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Output regulation for a class of nonlinear systems with unknown sinusoidal disturbances

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***Abstract*— A stabilizable nonlinear system in output feedback form subject to unknown sinusoidal disturbances generated by an unknown linear exosystem is considered: only an upper bound on the exosystem's order is supposed to be known. A nonlinear output error feedback control is designed to achieve output regulation. An application to a mechanical system is presented to illustrate the effectiveness the proposed control technique.**

I. Introduction and problem statement

The class of nonlinear systems

$$\begin{cases} \dot{x} = \Phi(y) + A_n x + bu + Dw \\ y = C_n x \end{cases} \quad (1)$$

$$\dot{w} = Rw \quad (2)$$

is considered, where $x \in \Re^n$ is the state, $u \in \Re$ is the control input, $w \in \Re^{2m+1}$ is the exosystem state, $y \in \Re$ is a measurable output to be regulated to zero, with $b = [-b_{n-1}, \ \ldots, \ -b_0]^T$ known constant vector, $\Phi(\cdot)$ ($\Phi(0) = 0$) known smooth vector function of its argument; $A_j \in \Re^j \times \Re^j$, and $C_j \in \Re^j$, are defined as $A_j = \begin{bmatrix} 0 & I_{j-1} \\ 0 & 0 \end{bmatrix}_{j \times j}$, $C_j = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1 \times j}$, j being a positive integer. The disturbances Dw to be rejected are generated by the unknown exosystem $\dot{w} = Rw$, with initial condition $w(0) = w_0$. The regulation problem consists in designing an output feedback compensator which drives exponentially to zero the output $y \in \Re$ of system (1), (2), on the basis of its measurement only. The following assumptions are made on system (1), (2):

(H1) The undisturbed system obtained by setting in system (1) $w = 0$ is globally exponentially stabilizable, i.e. there exists an output feedback controller

$$\dot{Y} = \bar{L}(Y, y); \ u_S = \bar{M}(Y, y); \quad (3)$$

with $Y \in \Re^s$; $s \geq 0$, such that the origin $x = 0$, $Y = 0$ of the closed loop undisturbed system obtained by setting in system (1) $w = 0$, and $u = u_S = \bar{M}(Y, y)$, is globally exponentially stable. **(H2)** The couple (D, R) is unknown and the exosystem $\dot{w} = Rw$ has $2m + 1$ simple distinct eigenvalues on the imaginary axis, i.e. the spectrum of R is $\{0, \ \pm j\omega_h, 1 \leq h \leq m\}$, with ω_h unknown distinct positive integers and m an unknown integer such that $0 \leq m \leq M$: M is a known upper bound. **(H3)** The polynomial $p_b(s) = b_{n-1}s^{n-1} + \ldots + b_1 s + b_0$ has no roots coinciding with eigenvalues of the matrix R.

In the last decades the problem of adaptive compensation of sinusoidal disturbances has attracted a considerable attention. As far as nonlinear systems are concerned, several results are available under the minimum phase assumption (MP): the semiglobal output regulation problem is addressed in [18] for systems with unknown parameters in the exosystem when bounds on the frequencies and the disturbances are known; a global robust state feedback control scheme is presented in [15] for systems affected by both structured unknown disturbances and an unknown noise, following earlier work in [14]. The global output tracking problem is studied in [1] for uncertain cascaded systems in lower triangular form coupled with a neutrally stable exosystem, the output regulation problem is addressed in [19] for a class of large-scale nonlinear interconnected systems perturbed by a neutrally stable exosystem via a decentralized error feedback controller. Global output feedback regulators for the same class of nonlinear systems considered in this paper under the MP assumption have been proposed in [3] following [2]; semiglobal output feedback regulators have been described in [17]. Preliminary results on the semiglobal regulation of non-minimum phase (NMP) systems can be found in [6], for classes of NMP systems which are more general than those considered in this paper, under the assumption of sinusoidal disturbances with known frequency. A global regulation algorithm is presented in [11] for the class of systems (1), (2) assuming that the number of frequencies disturbing the system is known, and a control strategy is described in [10] for linear systems affected by sinusoidal disturbances, removing the hypothesis that the number of frequencies is known. The contribution of this paper is to extend the ideas in [11] and [10] by presenting a global solution to the problem of designing an output feedback compensator for the class of nonlinear systems (1), (2) assuming that only an upper bound of the number of sinusoidal disturbances is known: the regulation problem for a mechanical system affected by sinusoidal disturbances is discussed as an illustrative application.

II. Regulator design

By virtue of (H2) and (H3), there exists a matrix (see [5]) $\Pi \in \Re^n \times \Re^{2m+1}$ and a row vector $\Gamma \in \Re^{2m+1}$ that are the solution of the regulator equations

$$\Pi R = A_n \Pi + b\Gamma + D = 0; \quad C_n \Pi = 0. \quad (4)$$

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The coordinate transformation $e = x - \Pi w$, $u_r = \Gamma w$ yields an error system with disturbances u_r appearing in "matching condition"

$$\begin{array}{rcl} \dot{e} & = & \Phi(y) + A_n e + b(u - u_r) \\ y & = & C_n e. \end{array} \quad (5)$$

The disturbance input $u_r(t) = \Gamma w$ (recall (4)) is the sum of a bias and m sinusoids with unknown m and unknown constant frequencies ω_i, $1 \le i \le m$. Defining $\theta = [\theta_1,\ \theta_2,\ \ldots, \theta_m]^T$ so that

$$s^{2m} + \theta_1 s^{2m-2} + \ldots + \theta_{m-1} s^2 + \theta_m = \prod_{i=1}^{m} \left(s^2 + \omega_i^2\right), \quad (6)$$

there exists a linear invertible transformation such that the disturbance $u_r(t) = \Gamma w$ can be modelled in the new coordinates $\bar{w} \in \Re^{2m+1}$ as the output of the following $(2m+1)-$order linear exosystem with eigenvalues on the imaginary axis parametrized by the vector $\theta \in \Re^m$

$$\dot{\bar{w}} = \begin{bmatrix} S(\theta) & 0 \\ 0 & 0 \end{bmatrix} \bar{w}; \ u_r = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \end{bmatrix} \bar{w}, \quad (7)$$

with $S(\theta) = A_{2m} - [0,\ \theta_1,\ 0,\ \theta_2,\ 0,\ \ldots, \theta_m]^T C_{2m}$.

