v^T \Sigma v$, and when $\Sigma = I$, the identity matrix, we skip the I in $|v|_I^2$.

II. BEHAVIORS AND DISSIPATIVITY

A linear differential controllable behavior $\mathcal{B}$ is the set of those trajectories $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ that are in the image of some matrix differential operator $M(\frac{d}{dt})$, where $M(\xi)$ is a polynomial matrix with $\mathtt{w}$ rows. More precisely,

$$\mathcal{B} = \{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \quad \text{there exists } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\ell) \\ \text{such that } w = M(\tfrac{d}{dt})\ell \}. \tag{1}$$

The set of such controllable behaviors with $\mathtt{w}$ components is denoted by $\mathfrak{L}_{\text{cont}}^{\mathtt{w}}$. For the purpose of this paper, the easiest way to define the input cardinality of a behavior $\mathcal{B} \in \mathfrak{L}_{\text{cont}}^{\mathtt{w}}$ is the rank of the polynomial matrix $M(\xi)$ in the above equation. We denote the input cardinality of $\mathcal{B}$ by $\mathtt{m}(\mathcal{B})$. We refer the reader to [3] for a good exposition on the behavioral approach to systems and control.

We also deal with bilinear and quadratic forms on the elements of a behavior. In this context we deal with polynomial matrices in two variables. Induced by $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$, we have the *bilinear differential form* $L_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined as follows. Let $\Phi(\zeta, \eta)$ be written as a (finite) sum $\Phi(\zeta, \eta) = \Sigma_{k,\ell \in \mathbb{Z}_+} \Phi_{k\ell} \zeta^k \eta^\ell$ with

$\Phi_{k\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$. Define $L_\Phi(w, v)$ by

$$L_\Phi(w, v) := \sum_{k,\ell \in \mathbb{Z}_+} \left(\tfrac{\mathrm{d}^k}{\mathrm{d}t^k} w\right)^T \Phi_{k\ell} \left(\tfrac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} v\right).$$

Using the bilinear differential form L_Φ induced by $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$, we define the quadratic differential form $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ by

$$Q_\Phi(w) := L_\Phi(w, w) = \sum_{k,\ell \in \mathbb{Z}_+} \left(\tfrac{\mathrm{d}^k}{\mathrm{d}t^k} w\right)^T \Phi_{k\ell} \left(\tfrac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} w\right).$$

In this paper we also consider integrals of quadratic differential forms. Given $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$, we call $\mathcal{B} \in \mathfrak{L}_{\mathrm{cont}}^{\mathtt{w}}$ Φ-*dissipative* if $\int_{\mathbb{R}} Q_\Phi(w) \, \mathrm{d}t \geqslant 0$ for all $w \in \mathcal{B} \cap \mathfrak{D}$. ($\mathcal{B} \cap \mathfrak{D}$ is the subspace of those trajectories in $\mathcal{B}$ which are compactly supported.) In this context of dissipativity, Φ is said to induce the *supply rate* Q_Φ.

$\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ is called symmetric if $\Phi(\zeta, \eta) = \Phi^T(\eta, \zeta)$. Notice that when considering the quadratic differential form induced by Φ, we can assume that Φ is symmetric without loss of generality.

Let $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. Define $\partial \Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ by $\partial \Phi(\xi) = \Phi(-\xi, \xi)$. It is easy to see that if Φ is symmetric then the the complex matrix $\partial \Phi(i\omega)$ is Hermitian for all $\omega \in \mathbb{R}$.

In this paper we consider the problem of synthesis of dissipative behaviors. In this context we require the relation between the input cardinality of the behavior and the *signature* of the polynomial matrix that induces the supply rate. (The signature of a nonsingular symmetric constant matrix M, denoted by $\mathrm{sign}(M)$, is defined as $(\sigma_-(M), \sigma_+(M))$, where $\sigma_-(M)$ is the number of negative eigenvalues of M, and $\sigma_+(M)$ is the number of positive eigenvalues of M.) In order to make an analogous definition for the signature of a two variable polynomial matrix $\Phi(\zeta, \eta)$, we make certain assumptions on Φ, and these assumptions remain for the rest of this paper.

Assumption 1: : Let $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ be symmetric. Assume that $\partial \Phi$ is nonsingular and that $\partial \Phi$ admits a J-spectral factorization, i.e. $\partial \Phi(\xi) = F^T(-\xi)JF(\xi)$ for some $F \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ and $J \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ of the form

$$J = \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix}.$$

Under these assumptions, we define $(\sigma_-(\partial \Phi), \sigma_+(\partial \Phi)) = \mathrm{sign}(\partial \Phi) := \mathrm{sign}(J)$.
It is well known that J-spectral factorizability of $\partial \Phi$ is equivalent to $\partial \Phi(i\omega)$ having constant signature for almost all $\omega \in \mathbb{R}$ (see [4]). We are now ready to state the dissipativity synthesis problem (DSP).

