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State Prediction for a Class of Time-Delay Unstable Linear Systems

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Abstract— **This papers deals with the state prediction of a class of linear time delay systems with delays at the input and at the state and considering that the open loop system is unstable. In order to deal with this problem, it is proposed a Smith predictor compensator type that results in a generalization of a previously reported compensator used to obtain the approximate prediction of the state of a system of the same class that it is assumed to be open loop stable. In the present case, of unstable plants, it is shown that the proposed compensator allows also to approximately estimate the future values of the estate and that the estimation error converges to zero due to the consideration of an additional internal feedback. The performance of the proposed compensator is evaluated by means of numerical simulations.**

I. Introduction

The study of classical control problems for retarded differential systems has been a very important research issue in the control community for the last decades. The necessity of practical implementation of control compensators for this class of systems has lead to the study of the conditions under which the existence of a causal solution could be assured.

The characterization of causal solutions for classical control problems for linear time-delay systems has been studied, for instance, in the linear case in [1], [2], [3] or in the nonlinear context in [4], [5], [6]. Due to the fact that the existences of these causal solutions are subject to restrictive conditions, the study of approximations techniques in order to obtain simplified (approximate) models free of delays has also been considered. This later methodology allows handling time-delay systems with the usual techniques proposed for free-delays systems [7].

Another attempt to solution to classical control problems is based on the estimation of the future values of the state of the system. In this sense, several proposals have been made by considering the so-called Smith predictor compensator [8].

The original Smith compensator has been subject to several modifications, in the linear SISO case see for example [9], [10], or in the case of linear multivariable systems [11]. Also, the Smith predictor has been analyzed in the presence of disturbances [12], where some attenuations properties have been established. Also, some modifications to the classical smith predictor compensator has been proposed to control integrator process and long dead-time [9], [13] or even unstable systems [14].

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Some other works related with the Smith predictor include a special class of nonlinear systems [15].

In most of the aforementioned works the time delay associated to the system is considered to affect the input and/or the output signal.

Regarding retarded systems that includes time-delays at the input signal and also at the state; in [16] a compensator of the Smith-predictor type is proposed in order to estimate the futures values of the estate of a linear time-delay system of this class. It is assumed in this work, as is classical on Smith-predictor compensators, that the system is open loop asymptotically stable, and it is shown that it is possible to approximately estimate the future value of the state. Based on this idea, in the present work, it is presented a prediction compensator scheme that allows the considerations of unstable linear time-delay systems with delays at the input and at the state, showing that the future estimate value converge approximately to the real value.

The main difference between the classical Smith-Predictor [8] and the modification proposed [16] is related with the effect that the time-delay on the state produce over the general dynamics of the systems where the stability is a very important issue.

The paper is organized as follows: In Section 2, the class of systems considered is presented and the result of [16] is recalled for the sake of completeness in Section 3. In Section 4, the compensator that allows the considerations of unstable plants is presented, showing the appropriate convergence of the estimation error. In Section 5, the performance of the proposed scheme is shown by means of digital simulations, while in Section 6 some conclusions are given.

II. Class of systems

Consider the time-delay system given by,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1x(t-h) + Bu(t-k) \\ y(t) &= Cx(t) \\ x(t_0+\varphi) &= \phi(\varphi) \end{aligned} \tag{1}$$

where $h \geq 0$ is a constant time delay associated to the state, $k \geq 0$ is the constant time-delay associated to the input, $\phi(\varphi)$ is a continuous function of initial conditions with $-h \leq \varphi \leq 0$. The state $x \in R^n$, the input $u \in R$ and the output $y \in R$. Finally, the matrices A, $A_1 \in R^{n\times n}$, $B \in R^{n\times 1}$ and $C \in R^{1\times n}$ are supposed to be known.

This class of systems produce a characteristic equation described by a quasipolynomial, this is, there exists a dependence of its characteristic polynomial on the time delay associated with the state. These systems are represented by differential-difference equations [7]. This fact represents a

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great difference with respect to systems with time-delays only at the input signal.

For the class of systems (1) a feedback control law design to solve problems like path tracking, disturbance rejection, etc., requires in the general situation the knowledge of future values of the state [1]. This fact represents a motivation to try to find a compensator that can solve the future state estimation of a time delay systems with delays at the input and the state.

In order to obtain the desired future estimation for unstable systems of the form (1) a compensator of the Smith-Predictor type will be proposed to approximately obtain these future values. This compensator represent a generalization of the one proposed in [16] for stable systems of the same class.

The convergence of the error estimation will be analyzed by considering the error between the exact future value and the approximate estimation.

