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An application of the Sum of Squares Decomposition to the $\mathcal{L}_2$ gain computation for a class of non linear systems

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Abstract— **This paper presents a new approach for determining the $\mathcal{L}_2$ gain of affine non-linear systems with polynomial vector fields within the sum of squares framework. The main feature of the proposed approach is that it does not require solving a Hamilton Jacobi Inequality (HJI). The solution to the HJI is done indirectly by solving another inequality augmented with slack variables. This new inequality is much easier to solve than the original HJI since it is linear in the Lyapunov function parameters. Numerical examples are given to illustrates the proposed approach.**

Index Terms— **$\mathcal{H}_\infty$ Linear Matrix Inequalities (LMIs), Nonlinear Systems, $\mathcal{L}_2$ gain.**

I. Introduction

The determination of an upper bound on the $\mathcal{L}_2$ gain of an affine non-linear system is related to the existence of a solution of a Hamilton-Jacobi-Issacs inequality (HJI) [7], [17]. In the case of linear systems, it is well-known that these inequalities reduce to a set of matrix Riccati algebraic inequalities whose solutions can be obtained algebraically [4], [18] (or as the solutions of linear Matrix Inequality optimization problem [1]). It is well-known that HJIs can be very difficult to solve [6], [17]. Series solution approaches to the Hamilton-Jacobi equalities have been reported to be useful in practice [14], [16]. However, the series approach presents some drawbacks such as lack of convergence properties and lack of efficient algorithms for checking the positive definiteness of the solution.

Recently, there has been some research on the positivity of multivariate polynomials (see [5] and the references therein). For instance, the decomposition of a polynomial as a sum of squares, when such decomposition exists, can be performed via semi-definite programming [9]. This idea has been exploited for the search of polynomial Lyapunov functions (or density functions) for proving the stability of nonlinear systems with polynomial vector fields [9], [12].

In a similar way, this paper addresses the case of nonlinear systems with polynomial vector fields within the sum of squares framework. The paper establishes sufficient conditions for the simultaneous search of a polynomial Lyapunov function and of an upper bound on the system $\mathcal{L}_2$ gain.

The paper is structured as follows. Two new results are given in Section III. Section IV outlines briefly an algorithm for the numerical determination of a Lyapunov function that proves the system stability along with an upper bound on the system gain. Three numerical examples are given in Section V. Section VI ends the paper with some conclusions.

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The notation used in this paper is standard: $\mathbb{R}^{m\times n}$ denotes the set of real $m \times n$ matrices, I_n is the $n \times n$ identity matrix, $A \geq 0$ means that A is symmetric and positive semidefinite, $\nabla V = \left[\frac{\partial V}{\partial x_1} \dots \frac{\partial V}{\partial x_n}\right]$, $V : \mathbb{R}^n \to \mathbb{R}$. The $\mathcal{L}_p$ norm of a vector-valued signal $u(t)$, denoted $\|u\|_p$ is defined as $\|u\|_p = \left(\int_0^\infty \|u(t)\|^p dt\right)^{1/p}$, $1 \leq p < \infty$.

II. Preliminaries

Definition 1: The space

$$\mathcal{L}_{p,e} = \{P_T x(t) \in \mathcal{L}_p, \quad \forall T < \infty\}$$

and where P_T denotes the truncation operator

$$P_T x(t) = \left(\begin{array}{ll} x(t), & t \leq T \\ 0, & t > T \end{array} \right. .$$

Definition 2: Finite gain stability A causal operator $H(.)$ is finite gain stable if

$$\|H(x)\|_p \leq \gamma \|x\|_p + \beta \quad \forall x \in \mathcal{L}_{p,e}$$

for some positive scalars γ and β.

Let us consider the autonomous affine nonlinear system

$$\dot{x} = f(x) + G(x)u, \quad x(0) = x_0 \tag{1}$$
$$y = h(x) \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuous over $\mathbb{R}^n$. Also, we assume $f(0) = 0$ and $h(0) = 0$.