Dissipativity synthesis problem (DSP): Assume $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ satisfies assumption 1 and let $\mathcal{N}, \mathcal{P} \in \mathfrak{L}_{\mathrm{cont}}^{\mathtt{w}}$ be two given controllable behaviors satisfying $\mathcal{N} \subseteq \mathcal{P}$. Find conditions under which there exists a behavior $\mathcal{K} \in \mathfrak{L}_{\mathrm{cont}}^{\mathtt{w}}$ satisfying

1) $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$,
2) $\mathcal{K}$ is Φ-dissipative, and
3) $\mathtt{m}(\mathcal{K}) = \sigma_+(\partial \Phi)$.

$\mathcal{K}$ is called the controlled behavior, while $\mathcal{N}$ and $\mathcal{P}$ are called the hidden and the plant behaviors respectively. Each of the three conditions above have important implications in systems theory, and more on this can be found in [7]. In [7], however, the problem was solved there only for the case that Φ is a constant matrix. (This means that the supply rate does not depend on *derivatives* of the concerned variables.) Another important difference between the above problem and the one studied in [7] is that the dissipativity there was required to hold on the *half-line*; in this paper we have relaxed this requirement and this relaxation makes the problem simpler to some extent.

III. MAIN RESULTS

Before we provide necessary and and sufficient conditions for the existence of a $\mathcal{K}$ satisfying the requirements of the DSP, we state and prove the following lemma which shows how the input cardinality condition in the DSP is a maximality requirement.

Lemma 2: : Assume $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ satisfies assumption 1. Let $\mathcal{K} \in \mathfrak{L}_{\mathrm{cont}}^{\mathtt{w}}$ be Φ-dissipative. Then $\mathtt{m}(\mathcal{K}) \leqslant \sigma_+(\partial \Phi)$. The above lemma is an analogue of proposition 2 of [7] which had a similar result for the case that Φ is a constant. We now need the notion of orthogonal complement of a behavior. Consider a behavior $\mathcal{B} \in \mathfrak{L}_{\mathrm{cont}}^{\mathtt{w}}$, the orthogonal complement $\mathcal{B}^\perp$ of $\mathcal{B}$ is defined as

$$\mathcal{B}^\perp := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \int_{\mathbb{R}} w^T v \, \mathrm{d}t = 0 \text{ for all } v \in \mathcal{B} \cap \mathfrak{D}\}$$

where $\mathfrak{D}$ is the subspace of compactly supported trajectories in $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. The orthogonal complement $\mathcal{B}^\perp$ of a controllable behavior $\mathcal{B}$ turns out to be a controllable behavior too. These facts, together with other relations about orthogonality and their proofs, can be found in [1]. We also require the notion of orthogonal complement with respect to a nonsingular $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. The Φ-orthogonal complement $\mathcal{B}^{\perp_\Phi}$ of $\mathcal{B}$ is defined as $(\partial \Phi(\tfrac{\mathrm{d}}{\mathrm{d}t})\mathcal{B})^\perp$. One can show that $\mathcal{B}^{\perp_\Phi}$ is the largest controllable behavior such that

$$\int_{\mathbb{R}} L_\Phi(w, v) \, \mathrm{d}t = 0 \text{ for all } w \in \mathcal{B} \text{ and } v \in \mathcal{B}^{\perp_\Phi} \cap \mathfrak{D}.$$

Thus $\mathcal{B}^{\perp}$ is nothing but $\mathcal{B}^{\perp_I}$, the orthogonal complement of $\mathcal{B}$ with respect to I, the identity matrix. Using this notion of the Φ-orthogonal complement of a behavior we are ready to state the main result of this paper: necessary and sufficient conditions for the solvability of the DSP.

Theorem 3: **:** Suppose $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ satisfies assumption 1 and let $\mathcal{N}, \mathcal{P} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ with $\mathcal{N} \subseteq \mathcal{P}$. There exists a behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ satisfying

1) $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$,
2) $\mathcal{K}$ is Φ-dissipative, and
3) $\mathtt{m}(\mathcal{K}) = \sigma_+(\partial\Phi)$.

if and only if

1) $\mathcal{N}$ is Φ-dissipative, and
2) $\mathcal{P}^{\perp_\Phi}$ is $(-\Phi)$-dissipative.

Notice the similarity in the conditions for the solvability of the DSP to those in the main result of [7] (theorem 5). The conditions are similar except for the absence of a third condition that suitably couples the dissipativities of $\mathcal{N}$ and $\mathcal{P}^{\perp_\Phi}$. This coupling condition was an outcome of the half-line dissipativity requirement on $\mathcal{K}$. An important difference in our paper is that the proof does not resort to any state-space representations of the various behaviors. Moreover, the result above is more general in the sense that derivatives of the to-be-controlled variables w are allowed to occur in the weighting functional and this accomodates the weighted $\mathcal{L}_\infty$-control problem as a special case (see [2]). The proof of the above result will be done for an analogous problem that we consider in the following section.

IV. A LINEAR ALGEBRAIC FORMULATION OF DSP

In this section we deal with a problem concerning only finite dimensional spaces (unlike the previous sections where the behaviors could be infinite dimensional subspaces of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$). Consider a real vector space $\mathbb{V}$ of dimension, say, $\mathtt{v}$ and a symmetric nonsingular matrix $\Sigma \in \mathbb{R}^{\mathtt{v} \times \mathtt{v}}$. A subspace $\mathbb{B}$ is said to be Σ-positive if $|v|^2_\Sigma > 0$ for all nonzero $v \in \mathbb{B}$, and $\mathbb{B}$ is called Σ-neutral if $|v|^2_\Sigma = 0$ for all $v \in \mathbb{B}$. Similarly, we have the obvious definitions for Σ-non-negativity, Σ-nonpositivity and Σ-negativity of a subspace.