A. A global filtered transformation

The aim of this section is to transform (5) and (7) into an "adaptive observer form" (see [7]). The first step is to introduce a nonlinear filter ($z \in \Re^n$)

$$dz/dt \ = \ [A_n - aC_n]\, z + \Phi(y) + ay + bu, \quad (8)$$

where $a = [a_{n-1},\ a_{n-2}\ ,\ldots,\ a_0]^T$ and $a_i \in \Re^+$, $0 \le i \le n-1$, are design parameters such that the polynomial $p_a(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0$ has all its roots with negative real part. By setting $\bar{e} = e - z$, from (5), (8), we obtain the linear error dynamics

$$d\bar{e}/dt = [A_n - aC_n]\, \bar{e} - bu_r; \quad \bar{e}_1 = C_n \bar{e} = y - z_1. \quad (9)$$

The autonomous system (9), (7) with state $\left[\bar{e}^T,\ \bar{w}^T\right]^T \in \Re^{2m+n+1}$ by virtue of (H3) is observable from the available output $\bar{e}_1$ for every θ complying with (6); hence it is transformed into the observer canonical form ($\zeta \in \Re^{2m+n+1}$)

$$\begin{array}{rcl} d\zeta/dt & = & A_{n+2m+1}\zeta - \bar{a}_0\,[m]\,\bar{e}_1 - \sum_{i=1}^{m} \theta_i \bar{a}_i\,[m]\,\bar{e}_1, \\ \bar{e}_1 & = & C_{n+2m+1}\zeta \end{array} \quad (10)$$

with $\bar{a}_i\,[m] \in \Re^{n+2m+1}$, $i \in [0, m]$ given by

$$\begin{array}{l} \bar{a}_0\,[m] = [a_{n-1}, ..., a_0, 0, 0, 0, ..., 0, 0, 0]^T, \\ \bar{a}_1\,[m] = [0, 1, a_{n-1}, ..., a_0, 0, 0, ..., 0, 0]^T, \\ \bar{a}_2\,[m] = [0, 0, 0, 1, a_{n-1}, ..., a_0, 0, ..., 0]^T, \ldots, \\ \bar{a}_m\,[m] = [0, 0, 0, ..., 0, 1, a_{n-1}, ..., a_0, 0]^T, \end{array} \quad (11)$$

by a linear transformation (which is nonsingular for all θ complying with (6)) expressed in matrix form as $\zeta = T_m(\theta)\left[\bar{e}^T,\ \bar{w}^T\right]$ where the columns $t_i\,[m]$, $i \in [1, n+2m+1]$ of the matrix $T_m(\theta) = [t_1\,[m],\ t_2\,[m], \ldots, t_{n+2m+1}\,[m]]$ are given by

$$\left\{ \begin{array}{l} t_1\,[m] = [1, 0, \theta_1, 0, \theta_2, 0, ..., 0, \theta_m, 0, ..., 0]^T, \\ t_2\,[m] = [0, 1, 0, \theta_1, 0, \theta_2, ..., 0, \theta_m, 0, ..., 0]^T, \ldots, \\ t_n\,[m] = [0, ..., 0, 1, 0, \theta_1, 0, \theta_2, ..., 0, \theta_m, 0]^T, \\ t_{n+1}\,[m] = [0, b_{n-1}, b_{n-2}, ..., b_0, 0, 0, ..., 0]^T, \\ t_{n+2}\,[m] = [0, 0, b_{n-1}, b_{n-2}, ..., b_0, 0, ..., 0]^T, \ldots, \\ t_{n+2m}\,[m] = [0, ..., 0, b_{n-1}, b_{n-2}, ..., b_0, 0]^T, \\ t_{n+2m+1}\,[m] = t_{n+1}\,[m] + \sum_{i=1}^{m-1} \theta_i t_{n+2i+1}\,[m] \\ +\theta_m\,[0, 0, \ldots, 0, 0, b_{n-1}, b_{n-2}, \ldots, b_0]^T. \end{array} \right. \quad (12)$$

Set $\bar{d}\,[m] = [d_1, \ldots, d_{n+2m}]^T \in \Re^{n+2m}$ where $d_i \in \Re^+$, $1 \le i \le n+2m$, are positive real numbers such that all the roots of $p_d(s) = s^{n+2m} + d_1 s^{n+2m-1} + \cdots + d_{n+2m}$ have negative real part. Define as in [7] the filters ($\xi_i \in \Re^{n+2m}$, $\mu_i \in \Re$, $1 \le i \le m$)

$$\begin{array}{rcl} d\xi_i/dt & = & \left[A_{n+2m} - \bar{d}\,[m]\,C_{n+2m}\right]\xi_i \\ & & -[0,\ I_{n+2m}]\,\bar{a}_i\,[m]\,\bar{e}_1, \\ \mu_i & = & C_{n+2m}\xi_i. \end{array} \quad (13)$$