Up to the best knowledge of the author, this work represent the first attempt to estimate the state future values of an open loop unstable system of the form (1).

III. Smith-predictor for input and state delayed systems

In this section the results presented reported in [16], where a prediction scheme is given for the case in which the system (1) is open loop stable will be briefly recalled.

Theorem 3.1: [16]Consider system (1) and the compensator

$$\begin{array}{rcl} \dot{z}_1(t) & = & Az_1(t) + A_1 z_1(t-h) + Bu(t-k) \\ \dot{z}_2(t) & = & Az_2(t) + A_1 z_2(t-h) + Bu(t) \\ \bar{x}(t+k) & = & \tilde{\Phi}(t+k,t)\left[x(t) - z_1(t)\right] + z_2(t) \end{array} \qquad (2)$$

with the initial conditions, $z_1(t_0+\tau) = \phi_1(\tau) = 0$ and $z_2(t_0+\tau) = \phi_2(\tau) = Bu(\tau)$, $\forall - h \leq \tau \leq t_0$. Assume that system (1) is stable and that $h \geq k$. Then, under these conditions, $\bar{x}(t+k)$ gives an approximate prediction of the state $x(t+k)$ of system (1). ■

The compensator proposed in Theorem 3.1 is shown in Figure 1 where a classical control scheme is depicted.

Remark 3.2: On Theorem 3.1, $\tilde{\Phi}(t+k,t)$ represents an approximation of the fundamental matrix $\Phi(t+k,t)$ of system (1), this is,

$$\Phi(t,\tau) = \tilde{\Phi}(t,\tau) + o(t,\tau,h^2)$$

where the matrix $\tilde{\Phi}(t,\tau)$ can be given in the case of system (1) as

$$\tilde{\Phi}(t,\tau) = e^{[I+hA_1]^{-1}[A+A_1](t-\tau)}.$$

The proof of this fact is also given in [16] and it is done by considering a result on fundamental matrices of [7].

IV. Smith-predictor for input and state delayed unstable systems

In this section, it will be shown that under appropriate modification of the compensator described in equation (2), a larger class of systems that takes into account unstable plants can be considered. With this purpose, it is proposed the modified Smith-compensator type,

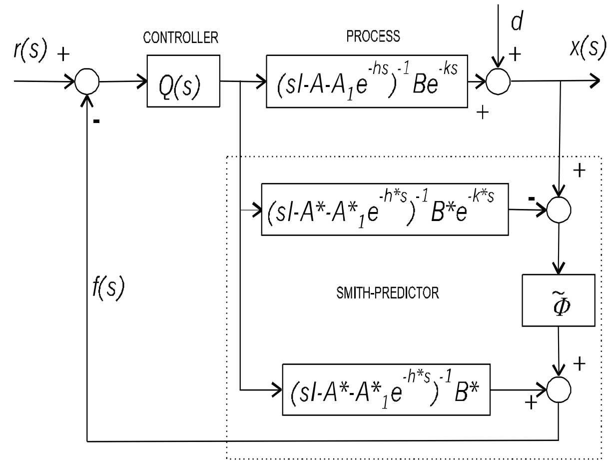

Fig. 1. Smith-Predictor compensator for stable plants

$$\begin{array}{rcl} \dot{z}_1(t) & = & Az_1(t) + A_1 z_1(t-h) \\ & & +L_1(x(t) - z_1(t)) + Bu(t-k) \\ \dot{z}_2(t) & = & Az_2(t) + A_1 z_2(t-h) + Bu(t) + J(t) \\ \bar{x}(t+k) & = & \tilde{\Phi}(t+k,t)\left[x(t) - z_1(t)\right] + z_2(t) \end{array} \qquad (3)$$

Where $J(t)$ will be appropriately defined to allow compensator (3) to estimate the desire future values of the state.

Before presenting the main result of this work note that the particular dynamics of the error signal $x(t) - z_1(t)$, can be described defining $\xi(t) = x(t) - z_1(t)$ as,

$$\dot{\xi}(t) = (A - L_1)\xi(t) + A_1\xi(t-h). \qquad (4)$$

The estimation $\bar{x}(t+k)$ of $x(t+k)$ can now be written as,

$$\bar{x}(t+k) = \tilde{\Phi}(t+k,t)\xi(t) + z_2(t)$$

from where, in the case of a constant k, it is obtained,

$$\begin{aligned} \dot{\bar{x}}(t+k) = {} & \tilde{\Phi}(t+k,t)\{(A-L_1)\xi(t) \\ & + A_1\xi(t-h)\} + Az_2(t) \\ & + A_1 z_2(t-h) + Bu(t) + J(t). \end{aligned}$$