We denote the dimension of a subspace $\mathbb{B}$ by $\mathtt{m}(\mathbb{B})$. The reason behind the choice of the notation for the dimension of a subspace $\mathbb{B}$, notwithstanding the obvious confusion with the input cardinality $\mathtt{m}(\mathcal{B})$ of a behavior $\mathcal{B}$, becomes clear after the following easy result whose proof is straightforward, and can be easily found. $\sigma_+(\Sigma)$ below stands for the number of positive eigenvalues of Σ and this definition of σ_+ is a special case of the definition in assumption 1.

Proposition 4: **:** Assume $\Sigma \in \mathbb{R}^{\mathtt{v} \times \mathtt{v}}$ is symmetric and nonsingular. Let $\mathbb{B}$ be a subspace of $\mathbb{R}^{\mathtt{v}}$. If $\mathbb{B}$ is Σ-non-negative, then $\mathtt{m}(\mathbb{B}) \leqslant \sigma_+(\Sigma)$.

Remark 5: **:** The similarities between the property of non-negativity of a subspace $\mathbb{B}$ of $\mathbb{V}$ and dissipativity of a behavior $\mathcal{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, and the related analogy between the input cardinality condition (lemma 2) and the dimension condition (proposition 4) become more obvious when we consider the following argument. The number of elements in any basis for $\mathbb{B}$ is the dimension of $\mathbb{B}$, $\mathtt{m}(\mathbb{B})$. Analogously, consider equation (1), and interpret the columns of the matrix $M(\frac{\mathrm{d}}{\mathrm{d}t})$ as 'generating' the behavior $\mathcal{B}$. The number of 'independent' columns of $M(\xi)$ is the input cardinality of $\mathcal{B}$, $\mathtt{m}(\mathcal{B})$. The independence here is over the field of rational functions $\mathbb{R}(\xi)$.

Keeping this analogy in mind, we define in this section the related notions like orthogonality for finite dimensional subspaces. We now formulate the problem analogous to the DSP. We call this new problem DSP2.

(DSP2): Let $\Sigma \in \mathbb{R}^{\mathtt{v} \times \mathtt{v}}$ be symmetric and nonsingular satisfying $\Sigma = \Sigma^T$ and assume $\mathbb{N}, \mathbb{P}$ are subspaces of $\mathbb{V}$ such that $\mathbb{N} \subseteq \mathbb{P}$. Find conditions under which there exists a subspace $\mathbb{K}$ of $\mathbb{V}$ satisfying

1) $\mathbb{N} \subseteq \mathbb{K} \subseteq \mathbb{P}$,
2) $\mathbb{K}$ is Σ-non-negative, and
3) $\mathtt{m}(\mathbb{K}) = \sigma_+(\Sigma)$.

Let $\Sigma \in \mathbb{R}^{\mathtt{v} \times \mathtt{v}}$ be symmetric and nonsingular. (This is a standing assumption throughout this paper, and is the assumption analogous to assumption 1.) Subspaces $\mathbb{B}_1, \mathbb{B}_2 \subseteq \mathbb{V}$ are called orthogonal with respect to Σ (or Σ-orthogonal) if $v_1^T \Sigma v_2 = 0$ for all $(v_1, v_2) \in \mathbb{B}_1 \times \mathbb{B}_2$. Given a subspace $\mathbb{B}$, we define the Σ-orthogonal complement $\mathbb{B}^{\perp_\Sigma}$ as follows

$$\mathbb{B}^{\perp_\Sigma} := \{v \in \mathbb{V} \mid v^T \Sigma w = 0 \text{ for all } w \in \mathbb{B}\}.$$

Obviously, $\mathbb{B}^{\perp_\Sigma}$ is also a subspace of $\mathbb{V}$. When the orthogonal complement is taken with respect to $\Sigma = I$, the identity matrix, then we skip the Σ and write simply $\mathbb{B}^{\perp}$ to denote the orthogonal complement. Notice that $\mathbb{B}^{\perp_\Sigma} = (\Sigma\mathbb{B})^{\perp} = \Sigma^{-1}\mathbb{B}^{\perp}$ and that $\mathbb{B}^{\perp_\Sigma}$ is Σ-non-negative (positive) if and only if $\mathbb{B}^{\perp}$ is Σ^{-1} non-negative (positive, respectively). Moreover, $(\mathbb{B}^{\perp_\Sigma})^{\perp_\Sigma} = \mathbb{B}$. We have the following analogous and main result.

Theorem 6: **:** $\mathbb{K}$ satisfying the requirements of the DSP2 above exists if and only if the following two conditions are satisfied:

1) $\mathbb{N}$ is Σ-non-negative, and

2) $\mathbb{P}^{\perp_\Sigma}$ is Σ-nonpositive.

Remark 7: **:** The motivation behind studying the above linear algebra problem is clear from the new problem formulation and its solution. In the rest of this paper we outline a proof of the above result. The proof of DSP1 and DSP2 are the same and the methods differ only to the extent as pointed out in the above remarks and what is explained below.