According to [7] the filtered transformation $\bar{\zeta} = \zeta - \begin{bmatrix} 0 \\ \sum_{i=1}^{m} \xi_i \theta_i \end{bmatrix}$, with $\bar{\zeta} \in \Re^{n+2m+1}$, mapping the state vector ζ into a new state vector $\bar{\zeta}$, transforms system (10) into an "adaptive observer" form

$$\begin{array}{rcl} d\bar{\zeta}/dt & = & A_{n+2m+1}\bar{\zeta} - \bar{a}_0\,[m]\,\bar{e}_1 + \begin{bmatrix} 1 \\ \bar{d}\,[m] \end{bmatrix} \mu^T \theta; \\ \bar{e}_1 & = & C_{n+2m+1}\bar{\zeta}, \end{array} \quad (14)$$

where $\mu = [\mu_1,\ \mu_2,\ \ldots \mu_m]^T$. The transformation from $\bar{\zeta}$ to $\left[\bar{e}^T\ \bar{w}^T\right]^T$ is given by

$$\begin{pmatrix} \bar{e} \\ \bar{w} \end{pmatrix} = T_m^{-1}(\theta)\left(\bar{\zeta} + \begin{bmatrix} 0 \\ \sum_{i=1}^{m} \xi_i \theta_i \end{bmatrix}\right). \quad (15)$$

Notice that the map $T_m^{-1}(\theta)$ is well defined since by (H2) and (H3) the matrix $T_m(\theta)$ is invertible for all θ complying with (6). If $t_m^*(\theta)$ denotes the sum of the $(n+1)$-th and the $(n+2m+1)$-th rows of the adjoint of the matrix $T_m(\theta)$, then the sinusoidal disturbance $u_r(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \end{bmatrix} \bar{w}$ by (15) can be expressed as

$$u_r(t) = \frac{1}{\det T_m(\theta)} t_m^*(\theta)\left(\bar{\zeta}(t) + \begin{bmatrix} 0 \\ \sum_{i=1}^{m} \xi_i(t)\theta_i \end{bmatrix}\right). \quad (16)$$

B. How to detect the number of excited frequencies

A key step in the regulator design is to build a dynamical system whose residual outputs are related to the number of excited frequencies. To this purpose, three cascaded filters are introduced: the first filter is defined as

$$\left\{ \begin{array}{rcl} d\eta/dt & = & A_\eta \eta + [0, \ldots, 0, 1]^T \bar{e}_1 \\ \nu_i & = & \eta_{2M-2i+4},\ 1 \le i \le M+1 \end{array} \right. \quad (17)$$

with state $\eta = [\eta_1,\ \eta_2,\ \ldots \eta_{2M+2}]^T \in \Re^{2M+2}$, initial condition $\eta(0) \in \Re^{2M+2}$, input $\bar{e}_1(t)$ given in (9), output

$\nu = [\nu_1,\ \nu_2,\ \ldots \nu_{M+1}]^T$, where $A_\eta = A_{n+2M+2} - [0,\ldots,0,1]^T [\alpha_0,\ \alpha_1,\ldots,\alpha_{2M+1}]$, and the design parameters α_i, $0 \le i \le 2M+1$ are such that the polynomial $p_\alpha(s) = s^{2M+2} + \alpha_{2M+1}s^{2M+1} + ... + \alpha_1 s + \alpha_0$ has all its roots with negative real part. The dynamical system (17) is introduced to generate the output $\nu \in \Re^{M+1}$ whose first i entries $\bar{\nu}_i = [\nu_1(t),\ \nu_2(t),\ \ldots,\nu_i(t)]$ with $i \in [1, M+1]$ can be shown to be persistently exciting if $1 \le i \le m$ and not persistently exciting if $m+1 \le i \le M+1$. The vector $\nu(t) \in \Re^{M+1}$ is the input to the second filter

$$\left\{ \begin{array}{lcl} \frac{d\Omega}{dt} & = & -c_1\Omega + \nu\nu^T,\ \Omega(0) > 0, \\ q_i & = & \left|\det(\Omega_i)\right|^{\frac{1}{i}},\ 1 \le i \le M+1 \end{array} \right. \quad (18)$$

with state $\Omega \in \Re^{M+1} \times \Re^{M+1}$ symmetric and positive definite initial condition $\Omega(0) > 0$, outputs $q_i(t)$, $1 \le i \le M+1$, where $\Omega_i \in \Re^i \times \Re^i$, $1 \le i \le M+1$, denotes the matrix collecting the first $i \times i$ entries of Ω and $c_1 \in \Re^+$ is a design parameter. Notice that Ω is symmetric and (18) can be implemented by a filter whose dimension is $\left(M^2 + 3M + 2\right)/2$.

It can be shown by virtue of the persistency of excitation condition that the outputs $q_i(t)$ with $1 \le i \le M+1$ of system (18) comply with the property that

$$q_i(t) \ge q_M > 0 \text{ for } 1 \le i \le m, \quad (19)$$

where q_M is a suitable positive real while $q_i(t)$ with $m+1 \le i \le M+1$ are exponentially vanishing. The outputs $q_i(t)$ of filter (18) are the inputs of the third filter

$$d\chi_i/dt \quad = \quad -\left[\sigma_i(q_i) + \psi(\chi_i)\right]\chi_i + \tilde{\sigma}_i(q_{M+1}), \quad (20)$$

with $1 \le i \le M$, state $\chi = (\chi_1, \ldots, \chi_M)^T$, in which $\chi_i(0) > 0$; $\sigma_i(\cdot)$ and $\tilde{\sigma}_i(\cdot)$ with $1 \le i \le M$ are suitable class $\mathcal{K}$ functions. The function $\psi(\chi_i)$ depends on a design parameter $\chi_0 \in \Re^+$ and is defined as $\psi(\chi_i) = 0$ if $\chi_i \le \chi_0$; $\psi(\chi_i) = 4\left(\chi_i - \chi_0\right)^2/\chi_i^2$ if $\chi_0 \le \chi_i \le 2\chi_0$ and $\psi(\chi_i) = 1$ if $2\chi_0 \le \chi_i$. It can be shown that $\chi_i(t)$ for $1 \le i \le m$ are globally vanishing functions and $\chi_i(t) \ge \bar{\chi} > 0$ for all $t \ge 0$ and $m+1 \le i \le M$, where $\bar{\chi}$ is a suitable positive real: hence the cascaded filters allow us to asymptotically determine the number $m \in [0, M]$ of excited frequencies.