Theorem 1: e.g. [7], [17]: Let γ be a positive number and suppose there is a continuously differentiable, positive semidefinite function $V(x)$ that satisfies the inequality

$$\begin{aligned} \mathcal{H} : \quad = \quad & \nabla V.f(x) + \frac{1}{2\gamma^2} \nabla V.G(x)G^T(x)(\nabla V)^T \\ + \quad & \frac{1}{2} h^T(x)h(x) \leq 0 \end{aligned} \tag{3}$$

for all $x \in \mathbb{R}^n$. Then the system (1)-(2) is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ.

Inequality (3) is the so-called Hamilton-Jacobi inequality which can be very difficult to solve.

III. MAIN RESULTS

A. $\mathcal{L}_2$ Gain

Theorem 2: Consider the non-linear system (1)-(2) and suppose there is a continuously differentiable positive semidefinite function $V(x)$ that satisfies the inequality

$$\begin{aligned} \mathcal{I} &= \nabla V.\left(f(x) + 2G(x)\xi\right) - 2\gamma^2\xi^T\xi \\ &+ \frac{1}{2}h^T(x)h(x) \le 0 \end{aligned} \quad (4)$$

for all $x \in \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^m$. Then, for any $x_0 \in \mathbb{R}^n$, the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less or equal to γ.

Proof: Note that $\mathcal{H} \le 0$ implies $\nabla V.f(x) + \frac{1}{2}h^T(x)h(x) \le 0$. Using a standard Schur complement argument one gets

$$\mathcal{H} \le 0 \Leftrightarrow \begin{pmatrix} -2\gamma^2 I_m & G(x)^T(\nabla V)^T \\ \nabla V.G(x) & \nabla V.f(x) + \frac{1}{2}h^T(x)h(x) \end{pmatrix} \le 0 \quad (5)$$

Then, using the homogenization Lemma given in the appendix, (5) holds if and only if, for all $x \in \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^m$

$$\begin{bmatrix} \xi \\ 1 \end{bmatrix}^T \begin{pmatrix} -2\gamma^2 I_m & G(x)^T(\nabla V)^T \\ \nabla V.G(x) & \nabla V.f(x) + \frac{1}{2}h^T(x)h(x) \end{pmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} \le 0 \quad (6)$$

The development of (6) leads to (4). This completes the proof. ■

The key feature of Theorem 2 is that (4) is a linear and thus a convex inequality in ∇V and γ^2.

B. Scaled $\mathcal{L}_2$ Gain

Let us consider the following system that has as many inputs as outputs (i.e. $m = p$)

$$\dot{x} = f(x) + G(x)T^{-1}u, \quad x(0) = 0 \quad (7)$$
$$y = Th(x), \quad (8)$$

where T is a positive-definite diagonal matrix. The scaled $\mathcal{L}_2$ gain is defined as

$$\alpha = \inf_{\substack{T > 0, \\ T \quad diagonal}} \sup_{\|u\|_2 \neq 0} \frac{\| y \|_2}{\| u \|_2}$$

Theorem 3: (Scaled $\mathcal{L}_2$ gain). The scaled $\mathcal{L}_2$ gain is guaranteed to be less than γ if there exist $V(x) > 0$ and $S \in \mathbb{R}^{p\times p}$, $S > 0$, S diagonal, which satisfy, for all $x \in \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^p$

$$\mathcal{I}_s := \nabla V.\left(f(x) + 2G(x)\xi\right) + \frac{1}{2}h^T(x)Sh(x) - 2\gamma^2\xi^T S\xi \le 0 \quad (9)$$

Proof: The proof is similar to the proof of theorem 1 and it is therefore omitted. ■

Clearly, for a given $\gamma > 0$, the inequality $\mathcal{I}_s$ is linear in S and ∇V.