The rest of this section consists of some properties of $\mathbb{B}$ and its Σ-orthogonal complement, which is necessary in the proof of the main result. We mention briefly about the similarity between

1. Subspace $\mathbb{B} \subseteq \mathbb{V}$ and its non-negativity with respect to a symmetric nonsingular $\Sigma \in \mathbb{R}^{v \times v}$, and

2. A controllable behavior $\mathcal{B} \in \mathfrak{L}_{\mathrm{cont}}^{w}$ ($\mathcal{B} \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$) and its dissipativity with respect to a $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$ that satisfies assumption 1.

The intersection and sum of two finite dimensional subspaces $\mathbb{B}_1, \mathbb{B}_2 \subseteq \mathbb{V}$ are subspaces and the concerned dimensions satisfy

$$\mathfrak{m}(\mathbb{B}_1 + \mathbb{B}_2) + \mathfrak{m}(\mathbb{B}_1 \cap \mathbb{B}_2) = \mathfrak{m}(\mathbb{B}_1) + \mathfrak{m}(\mathbb{B}_2).$$

For the case of behaviors, the intersection of two controllable behaviors, $\mathcal{B}_1$ and $\mathcal{B}_2$, is a behavior, however, it may not be controllable. With the suitable generalization of the definition of input cardinality for behaviors that are not controllable (see [1]), we have

$$\mathfrak{m}(\mathcal{B}_1 + \mathcal{B}_2) + \mathfrak{m}(\mathcal{B}_1 \cap \mathcal{B}_2) = \mathfrak{m}(\mathcal{B}_1) + \mathfrak{m}(\mathcal{B}_2).$$

Moreover, as far as the proof of the main result (theorems 3/6) goes, when we encounter an uncontrollable behavior $\mathcal{B}$ (due to intersection, for example), we continue with the 'controllable part' of $\mathcal{B}$, which is defined as the largest controllable behavior contained in $\mathcal{B}$, and is denoted by $\mathcal{B}_{\mathrm{cont}}$. Moreover, $\mathcal{B}$ and $\mathcal{B}_{\mathrm{cont}}$ have the same input cardinality. Detailed exposition on this together with proofs about these claims can be found in [1, chapter 2].

Having noted the important similarities and differences in the two dissipativity synthesis problems, we move on to state explicitly the results only for the case of finite dimensional subspaces, the analogous behavioral ones being true too. The following relation is easily verified.

$$(\mathbb{B} \cap \mathbb{B}^{\perp_\Sigma})^{\perp_\Sigma} = (\mathbb{B}^{\perp_\Sigma}) + (\mathbb{B}^{\perp_\Sigma})^{\perp_\Sigma} = \mathbb{B} + \mathbb{B}^{\perp_\Sigma} \quad (2)$$

The equation above implies that $(\mathbb{B} \cap \mathbb{B}^{\perp_\Sigma}) \perp_\Sigma (\mathbb{B} + \mathbb{B}^{\perp_\Sigma})$. A second fact that is easily proved is $\mathfrak{m}(\mathbb{B}^{\perp_\Sigma}) + \mathfrak{m}(\mathbb{B}) = v$.

Let $\mathbb{B}$ be a Σ-non-negative subspace. Define $\mathbb{B}_L$ as the set of all elements in $\mathbb{B}$ that are Σ-neutral. The following lemma

brings out a few properties about $\mathbb{B}_L$ that are essential for the proof of the main result of this paper.

Lemma 8: **:** Let $\Sigma \in \mathbb{R}^{v \times v}$ and let $\mathbb{B}$ be a Σ-non-negative subspace of $\mathbb{V}$. Then,

1) $\mathbb{B}_L = \mathbb{B} \cap \mathbb{B}^{\perp_\Sigma}$,
2) $\mathbb{B}_L = 0 \Leftrightarrow \mathbb{B}$ is Σ-positive,
3) $\Sigma\mathbb{B}_L \oplus (\mathbb{B} + \mathbb{B}^{\perp_\Sigma}) = \mathbb{V}$,
4) $\Sigma\mathbb{B}_L \subseteq \mathbb{B}^\perp$, and
5) $\Sigma\mathbb{B}_L$ is Σ^{-1} neutral.

In this context, notice that once we have $\mathbb{B}_L$, the Σ-neutral part of $\mathbb{B}$ (supposing $\mathbb{B}$ is Σ non-negative), there exists a (possibly non-unique) subspace $\mathbb{B}_+$ within $\mathbb{B}$ such that $\mathbb{B}_L \oplus \mathbb{B}_+ = \mathbb{B}$. Clearly, $\mathbb{B}_+$ is Σ-positive. In this paper, we often need to construct $\mathbb{B}_+$ explicitly; we define $\mathbb{B}_+ := \mathbb{B}_L^\perp \cap \mathbb{B}$.

