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# Robust Performance Analysis for Linear Slowly Time-Varying Perturbations

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*Abstract*— **Robust performance analysis problem is considered for LTI systems subject to block-diagonal structured and bounded LTV perturbations with specified maximal variation rates. Analysis methods are developed in terms of semi-definite programming problems for the computation of upper and lower bounds for optimum robust performance levels. Upper bound computation is based on an IQC test developed using a generalized version of the so-called swapping lemma. Lower bound computation method employs a power-distribution-theorem-based result from the literature together with a generalized version of the KYP lemma and serves as a means to assess the conservatism of the computed upper bounds in the case of dynamic LTV perturbations.**

## I. Introduction

The philosophy of viewing control design problem as a design that is to be performed for a nominal system amenable to some perturbations that are restricted to be from a specified set lead to a wide domain of efforts commonly referred to as robust control, with the basic analysis problems known as robust stability and performance analysis (see e.g. [1]). Although there have been various approaches to stability analysis, it is especially the structured singular value analysis approach (see [2]) that made a better treatment of robust stability analysis problem for complex systems possible, by making significant use of structural information about the uncertainties. The research in this direction has also provided numerically efficient ways to test stability by employing suitable relaxations known as D-G scaling. Since it is well-established by the so-called Main Loop Theorem that the robust performance analysis problem can also be viewed as a robust stability analysis problem (by a simple extension of the uncertainty structure), tools from structured singular value analysis can as well be utilized for robust performance analysis. Combining the foundations of the methods of stability analysis that emerged during the robust control era with the classical input/output and absolute stability theory, a unifying approach was developed by [3] for robust stability analysis. Within this approach, the information on the structure and nature of the perturbations is transformed into an integral quadratic constrain (IQC) that is to be satisfied by the uncertainties (by employing suitable relaxations whenever unavoidable) and the stability test is provided in terms of an IQC that is to be satisfied by the plant. This method admits a minor extension to robust performance analysis by a suitable application of the so-called S-procedure. The numerical implementations of the tests developed within the IQC framework can be realized by using the Kalman-Yakubovich-Popov (KYP) lemma to transform the IQC's (expressed usually in terms of frequency domain inequalities) to linear matrix inequalities (LMI's) [4] and then employing the dedicated solvers (e.g. [5]). In this fashion, robust stability/performance analysis or optimization problems are typically transformed into semi-definite programming problems. In fact, it has been a crucial observation that a variety of robust analysis (as well as synthesis) problems can be re-expressed as robust LMI problems. Among various other benefits, this also allowed -to some extent- to analyze the conservatism of the relevant analysis/optimization methods, via the elegant Lagrange duality theory for optimization and in particular for semi-definite programming (see [6] for a recent exposition).

There have been various recent works on robust stability and performance analysis in the case of linear (slowly) time-varying (LTV) perturbations. A crucial discovery concerning the robust stability/performance analysis was the exactness of constant D-scaling test for arbitrary LTV perturbations [7], [8], [9]. These results were providing perfect justification for using the constant D-scaling test for analysis against arbitrary LTV perturbations, whereas on the other hand it was well-known from structured singular value analysis that the frequency-dependent D-scaling test is not generically exact in robust performance analysis for linear time-invariant (LTI) perturbations, except in some relatively simple cases [2]. The justification for using the frequency-dependent D-scaling test for LTI perturbations was strengthened by [10], which proved, using a power distribution theorem, that, if the uncertainties are allowed to vary arbitrarily slowly, then the frequency-dependent D-scaling test will provide exact analysis results. The results of [8] and [10] were merged by [11] to show that mixed constant/frequency-dependent D-scaling can be employed as an exact analysis test for mixed arbitrarily fast/slow LTV perturbations. The robust stability analysis problem was considered by [12], [13] for scalar repeated parametric uncertainties, and a result referred to as swapping lemma in the adaptive control literature was utilized to develop an IQC-based robust stability analysis test, in a way to recover the well-known analysis tests of frequency-dependent and constant D-G scaling in the limiting cases of LTI and arbitrary LTV uncertainties respectively.

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An improvement to this was reported by [14]. The problem was also studied by [15], [16] using multiplier methods for parametric uncertainties with a description that is taking into account the coupling effect between the norm and rate-of-variation bounds.

Intrigued by the results discussed above, [17] generalized the stability analysis method developed by [12], [13] in a way to facilitate robust stability analysis for general block-diagonal structured LTV uncertainties with bounded norms and rates-of-variation. As a natural extension of this work, we consider in this paper the robust performance analysis problem for LTI systems subject to block-diagonal structured and bounded LTV perturbations with specified maximal variation rates. After providing some preliminaries, we give in Section III the precise statement of the considered robust performance analysis problems. In Section IV, we develop an algorithm for the computation of upper bounds for the optimum robust performance level. As a means for assessing the conservatism of the computed upper bounds in the case of dynamic LTV perturbations, we provide in Section V an optimization problem that provides guaranteed lower bounds for the optimum level of robust performance. We conclude by some final remarks on our contributions.