Considering that

$$z_2(t) = \bar{x}(t+k) - \tilde{\Phi}(t+k,t)\xi(t),$$

it is possible to obtain,

$$\begin{aligned} \dot{\bar{x}}(t+k) = {} & \tilde{\Phi}(t+k,t)\{(A-L_1)\xi(t) \\ & + A_1\xi(t-h)\} \\ & + A\left\{\bar{x}(t+k) - \tilde{\Phi}(t+k,t)\xi(t)\right\} \\ & + A_1\{\bar{x}(t+k-h) \\ & - \tilde{\Phi}(t-h+k,t-h)\xi(t-h)\} \\ & + Bu(t) + J(t), \end{aligned}$$

that can be rewritten as,

$$
\begin{aligned}
\dot{\bar{x}}(t+k) &= A\bar{x}(t+k) + A_1\bar{x}(t+k-h) \\
&+ \{\tilde{\Phi}(t+k,t)A - A\tilde{\Phi}(t+k,t) \\
&- \tilde{\Phi}(t+k,t)L_1\}\xi(t) \\
&+ \{\tilde{\Phi}(t+k,t)A_1\xi \\
&- A_1\tilde{\Phi}(t-h+k,t-h)\}\xi(t-h) \\
&+ Bu(t) + J(t).
\end{aligned}
$$

Considering the fact that [16],

$$\tilde{\Phi}(t-h+k,t-h) = \tilde{\Phi}(t+k,t) = \tilde{\Phi}(k,0) = \tilde{\Phi}$$

the above expression can be written equivalently in the form,

$$
\begin{aligned}
\dot{\bar{x}}(t+k) &= A\bar{x}(t+k) + A_1\bar{x}(t+k-h) \\
&+ \left\{\tilde{\Phi}A - A\tilde{\Phi} - \tilde{\Phi}L_1\right\}\xi(t) \\
&+ \left\{\tilde{\Phi}A_1 - A_1\tilde{\Phi}\right\}\xi(t-h) \\
&+ Bu(t) + J(t),
\end{aligned}
$$

that defining

$$R = \tilde{\Phi}A - A\tilde{\Phi},\ R_1 = \tilde{\Phi}A_1 - A_1\tilde{\Phi}$$

produces,

$$
\begin{aligned}
\dot{\bar{x}}(t+k) &= A\bar{x}(t+k) + A_1\bar{x}(t+k-h) \\
&\{R - \tilde{\Phi}L_1\}\xi(t) + R_1\xi(t-h) \\
&+ Bu(t) + J(t).
\end{aligned}
$$

Remark 4.1: From the above developments, the Compensator (3) can be rewritten directly in terms of the dynamics of the advanced variable $\bar{x}(t+k)$, this is,

$$
\begin{array}{rcl}
\dot{\xi}(t) &=& (A-L_1)\xi(t) + A_1\xi(t-h) \\
\dot{\bar{x}}(t+k) &=& A\bar{x}(t+k) + A_1\bar{x}(t+k-h) \\
&& +\{R - \tilde{\Phi}L_1\}\xi(t) + R_1\xi(t-h) \\
&& +Bu(t) + J(t)
\end{array}
\tag{5}
$$

with,

$$\xi(t) = x(t) - z_1(t)$$

A. Main result

Before presenting the main result, it is necessary to recall some standard results concerning the stability of a retarded system. Consider an autonomous systems of the form (1), this is,

$$\dot{x}(t) \quad = \quad Ax(t) + A_1x(t-h) \tag{6}$$

it is known that,

Lemma 4.2: [17]System (6) is stable if and only if the solutions of its characteristic equation

$$\det(sI - A - A_1e^{-hs}) = 0$$

are in the open left-half complex plane.

A simple sufficient condition to establish the stability of system (6) is given in the following proposition,

Proposition 4.3: [18]If

$$l_1 = \mu(A) + \|A_1\| < 0$$

then system (6)is asymptotically stable for all h.

In the above proposition it was considered

$$\mu(A) = \frac{1}{2}\max\left\{\lambda_i\left[A + A^T\right] : i = 1,2,...,n\right\}$$

and $\|A_1\|$ representing the induced norm of matrix A_1.