Further, an important fact is that $\mathbb{B}_L \oplus \Sigma\mathbb{B}_L$ is Σ-indefinite, except under trivial conditions. Addressing this issue turns out to be central in proving the main result. Another issue that turns out to make the proof complicated is as follows. Suppose $\mathbb{N}$ is Σ-non-negative. $\mathbb{N} \subseteq \mathbb{P}$ implies that $\mathbb{N}_L \subseteq \mathbb{P}$. However, in general $\Sigma\mathbb{N}_L \not\subseteq \mathbb{P}$ and this makes it necessary to decompose $\Sigma\mathbb{N}_L$ into the part contained in $\mathbb{P}$ and a complement (defined as $\mathbb{N}_1$ and $\mathbb{N}_2$, respectively; see the table of definitions of all these subspaces). Exploring carefully the interlinking of the spaces related to $\mathbb{N}$ and $\mathbb{P}$ is essential for the proof.

In this context, we need the following result concerning the interplay of two given subspaces and their orthogonal complements. We use this result extensively in the proof of the main result.

Lemma 9: **:** Suppose A and B are subspaces of $\mathbb{V}$. Then, $(A \cap B)^\perp \cap B = (A + B) \cap A^\perp \subseteq A^\perp$.

V. PROOF OF THEOREM 6: AN OUTLINE

This section contains the proof of theorem 6, and this proof requires certain auxiliary results that we formulate and prove when we need them below.

Proof of 'only if part' of theorem 6: Suppose there exists $\mathbb{K}$ satisfying the conditions of the DSP2. Since $\mathbb{N} \subseteq \mathbb{K}$, we have that $\mathbb{N}$ is Σ-non-negative. Further, using $\mathfrak{m}(\mathbb{K}) = \sigma_+(\Sigma)$ and lemma A-3 from [6] (the statement of the lemma is reproduced below as proposition 10), we have $\mathbb{K}^{\perp_\Sigma}$ is Σ-nonpositive. $\mathbb{K} \subseteq \mathbb{P}$ results in $\mathbb{P}^{\perp_\Sigma} \subseteq \mathbb{K}^{\perp_\Sigma}$ and this means that $\mathbb{P}^{\perp_\Sigma}$ is also Σ-nonpositive. The following proposition is rephrased in the notation of this paper.

Proposition 10: **:** Let $\mathbb{L}$ be a linear subspace of $\mathbb{R}^n$. Consider the quadratic form $x^T Q x$ on $\mathbb{R}^n$ with $Q = Q^T$

nonsingular. Assume that $\sigma_+(Q) = \mathfrak{m}(\mathbb{L})$. Then

- $a^T Q a > 0$ for all nonzero $a \in \mathbb{L}$ if and only if $b^T Q^{-1} b < 0$ for all nonzero $b \in \mathbb{L}^\perp$, and

- $a^T Q a \geqslant 0$ for all $a \in \mathbb{L}$ if and only if $b^T Q^{-1} b \leqslant 0$ for all $b \in \mathbb{L}^\perp$.

Proof of 'if part' of theorem 6: Let $\mathbb{N}$ and $\mathbb{P}$ be subspaces of $\mathbb{V}$ satisfying $\mathbb{N} \subseteq \mathbb{P}$. and let $\mathbb{N}$ be Σ-non-negative and let $\mathbb{P}^{\perp\Sigma}$ be Σ-nonpositive. Define $\mathbb{N}_L := \mathbb{N} \cap \mathbb{N}^{\perp\Sigma}$ and $\mathbb{P}_L := \mathbb{P} \cap \mathbb{P}^{\perp\Sigma}$. Then, using statement 3 of lemma 8, we have $(\mathbb{N} + \mathbb{N}^{\perp\Sigma}) \oplus \Sigma\mathbb{N}_L = \mathbb{V}$ and $(\mathbb{P} + \mathbb{P}^{\perp\Sigma}) \oplus \Sigma\mathbb{P}_L = \mathbb{V}$. We now study some properties interlacing these behaviors. The following lemma is one such property.

Lemma 11: : $\Sigma\mathbb{N}_L \cap \mathbb{P}_L = 0$.

We now continue with the proof of theorem 6. Let $\mathbb{L}_{\mathbb{NP}} := \mathbb{N}_L \cap \mathbb{P}_L$. Notice that $\mathbb{P} + \mathbb{N}^{\perp\Sigma} = (\mathbb{L}_{\mathbb{NP}})^{\perp\Sigma}$. This is true because $\mathbb{P} + \mathbb{P}^{\perp\Sigma} = (\mathbb{P}_L)^{\perp\Sigma}$ and $\mathbb{N} + \mathbb{N}^{\perp\Sigma} = (\mathbb{N}_L)^{\perp\Sigma}$, and hence

$$\mathbb{P} + \mathbb{P}^{\perp\Sigma} + \mathbb{N} + \mathbb{N}^{\perp\Sigma} = \mathbb{P}_L^{\perp\Sigma} + \mathbb{N}_L^{\perp\Sigma}$$
$$= (\mathbb{P}_L \cap \mathbb{N}_L)^{\perp\Sigma} = (\mathbb{L}_{\mathbb{NP}})^{\perp\Sigma}.$$

Now, since $\mathbb{N} \subseteq \mathbb{P}$, we have $\mathbb{P}^{\perp\Sigma} \subseteq \mathbb{N}^{\perp\Sigma}$, and this simplifies the left-hand-side above to give $\mathbb{P} + \mathbb{N}^{\perp\Sigma} = (\mathbb{L}_{\mathbb{NP}})^{\perp\Sigma}$.