Lemma 1: Consider the cascaded interconnection of the filters (17), (18), (20) with input $\bar{e}_1 \in \Re$ given by (9), state $\eta \in \Re^{2M+2}$, $\Omega \in \Re^{M+1} \times \Re^{M+1}$, $\chi \in \Re^M$, and outputs

$$\left\{ \begin{array}{lll} \beta_i(t) = 1 & \text{if} & q_i(t) > c_2\chi_i(t) \\ \beta_i(t) = q_i/(c_2\chi_i) & \text{if} & q_i(t) \le c_2\chi_i(t) \end{array} \right. \quad (21)$$

where $i \in [1, M]$ and $c_2 \in \Re^+$ is a design parameter. Then the following holds: (i) the state trajectories are bounded for any $\eta(0) \in \Re^{2M+2}$, $\Omega(0) \in \Re^{M+1} \times \Re^{M+1}$ such that $\Omega(0) > 0$, and any $\chi_i(0) > 0$, $1 \le i \le M$; (ii) the functions $\beta_i(t)$, $i \in [1, M]$, are such that $0 \le \beta_i(t) \le 1$, for all $t \ge 0$, and

$$\left\{ \begin{array}{lll} \lim\limits_{t\to\infty} \beta_i(t) & = 1 & \text{for } 1 \le i \le m, \\ \lim\limits_{t\to\infty} \beta_i(t) & = 0 & \text{for } m+1 \le i \le M, \end{array} \right. \quad (22)$$

where the functions $\beta_i(t)$, $i \in [1, M]$, tend exponentially to their limits.

C. Adaptive regulator design with frequencies identification

In this section we design an adaptive observer for system (14) and determine an estimate $\widehat{u}_r(t)$ of the disturbance $u_r(t)$ via (16); the control input u is the sum of $\widehat{u}_r(t)$ and the function $\bar{M}(Y, y)$ in (3). To this purpose we define the diagonal matrix $\bar{U}(t) \in \Re^{n+2M} \times \Re^{n+2M}$, with entries $\bar{U}_{i,i}(t) = 1$ for $1 \le i \le n$ and $\bar{U}_{i,i}(t) = \beta_k(t)$, with $k = \left\lceil \frac{i-n}{2} \right\rceil$ for $n+1 \le i \le n+2M$. Consider the vector $\bar{\beta}(t) = \left[\bar{\beta}_0(t), \ldots, \bar{\beta}_M(t)\right]$ defined as

$$\left\{ \begin{array}{lcl} \bar{\beta}_0(t) & = & (1 - \beta_1(t));\ \bar{\beta}_M(t) = \beta_M(t) \\ \bar{\beta}_i(t) & = & \beta_i(t)(1 - \beta_{i+1}(t)) \text{ for } 1 \le i \le M-1 \end{array} \right. \quad (23)$$

along with the vector $\bar{\delta}(t) = \bar{\beta}_0(t)\begin{pmatrix} \bar{d}[0] \\ 0 \end{pmatrix} + \ldots + \bar{\beta}_{M-1}(t)\begin{pmatrix} \bar{d}[M-1] \\ 0 \end{pmatrix} + \bar{\beta}_M(t)\begin{pmatrix} \bar{d}[M] \end{pmatrix}$, where the entries of the constant vectors $\bar{d}[i] \in \Re^{n+2i}$, $0 \le i \le M$ are design parameters such that the polynomials $s^{n+2i} + \left[s^{n+2i-1},\ s^{n+2i-2},\ \ldots, 1\right]\bar{d}[i]$ are Hurwitz. Notice that by (22)

$$\left\{ \begin{array}{ll} \lim\limits_{t\to\infty} \bar{\beta}_m(t) & = 1 \\ \lim\limits_{t\to\infty} \bar{\beta}_i(t) & = 0, \forall i \ne m \end{array} \right. \quad (24)$$

$$\lim_{t\to\infty} \bar{U}(t) = \left[\begin{array}{c|c} I_{n+2m} & 0 \\ \hline 0 & 0 \end{array} \right],\ \lim_{t\to\infty} \bar{\delta}(t) = \left[\begin{array}{c} \bar{d}[m] \\ 0 \end{array} \right] \quad (25)$$

where the entries of the matrix $\bar{U}(t)$ and of the vector $\bar{\delta}(t)$ tend exponentially to their limits. The matrix $\bar{U}(t)$ and the vector $\bar{\delta}(t)$ are tools to construct a generalization of the filters (13) that by virtue of (25) are adaptive with respect to the unknown number m: they are defined as

$$\left\{ \begin{array}{lcl} d\widehat{\xi}_i/dt & = & \bar{U}\left\{\left(A_{n+2M} - \bar{\delta}(t)C_{n+2M}\right)\widehat{\xi}_i \right. \\ & & \left. - \left([0,\ I_{n+2M}]\,\bar{a}_i[M]\right)\bar{e}_1\right\} \\ & & -c_3\left(I_{n+2M} - \bar{U}\right)\widehat{\xi}_i, \\ \widehat{\mu}_i & = & \beta_i C_{n+2M}\widehat{\xi}_i;\quad 1 \le i \le M \end{array} \right. \quad (26)$$

with state variables $\widehat{\xi}_i \in \Re^{n+2M}$, $1 \le i \le M$, arbitrary initial conditions $\widehat{\xi}_i(0) \in \Re^{n+2M}$, where $c_3 \in \Re^+$ is a positive design parameter and $\bar{a}_i[M]$, $1 \le i \le M$ are defined according to (11) with M in place of m. We consider now an observer for system (14) which is adaptive with respect to the unknown number m of excited frequencies