The preceding developments can be formally stated in the next theorem,

Theorem 4.4: Consider system (1) and the compensator

$$
\begin{array}{rcl}
\dot{z}_1(t) &=& Az_1(t) + A_1z_1(t-h) \\
&& +L_1(x(t) - z_1(t)) + Bu(t-k) \\
\dot{z}_2(t) &=& Az_2(t) + A_1z_2(t-h) + Bu(t) + J(t) \\
\bar{x}(t+k) &=& \tilde{\Phi}(t+k,t)\left[x(t) - z_1(t)\right] + z_2(t)
\end{array}
\tag{7}
$$

with $J(t) = L_2\left[x(t) + k\dot{x}(t) - \bar{x}(t+k)\right]$ and the initial conditions, $z_1(t_0+\tau) = \phi_1(\tau) = 0$, $z_2(t_0+\tau) = 0$, $\forall -h \le \tau \le t_0$. Assume that the matrices, $A - L_1$ and $A - L_2$ are such that,

$$
\begin{array}{c}
\mu(A-L_1) + \|A_1\| < 0 \\
\mu(A-L_2) + \|A_1\| < 0
\end{array}
\tag{8}
$$

Then, under these conditions, $\bar{x}(t+k)$ gives an approximate prediction of the state $x(t+k)$ of the unstable system (1).

Proof. Consider the prediction error,

$$e(t+k) = x(t+k) - \bar{x}(t+k). \tag{9}$$

Taking time derivatives, it is obtained from (5),

$$
\begin{aligned}
\dot{e}(t+k) &= Ax(t+k) + A_1x(t-h+k) + Bu(t) \\
&- \{A\bar{x}(t+k) + A_1\bar{x}(t+k-h) \\
&- \left[R - \tilde{\Phi}L_1\right]\xi(t) - R_1\xi(t-h) \\
&- Bu(t) - J(t)\}
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
\dot{e}(t+k) &= Ae(t+k) + A_1e(t+k-h) \\
&- \left[R - \tilde{\Phi}L_1\right]\xi(t) - R_1\xi(t-h) \\
&- L_2\left[x(t) + k\dot{x}(t) - \bar{x}(t+k)\right].
\end{aligned}
$$

Noting that,

$$x(t+k) = x(t) + k\dot{x}(t) + o(k^2)$$

it is possible to obtain,

$$
\begin{aligned}
\dot{e}(t+k) &= (A-L_2)\,e(t+k) + A_1e(t+k-h) \\
&- \left[R - \tilde{\Phi}L_1\right]\xi(t) - R_1\xi(t-h) \\
&+ o(k^2).
\end{aligned}
\tag{10}
$$

Considering now that $\mu(A-L_1) + \|A_1\| < 0$, the subsystem

$$\dot{\xi}(t) \quad = \quad (A-L_1)\xi(t) + A_1\xi(t-h)$$

is stable. Therefore considering the fact that $\mu(A-L_2) + \|A_1\| < 0$, it is possible to assure the stability of the differential equation (10) that produces the desired convergence of the error $e(t+k)$, that lead to the approximate estimation of $x(t+k)$. ■

Note that Theorem 4.4 has the drawback that the feedback term $J(t)$ has been defined on terms of the state of the system $x(t)$ and its time derivative $\dot{x}(t)$. In order to avoid this problem, in the next lemma it is shown that the estimation of $x(t+k)$ can also be obtained by considering estimated values of $x(t)$ and $\dot{x}(t)$.

Lemma 4.5: Consider system (1) and the compensator (7) with $J(t) = L_2\left[z(t) + k\dot{z}(t) - \bar{x}(t+k)\right]$. Under the same conditions of Theorem 4.4, $\bar{x}(t+k)$ is an approximate prediction of the state $x(t+k)$ for system (1).

Proof. Note that the error (9) can be written as,

$$\begin{aligned}\dot{e}(t+k) &= Ae(t+k) + A_1 e(t+k-h) \\ &\quad - \left[R - \tilde{\Phi}L_1\right]\xi(t) - R_1\xi(t-h) \\ &\quad - L_2\left[z(t) + k\dot{z}(t) - \bar{x}(t+k)\right]\end{aligned}$$

that after some simple manipulations produces,

$$\begin{aligned}\dot{e}(t+k) &= (A-\beta)\,e(t+k) + A_1 e(t+k-h) \qquad (11)\\ &\quad - \left[R - \tilde{\Phi}L_1\right]\xi(t) - R_1\xi(t-h) \\ &\quad + L_2\xi(t+k) - o(k^2),\end{aligned}$$

from where, under the assumptions of Theorem 4.4, the result follows. ■

V. Numerical results

In order to show the performance of the proposed estimator, consider the academic single–input, single-output system,

$$x(t) = ax(t) + a_1 x(t-h) + u(t-k) \qquad (12)$$

with

$$a = -0.45,\ a_1 = 0.5,\ h = 0.2,\ k = 0.3 \qquad (13)$$

It is possible to verify that system (12)-(13) is unstable [19] for the selected parameters. The estimation of the future values of $x(t)$, is obtained by means of the compensator