A similar argument using lemma 8 (statement 3) results in $(\mathbb{P} + \mathbb{N}^{\perp\Sigma}) \oplus (\Sigma\mathbb{L}_{\mathbb{NP}}) = \mathbb{V}$. The idea behind the rest of the proof is to obtain a direct sum decomposition of $\mathbb{P} + \mathbb{N}^{\perp\Sigma}$, and in turn of $\mathbb{V}$, and then to carefully choose the right subspaces to construct $\mathbb{K}$.

Define $\mathbb{N}_{L_r}$ to be the subspace defined by $\mathbb{N}_{L_r} := \mathbb{N}_L \cap (\mathbb{L}_{\mathbb{NP}})^\perp$. We thus have $\mathbb{N}_L = \mathbb{L}_{\mathbb{NP}} \oplus \mathbb{N}_{L_r}$. (In other words, $\mathbb{N}_{L_r}$ complements $\mathbb{L}_{\mathbb{NP}}$ in $\mathbb{N}_L$.) Similarly, define $\mathbb{P}_{L_r} := \mathbb{P}_L \cap (\mathbb{L}_{\mathbb{NP}})^\perp$. We correspondingly have $\mathbb{P}_L = \mathbb{L}_{\mathbb{NP}} \oplus \mathbb{P}_{L_r}$.

$\mathbb{N}_d := \mathbb{N} \cap (\mathbb{N}_L)^\perp$, i.e. $\mathbb{N} = \mathbb{N}_L \oplus \mathbb{N}_d$. One can show that $\mathbb{N}_d$ is Σ-positive. Similarly, $\mathbb{P}_d := \mathbb{P}^{\perp\Sigma} \cap (\mathbb{P}_L)^\perp$ resulting in $\mathbb{P}^{\perp\Sigma} = \mathbb{P}_L \oplus \mathbb{P}_d$, with $\mathbb{P}_d$ being Σ-negative.

We need to decompose $\Sigma\mathbb{N}_{L_r}$ into the part within $\mathbb{P}$ and the rest of it. Let $\Sigma\mathbb{N}_{L_r} \cap \mathbb{P} =: \mathbb{N}_1$. Define $\mathbb{N}_2 := \Sigma\mathbb{N}_{L_r} \cap \mathbb{N}_1^\perp$. We thus have $\Sigma\mathbb{N}_{L_r} = \mathbb{N}_1 \oplus \mathbb{N}_2$. Using lemma 9, we have $\mathbb{N}_2 \subseteq \mathbb{P}^\perp$.

In the construction of a Σ-non-negative $\mathbb{K}$, in addition to $\mathbb{N}$, we need to take a suitable part from $\mathbb{P} \cap \mathbb{N}^{\perp\Sigma}$. However, $\mathbb{P}_L$ and $\mathbb{N}_L$ will be contained in $\mathbb{P} \cap \mathbb{N}^{\perp\Sigma}$, which we will take into $\mathbb{K}$ anyway. The part in $\mathbb{P} \cap \mathbb{N}^{\perp\Sigma}$ outside $\mathbb{N}_L$ and $\mathbb{P}_L$ is what we are interested in. Define $\mathbb{F} := (\mathbb{P} \cap \mathbb{N}^{\perp\Sigma}) \cap \mathbb{N}_L^\perp \cap \mathbb{P}_L^\perp$.

We restrict Σ to $\mathbb{F}$ and decompose $\mathbb{F}$ into subspaces $\mathbb{F}_+$ and $\mathbb{F}_-$ such that $\mathbb{F}_+$ is Σ-positive, and $\mathbb{F}_-$ is Σ-negative; this can be done as follows. Let $\mathtt{f}$ be the dimension of $\mathbb{F}$ and suppose $F \in \mathbb{R}^{\mathtt{v} \times \mathtt{f}}$ is a matrix with full column rank whose image

is $\mathbb{F}$, i.e. $v = Fu$ is an image representation of $\mathbb{F}$. Construct $F^T \Sigma F$. Notice that $F^T \Sigma F$ is symmetric and nonsingular. Let $\mathtt{f}_+$ and $\mathtt{f}_-$ denote the positive and negative signatures, $\sigma_+(F^T \Sigma F)$ and $\sigma_-(F^T \Sigma F)$, of $F^T \Sigma F$, respectively. There exists a nonsingular matrix $U \in \mathbb{R}^{\mathtt{f} \times \mathtt{f}}$ partitioned suitably into $U = \begin{bmatrix} U_+ \\ U_- \end{bmatrix}$ such that

$$F^T \Sigma F = \begin{bmatrix} U_+^T & U_-^T \end{bmatrix} \begin{bmatrix} I_{\mathtt{f}_+} & 0 \\ 0 & -I_{\mathtt{f}_-} \end{bmatrix} \begin{bmatrix} U_+ \\ U_- \end{bmatrix}.$$

Now define $\mathbb{F}_+ \subseteq \mathbb{F}$ by $\mathbb{F}_+ := F(\text{kernel}(U_-))$, i.e.

$$\mathbb{F}_+ := \{ v \in \mathbb{V} \mid \text{ there exists } u \in \mathbb{R}^{\mathtt{f}} \text{ such that}$$
$$v = Fu \text{ and } U_- u = 0 \}.$$

$\mathbb{F}_-$ is defined analogously as $\mathbb{F}_- := F(\text{kernel}(U_+))$.