$$\left\{ \begin{array}{lcl} d\widehat{\zeta}/dt & = & U(t)\left\{\left(A_{n+2M+1} - K(t)C_{n+2M+1}\right)\widehat{\zeta} \right. \\ & & \left. + \left(K(t) - \bar{a}_0[M]\right)\bar{e}_1 + \delta(t)\sum_{i=1}^{M}\widehat{\mu}_i\widehat{\theta}_i\right\} \\ & & -c_4\left[I_{n+2M+1} - U(t)\right]\widehat{\zeta}; \\ d\widehat{\theta}_i/dt & = & g_i\widehat{\mu}_i(t)\left[\bar{e}_1 - \widehat{\zeta}_1\right] - \bar{g}_i\left[1 - \beta_i(t)\right]\widehat{\theta}_i; \\ & & 1 \le i \le M \end{array} \right. \quad (27)$$

with $\widehat{\zeta} \in \Re^{n+2M+1}$, $\widehat{\theta} = \left[\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_M\right]^T \in \Re^M$, arbitrary initial conditions $\widehat{\zeta}(0) \in \Re^{n+2M+1}$, $\widehat{\theta}(0) \in \Re^M$, in which $\widehat{\zeta}_1 = C_{n+2M+1}\widehat{\zeta}$, $g_i, \bar{g}_i \in \Re^+$, $c_4 \in \Re^+$ are design parameters and $\bar{a}_0[M]$ is defined according to (11) with M in place of m; the matrix $U(t) \in \Re^{n+2M+1} \times \Re^{n+2M+1}$ and the vector $\delta(t) \in \Re^{n+2M+1}$ are defined as $U(t) = \begin{bmatrix} 1 & 0 \\ 0 & \bar{U}(t) \end{bmatrix}$, $\delta(t) = \begin{bmatrix} 1 \\ \bar{\delta}(t) \end{bmatrix}$; the vector $K(t) \in \Re^{n+2M+1}$ is $K(t) = (A_{n+2M+1} + \lambda I_{n+2M+1})\,\delta(t)$ with $\lambda \in \Re^+$ design parameter. Let j be an integer such that $j \in [0, M]$; consider a partition of the vectors $\widehat{\xi}_i \in \Re^{n+2M}$ $1 \le i \le M$, $\widehat{\zeta} \in \Re^{n+2M+1}$, and $\widehat{\theta} \in \Re^M$ into subvectors whose dimension depends on the integer j as follows:

$$\widehat{\xi}_i = \begin{bmatrix} \widehat{\xi}_i^{[j]} \in \Re^{n+2j} \\ \breve{\xi}_i^{[j]} \in \Re^{2(M-j)} \end{bmatrix} 1 \le i \le M;$$
$$\widehat{\zeta} = \begin{bmatrix} \widehat{\zeta}^{[j]} \in \Re^{n+2j+1} \\ \breve{\zeta}^{[j]} \in \Re^{2(M-j)} \end{bmatrix};\ \widehat{\theta} = \begin{bmatrix} \widehat{\theta}^{[j]} \in \Re^{j} \\ \breve{\theta}^{[j]} \in \Re^{M-j} \end{bmatrix}. \tag{28}$$

By virtue of (25) it can be shown that the partition in (28) obtained by setting $j = m$ complies with the following properties.

Lemma 2: Consider the filters (26) and the observer (27). Set in (28) $j = m$. Then: (i) $(\xi_i - \widehat{\xi}_i^{[m]})$, $\breve{\xi}_i^{[m]}$, with $1 \le i \le m$, $(\bar{\zeta} - \widehat{\zeta}^{[m]})$, $\breve{\zeta}^{[m]}$, $(\theta - \widehat{\theta}^{[m]})$, $\breve{\theta}^{[m]}$, tend exponentially to zero for any initial condition of the systems (14), (26), (27); (ii) $\widehat{\xi}_i \in \Re^{n+2M}$, with $1 \le i \le M$, $\widehat{\theta} \in \Re^M$ and $\widehat{\zeta} \in \Re^{n+2M+1}$ are bounded.

By virtue of Lemmas 1 and 2 we construct an estimate of the disturbance $u_r(t)$ in (16). To this purpose, let $j \in [0, M]$ and consider the matrix $T_j(\widehat{\theta}^{[j]}) \in \Re^{n+2j+1} \times \Re^{n+2j+1}$ whose columns are obtained from (12) with j in place of m and $\widehat{\theta}^{[j]} \in \Re^j$ in place of $\theta \in \Re^m$. Let $t_j^*(\widehat{\theta}^{[j]})$ be the sum of the $(n+1)$-th and the $(n+2j+1)$-th rows of the adjoint of the matrix $T_j(\widehat{\theta}^{[j]})$ for $j \in [1, M]$ and $t_0^*(\widehat{\theta}^{[0]})$ be the $(n+1)$-th row of the matrix $T_0(\widehat{\theta}^{[0]})$. Set

$$\widehat{\rho}_j(t) = t_j^*(\widehat{\theta}^{[j]}) \left(\widehat{\zeta}^{[j]}(t) + \begin{bmatrix} 0 \\ \sum_{i=1}^{M} \widehat{\xi}_i^{[j]}(t)\widehat{\theta}_i(t) \end{bmatrix} \right), \tag{29}$$

with $j \in [0, M]$. Consider the filters

$$dp_j/dt = -c_5 p_j + c_6\left(1 - \bar{\beta}_j(t)\right) + c_7\left|\bar{e}_1 - \hat{\zeta}_1\right|, \tag{30}$$

with $j \in [0, M]$, state $p = [p_0,\ p_1, \ldots p_M] \in \Re^{M+1}$ driven by the inputs $\left(1 - \bar{\beta}_j(t)\right)$, $j \in [0, M]$, along with the estimation error $\left|\bar{e}_1 - \widehat{\zeta}_1\right|$, where c_5, c_6, c_7 are positive design parameters. Notice that if $p_j(0) > 0$ then all $p_j(t)$ with $j \ne m$ are greater than a positive lower bound, while $p_m(t)$ tends exponentially to zero as t goes to infinity. The estimate $\hat{u}_r$ for the disturbance $u_r(t)$ in (16) is defined by the adaptive saturation algorithm