IV. SUMS OF SQUARES

Consider a polynomial $p(x) : \mathbb{R}^n \to \mathbb{R}$ of degree $2k$. The polynomial is said to be positive semidefinite (PSD) if $p(x) \ge 0$ for all x. In general, it is extremely difficult to determine whether or not a given polynomial is PSD. A sufficient condition for a polynomial to be PSD is to be in the form

$$p(x) = \sum_{i=1}^{r} q_i(x)^2$$

for some polynomials q_i, with degree no more than k. A polynomial p that has this sum-of-squares form is called SOS. The condition that a polynomial p be SOS turns out to be equivalent to an Linear Matrix Optimization (LMI) problem, see [9], [10], [2] for details.

Assume that $f(x)$, $G(x)$ and $h(x)$ have multivariate polynomial entries. Also, assume that we are looking for a multivariate polynomial Lyapunov function $V(x)$ and a positive number γ that satisfy the polynomial inequality (4). Under these assumptions, the determination of $V(x)$ and of the smallest γ such that $V(x)$ and $-\mathcal{I}(x,\xi)$ are SOS is an LMI problem. Put in another way, given strictly positive numbers ϵ and β, one has to solve the following semi-definite optimization problem

$$\begin{aligned} &minimize \quad \gamma - \beta \\ &subject \quad to: \\ &\quad V(x) - \epsilon x^T x \ge 0 \\ &-\mathcal{I}(x,\xi) \ge 0 \end{aligned}$$

Such an optimization problem can be easily formulated and efficiently solved with the Matlab toolboxes Sostools 2.0 [11] and SeDuMi 1.05 [15]. The introduction of the positive numbers β and ϵ in the above problem is made to prevent numerical difficulties when solving the corresponding semi-definite program.

It worth noting that, in Theorem 2, $\mathcal{I}(x,\xi) = \mathcal{H}(x)$ for $\xi = \frac{1}{2\gamma^2}G(x)^T(\nabla V(x))^T$. This implies that if $\mathcal{I}(x,\xi)$ is SOS then the Hamilton Jacobi inequality $\mathcal{H}(x)$ is also SOS.

V. EXAMPLES

The following $\mathcal{L}_2$ gain examples were solved with the algorithm of Section IV. The numerical computations were carried out with Matlab 6.5 and with the Sum of Squares toolbox Sostools [11] on a 2GHz PC. For each problem, we indicate the number of decision variables n, the number of SeDuMi constraints m and the required CPU time. Also, for each example, the parameters β and ϵ are indicated.

A. Example 1

Let us consider the non-linear system taken from [13]

$$\begin{aligned} \dot{x} &= -x - x^3 + u \\ y &= x \end{aligned}$$

It can be shown that this system has a finite $\mathcal{L}_2$ gain $\gamma = 1$ [13]. In this example, we decided to search over the set of polynomial Lyapunov functions of order 4 since no solution was found with quadratic Lyapunov functions. Using the sum of squares algorithm of section IV, with $\epsilon = 0.25$ and $\beta = 0.1$, we got $\gamma = 1.0000$ and the following 4th order Lyapunov function

$$V(x) = 0.4999x^2 - 0.3733 \times 10^{-7}x^3 + 0.2556x^4$$

($n = 21$, $m = 20$, CPU time = 0.45s.)

B. Example 2

Let us consider the following numerical example adapted from [5]

$$\begin{aligned} \dot{x}_1 &= -x_1^3 - x_2x_3 - x_1 - x_1x_3^2 \\ \dot{x}_2 &= -x_1x_3 + 2x_1^2 - x_2 \\ \dot{x}_3 &= -x_3 + 2x_1^3 + u \\ y &= x_1. \end{aligned}$$

We decided to search over the set of polynomial quadratic Lyapunov functions $V(x) = x^T Px$, $P > 0$. In this case, the algorithm of section IV, with $\epsilon = 0.25$ and $\beta = 0.1$ returns the following Lyapunov function $V(x) = 0.25001(x_1^2+x_2^2+x_3^2)$ and $\gamma = 0.5$ ($n = 61$, $m = 36$, CPU time = 0.67s.)