We now show that $\mathbb{F}_+$ is Σ non-negative, and that $\mathbb{F}_+$ and $\mathbb{F}_-$ are Σ-orthogonal to each other. Consider $v_1 \in \mathbb{F}_+$ and suppose $u_1 \in \mathbb{R}^{\mathtt{f}}$ is such that $v_1 = Fu_1$. We have $|v_1|_\Sigma^2 = |U_+ u_1|^2 - |U_- u_1|^2 = |U_+ u_1|^2$, since $U_- u_1 = 0$ due to nonsingularity of U and by definition of $\mathbb{F}_+$. This shows that $\mathbb{F}_+$ is Σ non-negative. Again using the nonsingularity of U, one can further show easily that, in fact, $\mathbb{F}_+$ is Σ positive. Similarly, one also proves that $\mathbb{F}_-$ is Σ-negative, and that $\mathbb{F}_+$ and $\mathbb{F}_-$ are Σ-orthogonal.

The table below is a recapitulation of all the definitions made so far:

Subspace and its definition	dimension	$\subseteq \mathbb{P}$ or $\mathbb{P}^\perp$?
$\mathbb{N}_L$: the Σ-neutral part of $\mathbb{N}$	$m_2 + m_3$	$\subseteq \mathbb{P}$
$\mathbb{P}_L$: the Σ-neutral part of $\mathbb{P}^{\perp\Sigma}$	$m_3 + m_8$	$\subseteq \mathbb{P}$
$\mathbb{L}_{\mathbb{NP}} := \mathbb{N}_L \cap \mathbb{P}_L$	m_3	$\subseteq \mathbb{P}$
$\Sigma\mathbb{L}_{\mathbb{NP}}$	m_3	$\subseteq \mathbb{P}^\perp$
$\mathbb{P}_{L_r} := \mathbb{P}_L \cap \mathbb{L}_{\mathbb{NP}}^\perp$	m_6	$\subseteq \mathbb{P}$
$\Sigma\mathbb{P}_{L_r}$	m_6	$\subseteq \mathbb{P}^\perp$
$\mathbb{N}_{L_r} := \mathbb{N}_L \cap \mathbb{L}_{\mathbb{NP}}^\perp$	m_2	$\subseteq \mathbb{P}$
$\mathbb{N}_d := \mathbb{N} \cap \mathbb{N}_L^\perp$ (Σ +ve part of $\mathbb{N}$)	m_1	$\subseteq \mathbb{P}$
$\Sigma\mathbb{N}_{L_r}$ (see $\mathbb{N}_1$ and $\mathbb{N}_2$ right below)	m_2	
$\mathbb{N}_1 := \Sigma\mathbb{N}_{L_r} \cap \mathbb{P}$	m_{21}	$\subseteq \mathbb{P}$
$\mathbb{N}_2 := \Sigma\mathbb{N}_{L_r} \cap \mathbb{N}_1^\perp$	$m_2 - m_{21}$	$\subseteq \mathbb{P}^\perp$
$\mathbb{P}_d := \mathbb{P}^{\perp\Sigma} \cap \mathbb{P}_L^\perp$ (Σ-ve part of $\mathbb{P}^{\perp\Sigma}$)	m_7	$\subseteq \mathbb{P}^\perp$
$\mathbb{N}_r := \mathbb{N}^{\perp\Sigma} \cap \mathbb{N}_L^\perp$		
$\mathbb{F} := \mathbb{P} \cap \mathbb{N}^{\perp\Sigma} \cap \mathbb{P}_L^\perp \cap \mathbb{N}_L^\perp$	$m_4 + m_5$	$\subseteq \mathbb{P}$
$\mathbb{F}_+$: Σ +ve part of $\mathbb{F}$	m_4	$\subseteq \mathbb{P}$
$\mathbb{F}_-$: Σ -ve part of $\mathbb{F}$	m_5	$\subseteq \mathbb{P}$

Consider the following two direct sum decompositions of $\mathbb{V}$

$$\mathbb{V} = \mathbb{N}_L \oplus \mathbb{N}_d \oplus (\mathbb{N}^{\perp\Sigma} \cap \mathbb{N}_L^\perp) \oplus \Sigma\mathbb{N}_L$$
$$\mathbb{V} = \mathbb{P}_L \oplus \mathbb{P}_d \oplus (\mathbb{P} \cap \mathbb{P}_L^\perp) \oplus \Sigma\mathbb{P}_L .$$

Using the definitions of the various subspaces and the intertwining between them, we have

$$\mathbb{V} = \mathbb{L}_{\mathbb{NP}} \oplus \mathbb{N}_{L_r} \oplus \mathbb{N}_d \oplus \mathbb{P}_{L_r} \oplus (\mathbb{P} \cap \mathbb{N}^{\perp_\Sigma} \cap \mathbb{N}_L^\perp \cap \mathbb{P}_L^\perp) \oplus$$
$$\mathbb{P}_d \oplus \Sigma\mathbb{L}_{\mathbb{NP}} \oplus \Sigma\mathbb{N}_{L_r} \oplus \Sigma\mathbb{P}_{L_r}.$$