$$\widehat{u}_r(t) = \sum_{j=0}^{M} \left(\bar{\beta}_j(t)\widehat{u}_j(t)\right) \text{ where}$$
$$\widehat{u}_j(t) = \begin{cases} \frac{\widehat{\rho}_j(t)}{\det T_j(\widehat{\theta}^{[j]}(t))} & \text{if } \left|\det T_j(\widehat{\theta}^{[j]})\right| > p_j, \\ \frac{\widehat{\rho}_j(t) \det T_j(\widehat{\theta}^{[j]}(t))}{p_j^2} & \text{if } \left|\det T_j(\widehat{\theta}^{[j]})\right| \le p_j. \end{cases} \tag{31}$$

The task of the signals $p_j(t)$ is to avoid the singularities in which $\det T_j(\widehat{\theta}^{[j]}) = 0$, while the functions $\bar{\beta}_j(t)$ by virtue of (22) select the correct disturbance estimate $\widehat{u}_m(t)$ in the set $[\widehat{u}_0(t), \widehat{u}_1(t), \ldots, \widehat{u}_M(t)]$. The overall compensating control law, which is the sum of a stabilizing part and of a disturbance rejection part, is defined as

$$u = \bar{M}(Y, y) + \hat{u}_r(t). \tag{32}$$

At this point we are ready to state the main result.

Proposition 1: Consider system (1), (2). Assume that hypotheses (H1)-(H3) hold. Then the dynamic output feedback compensator (8), (17), (18), (20), (26), (27), (30), (31), (3), (32) whose state is $z \in \Re^n$ $\eta \in \Re^{2M+2}$, $\Omega \in \Re^{M+1} \times \Re^{M+1}$, $\chi \in \Re^M$, $\widehat{\xi}_i \in \Re^{n+2M}$, $1 \le i \le M$, $\widehat{\zeta} \in \Re^{n+2M+1}$, $\widehat{\theta} \in \Re^M$, $p \in \Re^{M+1}$, $Y \in \Re^s$, yields a closed loop system such that: its trajectories are bounded and the output $y(t)$ tends exponentially to zero as t goes to infinity for every initial condition $x(0) \in \Re^n$, $w(0) \in \Re^{2m+1}$ of system (1), (2), every initial condition of the dynamic compensator $z(0) \in \Re^n$ $\eta(0) \in \Re^{2M+2}$, $\Omega(0) \in \Re^{M+1} \times \Re^{M+1}$, $\chi(0) \in \Re^M$, $\widehat{\xi}_i(0) \in \Re^{n+2M}$, $1 \le i \le M$, $\widehat{\zeta}(0) \in \Re^{n+2M+1}$, $\widehat{\theta}(0) \in \Re^M$, $p(0) \in \Re^{M+1}$, $Y(0) \in \Re^s$ with $\det\Omega(0) > 0$, $p_j(0) > 0$, $j \in [0, M]$, $\chi_i(0) > 0$, $i \in [1, M]$.

III. SET POINT REGULATION OF LINK-SHAFT MECHANICAL SYSTEM

Consider a physical system modelled by two interconnected rigid bodies: a link and a cylindrical shaft represented in Figure 1. The link rotates about one of its ends around the shaft while the shaft is elastically coupled to a frame that provides a disturbance which is a sum of sinusoidal functions. The angular position of the link is controlled via an actuator that is placed between the link and the shaft. The torque provided by the motor is transmitted to the link via a gearbox. Let ϕ_1 and ϕ_2 be the angular positions with respect to a fixed reference frame of the link and of the shaft respectively. The system is described by two second order differential equations

$$\begin{cases} J_1 \frac{d^2\phi_1}{dt^2} &= -B\left(\frac{d\phi_1}{dt} - \frac{d\phi_2}{dt}\right) + \frac{T}{N} + M_l g l \sin\phi_1 \\ J_2 \frac{d^2\phi_2}{dt^2} &= B\left(\frac{d\phi_1}{dt} - \frac{d\phi_2}{dt}\right) - T - k_s\phi_2 + \Delta \end{cases} \tag{33}$$

where J_1 and J_2 are the inertias of the link and the shaft, B the viscous friction constant, k_s is the elasticity constant of the spring representing the elastic coupling between the shaft and the frame; M_l and l are respectively the mass and the position of the center of gravity of the link and

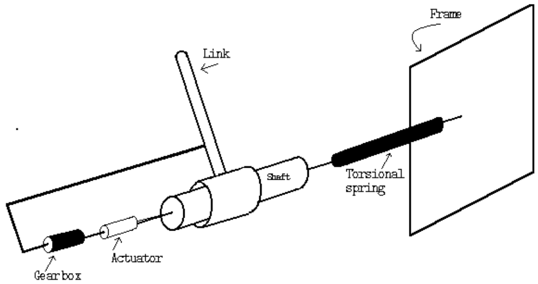

Fig. 1. The link-shaft system model.