C. Example 3

This example is taken from [8]. The system is a fifth order system which represents the model of an induction motor:

$$\begin{aligned} \dot{x}_1 &= 31.21(x_2x_5 - x_3x_4) - 0.667x_1 \\ \dot{x}_2 &= -7.66x_2 - 2x_1x_3 + 3.37x_4 \\ \dot{x}_3 &= 2x_1x_2 - 7.66x_3 + 3.37x_5 \\ \dot{x}_4 &= 127.14x_2 + 33.19x_1x_3 - 197.78x_4 + 17.73u_1 \\ \dot{x}_5 &= -33.19x_1x_2 + 127.14x_3 - 197.78x_5 + 17.73u_2 \\ y &= [x_1, x_4, x_5]^T, \end{aligned}$$

where x_1 is the rotor speed, x_2, x_3 are the rotor fluxes, x_4, x_5 are the stator currents and $u = [u_1,\ u_2]^T$ is the stator voltage control input.

In this case, we decided to search over the set of positive definite quadratic Lyapunov functions $V(x) = x^T Px$, $P > 0$. The algorithm of section IV, with $\epsilon = 0.3$ and $\beta = 0.1$, returned

$$P = \begin{pmatrix} 0.3167 & -0.0000 & -0.7151 & 0.0000 & -0.000 \\ -0.0000 & 0.3165 & 0.000 & -0.7127 & 0.0000 \\ -0.7151 & 0.0000 & 31.4331 & 0.000 & -0.000 \\ 0.0000 & -0.7127 & 0.000 & 31.1613 & 0.0000 \\ -0.0000 & 0.0000 & -0.000 & 0.000 & 0.3828 \end{pmatrix}$$

with $\gamma = 0.6980$ ($n = 77$, $m = 287$, CPU time=2.72s.)

VI. Conclusions

A new approach for the computation of the $\mathcal{L}_2$ gain of a nonlinear system with polynomial vector field has been presented. The approach is based on the sum of squares decomposition of multivariate polynomials which can be formulated as an LMI optimization problem. This new approach differs from other $\mathcal{L}_2$ gain estimation because it does not require to solve a Hamilton Jacobi inequality. The solution to the HJI is done indirectly trough a new, but equivalent, inequality which is linear in the derivative of the Lyapunov function. This important feature enables us to reformulate the determination of an upper bound on the system $\mathcal{L}_2$ gain as a linear semidefinite optimization problem. Despite the conservatism of the sum of squares decomposition, the tractability of the proposed approach is illustrated by three numerical examples. Following the ideas presented here, many other analysis results, available for linear systems, could be extended to affine polynomial non-linear systems.

However, more work is required to improve the numerical reliability of sum of squares algorithms. Sum of squares problems tend to be badly conditioned, essentially because they deal with polynomials. Another related issue is the the choice of parameters ϵ and β, in the algorithm of Section IV. This choice influences greatly the solution and it deserves further attention. Finally, the relationship between the Lyapunov function order and the polynomial degrees of the system vector field needs to be clarified.

VII. Appendix

Lemma 1: Schur complement (see e.g. [1]).

Suppose Q and R are symmetric and R^{-1} exists. The Condition

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \geq 0$$

is equivalent to $R \geq 0$, $Q - SR^{-1}S^T \geq 0$.

Lemma 2: (Homogenization)

Let $T = T^T$. The following two conditions are equivalent.

$$a) \quad \begin{bmatrix} \xi \\ 1 \end{bmatrix}^T \begin{pmatrix} T & u \\ u^T & v \end{pmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} \leq 0 \quad for \quad all \quad \xi$$

$$b) \quad \begin{pmatrix} T & u \\ u^T & v \end{pmatrix} \leq 0$$

Proof: The implication from (b) to (a) is trivial. (a) implies (b) can be found in [3]. ■

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