Notice that except $\Sigma\mathbb{L}_{\mathbb{NP}}$ above, all the subspaces belong to $\mathbb{P} + \mathbb{N}^{\perp_\Sigma}$. Using the above definition of $\mathbb{F}$, and also writing $\Sigma\mathbb{N}_{L_r} = \mathbb{N}_1 \oplus \mathbb{N}_2$, we rewrite $\mathbb{V}$ as a direct sum of the following subspaces

$$\mathbb{V} = (\mathbb{L}_{\mathbb{NP}} \oplus \mathbb{N}_{L_r} \oplus \mathbb{N}_1 \oplus \mathbb{N}_d \oplus \mathbb{F} \oplus \mathbb{P}_{L_r}) \quad \oplus \tag{3}$$
$$(\mathbb{N}_2 \oplus \Sigma\mathbb{L}_{\mathbb{NP}} \oplus \Sigma\mathbb{P}_{L_r} \oplus \mathbb{P}_d).$$

Subspaces in the first group belong to $\mathbb{P}$ while those in the second group belong to $\mathbb{P}^\perp$. We now write $\mathbb{P} = \mathbb{P} \cap \mathbb{V}$, and use the modular distributive property, together with $\mathbb{P} \cap \mathbb{P}^\perp = 0$, to obtain the direct sum decomposition of $\mathbb{P} = \mathbb{L}_{\mathbb{NP}} \oplus \mathbb{N}_{L_r} \oplus \mathbb{N}_1 \oplus \mathbb{N}_d \oplus (\mathbb{F}_+ \oplus \mathbb{F}_-) \oplus \mathbb{P}_{L_r}$.

It is obvious now as to which of the above subspaces ought to be taken into $\mathbb{K}$; define $\mathbb{K}$ as follows

$$\mathbb{K} := \mathbb{L}_{\mathbb{NP}} \oplus \mathbb{N}_{L_r} \oplus \mathbb{N}_d \oplus \mathbb{F}_+ \oplus \mathbb{P}_{L_r}. \tag{4}$$

The subspaces that have been added to form $\mathbb{K}$ are mutually Σ-orthogonal and each of them are either Σ-neutral or Σ-positive. We use the following lemma to conclude that $\mathbb{K}$ is Σ-non-negative.

Lemma 12: **:** Let $\mathbb{B}_1, \mathbb{B}_2 \subseteq \mathbb{V}$ be Σ-non-negative subspaces satisfying $\mathbb{B}_1 \perp_\Sigma \mathbb{B}_2$. Then, $\mathbb{B}_1 + \mathbb{B}_2$ is also Σ-non-negative.

Moreover, by construction we also have $\mathbb{N} \subseteq \mathbb{K} \subseteq \mathbb{P}$. The definition of $\mathbb{K}$ in equation (4) above shows that $\mathbb{K}$ is nothing but $\mathbb{N} + \mathbb{P}_L + \mathbb{F}_+$. However, in general $\mathbb{N}$ and $\mathbb{P}_L$ intersect nontrivially, and making sure that the dimension is sufficiently high for the case of nontrivial intersection of $\mathbb{N}$ and $\mathbb{P}_L$ is actually the difficult part of the proof. This is described below. Consider again the decomposition of $\mathbb{V}$ as in equation (3), with a reordering of the subspaces as $\mathbb{V} =$

$$\mathbb{N}_d \oplus \mathbb{N}_{L_r} \oplus \mathbb{N}_1 \oplus \mathbb{N}_2 \oplus \mathbb{L}_{\mathbb{NP}} \oplus \Sigma\mathbb{L}_{\mathbb{NP}} \oplus \mathbb{F}_+ \oplus \mathbb{F}_- \oplus \mathbb{P}_{L_r} \oplus \Sigma\mathbb{P}_{L_r} \oplus \mathbb{P}_d.$$

We are now in a position to conclude that $\mathbb{K}$ as defined in equation (4) satisfies the dimensionality requirement.

This is done using the Σ-positivity, Σ-negativity or the Σ-neutrality, and Σ-orthogonality of the concerned spaces. This essentially comes from the two equalities below that relate the dimensions of the above subspaces and the signature of Σ

$$\sigma_+(\Sigma) = m_1 + m_2 + m_3 + m_4 + m_6, \quad \text{and}$$
$$\sigma_-(\Sigma) = m_2 + m_3 + m_5 + m_6 + m_7$$

This ends the proof of theorem 6. $\qquad\square$

VI. Conclusion

We have formulated and proved the dissipativity synthesis problem. The problem in this paper is more general than the one in [7], [5] because dynamics are allowed in the weighting functional; also, we have relaxed the requirement on the controlled behavior as compared to the problem in [7], [5]. The analogous problem DSP2 was formulated and proved and the proof of the main result (theorem 3), which solves DSP1, follows along exactly similar lines (see remarks 5 & 7 above). It was due to posing the dissipativity synthesis problem in the behavioral framework that the connection with the analogous linear-algebraic problem DSP2 becomes easily tangible. Further, as outlined in section V, the solution of this linear algebraic problem is fairly tractable.

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