N , $0 < N < \infty$, is the gearbox transmission ratio. We assume these quantities to be known. The frame generates a torque disturbance Δ which is a sum of unknown sinusoidal disturbances. The system input is the torque T provided by the actuator on the shaft. Consider the problem of the set point regulation to the fixed reference ϕ_{1r} of the angular position of the link ϕ_1 assuming that the output $y = \phi_1 - \phi_{1r}$ is available for measurement, with control input $u = T + NM_l gl \sin\phi_{1r}$, when the shaft is affected by the sinusoidal disturbances Δ originating from the frame to which the shaft is connected. The regulation of system (33) can be interpreted in the framework of vibration isolation for single link robot arms, or for ship born cranes, antennas, or other devices that are required to point a prescribed direction under ship rolling. By introducing the coordinates

$$\begin{aligned} x_1 &= (\phi_1 - \phi_{1r}) \\ x_2 &= B\left(\frac{1}{J_1} + \frac{1}{J_2}\right)(\phi_1 - \phi_{1r}) + \frac{d\phi_1}{dt} \\ x_3 &= \left(\frac{k_s}{J_2}\right)(\phi_1 - \phi_{1r}) + \left(\frac{B}{J_2}\right)\frac{d\phi_1}{dt} + \left(\frac{B}{J_1}\right)\left(\frac{d\phi_2}{dt}\right) \\ x_4 &= \left(\frac{Bk_s}{J_1 J_2}\right)\left(\phi_1 - \phi_{1r} - \phi_2 + \frac{NM_l gl}{k_s}\sin\phi_{1r}\right) \\ &\quad + \left(\frac{k_s}{J_2}\right)\frac{d\phi_1}{dt} \end{aligned} \tag{34}$$

and by setting

$$\begin{aligned} \Phi_1(y) &= -B\left(\frac{1}{J_1} + \frac{1}{J_2}\right)y, \\ \Phi_2(y) &= -\frac{k_s}{J_2}y + \frac{M_l gl}{J_1}\left[\sin(y + \phi_{1r}) - \sin\phi_{1r}\right], \\ \Phi_3(y) &= -\frac{Bk_s}{J_1 J_2}y + \frac{BM_l gl}{J_1 J_2}\left[\sin(y + \phi_{1r}) - \sin\phi_{1r}\right], \\ \Phi_4(y) &= \frac{M_l gl k_s}{J_1 J_2}\left[\sin(y + \phi_{1r}) - \sin\phi_{1r}\right], \end{aligned}$$

the system (33) along with (34) can be seen as a special instance of a system in the form (1), (2) with $n = 4$, $\Phi(y) = [\Phi_1(y),\ \Phi_2(y),\ \Phi_3(y),\ \Phi_4(y)]^T$, $b = \left[0,\ \frac{1}{NJ_1},\ \left(\frac{B(1-N)}{NJ_1J_2}\right),\ \left(\frac{k_s}{J_1J_2N}\right)\right]^T$; the couple (D, R) along with the exosystem state w depend on the number and on the parameters values of the sinusoids in Δ. System (33) along with (34) is minimum phase if $N < 1$ and is non-minimum phase if $N > 1$. We simulate the performance of the proposed algorithm on a single link robot arm to regulate the link angular position $\phi_1(t)$ to the reference value $\phi_{1r} = \frac{\pi}{12}$, choosing the system parameters

$$\begin{array}{lllllll} J_1 &=& 10[Nms^2/rad], & k_s &=& 500[Nm/rad], \\ J_2 &=& 30[Nms^2/rad], & N &=& 20, \\ B &=& 20[Nms/rad], & M_l gl &=& 2[Nm/rad]. \end{array} \tag{35}$$

The disturbance $\Delta \in \Re$ is expressed by the function

$$\Delta(t) = \begin{cases} 150\,(\sin 2t) & \text{for } t \leq 120 \\ 150\,(\sin 2t) + 250\,(\sin 3t) & \text{for } 120 \leq t. \end{cases} \tag{36}$$

According to (H2) we set $M = 2$ so that any reference signal in (36) may be generated by an exosystem whose order is at most five. The unforced dynamics of system (33) obtained by setting $\Delta = 0$ are not globally exponentially stable; however the output feedback control law $u_s(t) = -200y$ globally exponentially stabilizes the system when $\Delta = 0$. We set in system (17) α_0 =64.00, α_1 =140.20 α_2 =145.56 α_3 =91.26 α_4 =36.39 α_5 =8.76. In (18) we set $c_1 = 4$, $Q(0) = 10^{-3} \cdot I$, in (20) we set $\sigma_1(q_1) = \arctan\left(510^3 q_1\right)$, $\sigma_2(q_2) = \arctan\left(510^3 q_2\right)$, $\tilde{\sigma}_1(q_3) = \arctan\left(10^3 q_3\right)$, $\tilde{\sigma}_2(q_3) = \arctan\left(10^2 q_3\right)$, in expression (21) we set $c_2 = 10^{-2}$, in (26) we set $c_3 = 1$,

$$\begin{aligned} &\bar{D}(t) = \bar{\beta}_0(t)\,[4.0,\ 6.2,\ 4.2,\ 1.0,\ 0,\ 0,\ 0,\ 0]^T \\ &+\bar{\beta}_1(t)\,[6.6,\ 18.4,\ 27.1,\ 22.3,\ 9.7,\ 1.7,\ 0,\ 0]^T \\ &+\bar{\beta}_2(t)\,[9.6, 40.1, 95.5, 141.5, 133.6, 78.5, 26.2, 3.8]^T. \end{aligned}$$