$$\begin{aligned}\dot{\xi}(t) &= (a - L_1)\xi(t) + a_1\xi(t-h) \\ \dot{\bar{x}}(t+k) &= a\bar{x}(t+k) + a_1\bar{x}(t+k-h) \\ &\quad + \left\{R - \tilde{\Phi}L_1\right\}\xi(t) + R_1\xi(t-h) \\ &\quad + Bu(t) + \beta\left[x(t) + k\dot{x}(t) - \bar{x}(t+k)\right]\end{aligned} \qquad (14)$$

with

$$L_1 = 0.1,\ \beta = 1.5.$$

The digital experiments were carried out by considering an input signal of the form

$$u(t) = 1 + \sin(t)$$

that it is shown in Figure 2.

In Figure 3 it is shown the response of the system (12) and the compensator (14), while in Figure 4 it is shown the convergence error $e(t) = x(t) - \bar{x}(t)$ at time t.

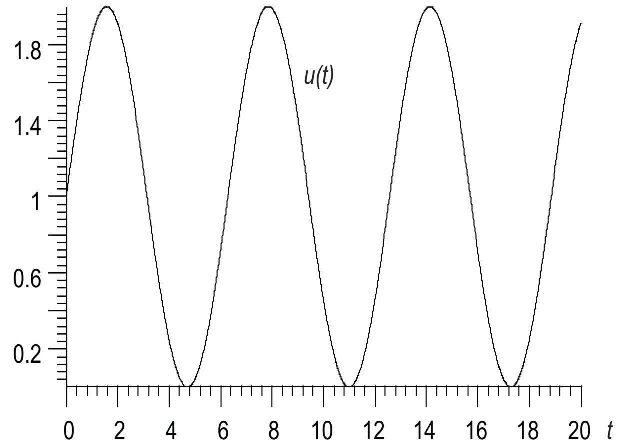

Fig. 2. Input signal $u(t)$

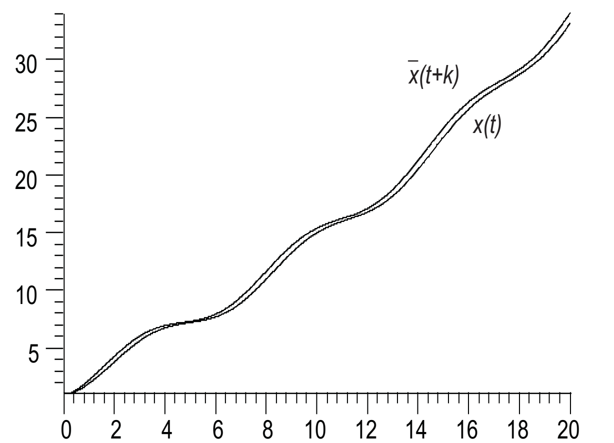

Fig. 3. State $x(t)$ and the estimate $\bar{x}(t+k)$

Remark 5.1: Notice that the error $e(t+k)$ it is shown at time t due to the fact that $x(t+k)$ it is not available for measurement. Therefore, the estimate $\bar{x}(t+k)$ is delayed k units of time to generate $e(t)$.

VI. Conclusions

In this paper a Smith-type compensator is proposed to estimate the future values of the state of a class of time delay system with delays at the input signal and at the state under the assumption that the systems is open loop unstable. The proposed compensator provides an approximate prediction of the future values the state of the system. It is shown that the estimation error converges to zero due to the consideration of an additional internal feedback that stabilized the error dynamics. The performance of the proposed compensator is evaluated by means of numerical simulations.

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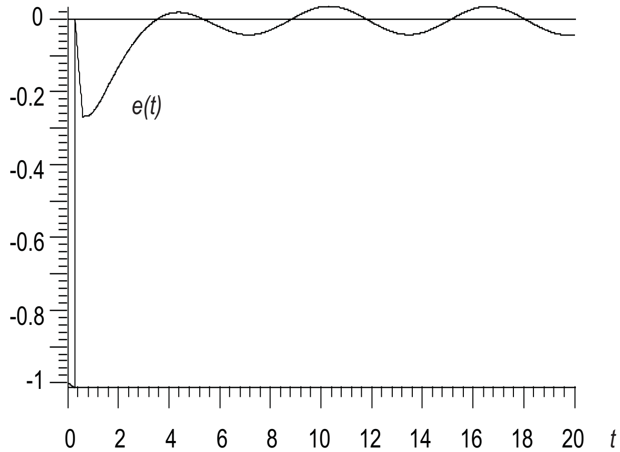

Fig. 4. Error signal $e(t)$

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