We set in system (27) $c_4 = 1$, $\lambda = 30$, $g_1 = 10^6$; $g_2 = 10^7$; $\bar{g}_1 = 10^4$; $\bar{g}_2 = 10^4$; in system (30), $c_5 = 2$, $c_6 = 2000$, $c_7 = 40$. The system behavior has been tested for $0 \leq t \leq 240$. Both the system and the compensator dynamics have been simulated starting from zero initial conditions except for $\phi_1(0) = \phi_{1r}$, $\phi_2(0) = NM_l gl \sin\phi_{1r}/k_s$, $\chi_0(0) = 1.0$, and $p_j(0) = 10$, $j \in [0, 2]$. The simulation results are reported in Figures 2-5 for $0 \leq t \leq 240$. Figure 2 shows the time history of the functions $\bar{\beta}_0(t)$, $\bar{\beta}_1(t)$, $\bar{\beta}_2(t)$ that comply with conditions (24). With both exosystem models the dynamic compensator detects how many sinusoids are affecting the system, and estimates the exosystem parameters, as shown in Figure 3. In fact, for $t \leq 120$ by (36) we have $m = 1$, with $\theta_1 = 4$, as a result $\bar{\beta}_0(t) \to 0$, $\bar{\beta}_1(t) \to 1, \bar{\beta}_2(t) \to 0$, and $\hat{\theta}_1(t)$ estimates $\theta_1 = 1$ while $\hat{\theta}_2(t)$ is set to zero. For $120 \leq t$ by (36) we have $m = 2$, with $\theta_1 = 13$, $\theta_2 = 36$; in this case $\bar{\beta}_0(t) \to 0$, $\bar{\beta}_1(t) \to 0, \bar{\beta}_2(t) \to 1$, and $\hat{\theta}_1(t)$, $\hat{\theta}_2(t)$ correctly estimate $\theta_1 = 13$, $\theta_2 = 36$ respectively. In Figure 4-(A) it is described the time history of $\Delta(t)$ given by (36). In Figure 4-(B) is reported the total control torque $T(t) = -NM_l gl \sin\phi_{1r} - 200(\phi_1(t) - \phi_{1r}) + \hat{u}_r(t)$, with $\hat{u}_r(t)$ given by (30). In Figure 5 is plotted the link angular position $\phi_1(t)$ regulated to the reference value $\phi_{1r} = \frac{\pi}{12}$ in the two operating conditions.

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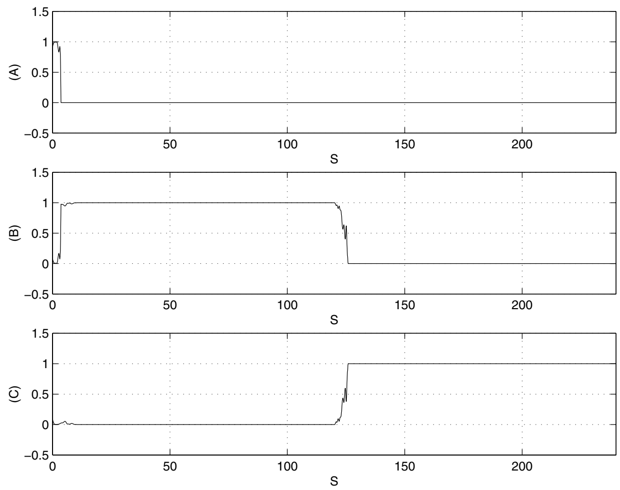

Fig. 2. In (A) is displayed the function $\bar{\beta}_0(t)$, in (B) is shown the function $\bar{\beta}_1(t)$, in (C) $\bar{\beta}_2(t)$.

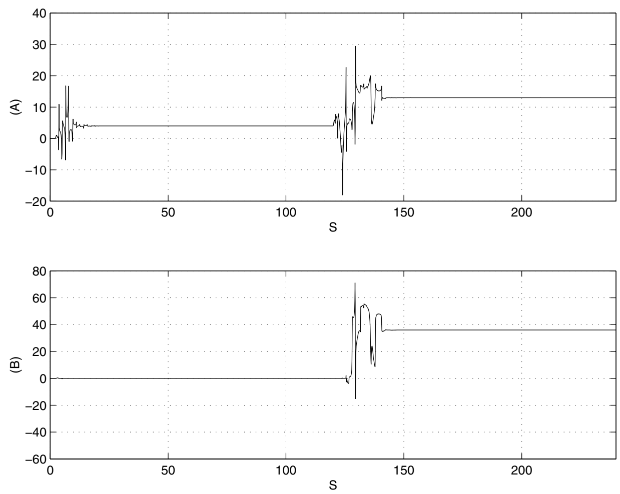

Fig. 3. in (A) is displayed the function $\hat{\theta}_1(t)$, in (B) is shown the function $\hat{\theta}_2(t)$.

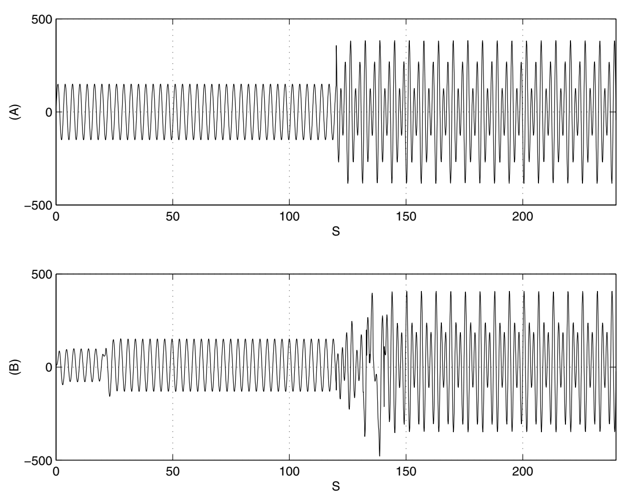

Fig. 4. In (A) is plotted the disturbance $\Delta(t)$, in (B) is displayed the control torque T, both expressed in $N \cdot m$.

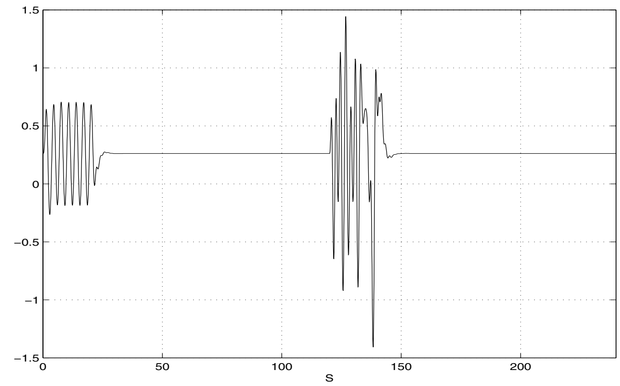

Fig. 5. The link angular position $\phi_1(t)$, expressed in rad.

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