## II. Notation and Preliminaries

We work in a discrete-time setting with square-summable sequences, $x \in \ell_2^n$, as signals, and $\ell_2^n$ into $\ell_2^m$ linear operators, $T \in \mathscr{L}^{m\times n}$, as systems. Occasionally we treat real-valued matrices, $X \in \mathbb{R}^{m\times n}$, and real-rational transfer functions, $H \in \mathscr{RL}_\infty^{m\times n}$, also as operators. In particular, we refer to the set of real-valued (complex-valued) symmetric matrices, $X = X^T$ $(X = X^*)$, by $\mathbb{S}^m$ $(\mathbb{H}^m)$; to the set of real-valued (complex-valued) positive-definite matrices, $X > 0$, by $\mathbb{S}_+^m$ $(\mathbb{H}_+^m)$; to the set of stable and causal transfer functions by $\mathscr{RH}_\infty^{m\times n}$; and to the set of causal and unity norm-bounded linear operators by $\mathscr{BL}_\mathrm{c}^{m\times n} \triangleq \{T \in \mathscr{L}_\mathrm{c}^{m\times n} : \|T\|_{\mathrm{i}2} \le 1\}$, where $\|\cdot\|_{\mathrm{i}2}$ denotes the induced $\ell_2$-norm defined as $\|T\|_{\mathrm{i}2} \triangleq \sup_{0\neq x\in\ell_2^n} \|Tx\|_2/\|x\|_2$, with $\|\cdot\|_2$ denoting the standard $\ell_2$-norm, $\|x\|_2^2 \triangleq \sum_t x^T(t)x(t)$. We denote the identity matrix of size $m$ by $I_m$; the zero matrix of size $m\times n$ by $0_{m\times n}$ and the identity matrix extended by zeros as $I_{m\times n} = [I_m\, 0_{m\times(n-m)}]$ for $n > m$ and $I_{m\times n} = [I_n\, 0_{n\times(m-n)}]^T$ for $m > n$, occasionally with the size descriptions dropped. For a given linear (possibly time-varying) operator $\Delta \in \mathscr{L}_\mathrm{c}^{m\times n}$, we define its **variation** as $\mathscr{V}_\Delta \triangleq \Delta Z^{-1} - Z^{-1}\Delta \in \mathscr{L}_\mathrm{c}^{m\times n}$, where $Z^{-1}$ denotes the **right shift (delay)** operator, and refer to $\|\mathscr{V}_\Delta\|_{\mathrm{i}2}$ $(\le 2\|\Delta\|_{\mathrm{i}2})$ as the **rate-of-variation of** $\Delta$. For a compact representation of structures, we denote the block-diagonal operators as $[T_i]_{i=1}^\kappa = [T_i]_{i\in\mathscr{I}} \triangleq \mathrm{diag}(T_1,\ldots,T_\kappa)$ $(\mathscr{I} = \{1,\ldots,\kappa\})$ and employ the Kronecker product to describe the commuting structures.

Working in an IQC setting, we need to briefly describe the following basic notions in inner product spaces. As is well-known, $\ell_2^n$ is a complete inner product space, with the inner product defined for $x, y \in \ell_2^n$ as $\langle x, y\rangle \triangleq \sum_t x^T(t)y(t)$. With $T \in \mathscr{L}^{m\times n}$ being a bounded linear operator, its (Hilbert) **adjoint**, $T^* \in \mathscr{L}^{n\times m}$, is defined as the unique operator for which $\langle Tx, y\rangle = \langle x, T^*y\rangle$, $\forall x \in \ell_2^n, \forall y \in \ell_2^m$. An operator, $T \in \mathscr{L}^{m\times m}$, is referred to as **self-adjoint** if $T^* = T$. A self-adjoint operator, $T \in \mathscr{L}^{m\times m}$, is described as **positive-definite**, $T > 0$ (**negative-definite**, $T < 0$) if there exists an $\varepsilon \in \mathbb{R}_+$ such that $\langle Tx, x\rangle \ge \varepsilon\|x\|_2^2$ $(\langle Tx, x\rangle \le -\varepsilon\|x\|_2^2)$, $\forall x \in \ell_2^m$. Within this setting, the following two lemmas play crucial roles in our development:

*Lemma 1 (Bounding Lemma):* Let $T_1, T_2 \in \mathscr{L}^{m\times n}$ and $X \in \mathbb{S}_+^m$. Then we have

$$\mp\mathfrak{He}(T_1^*T_2) \triangleq \mp(T_1^*T_2 + T_2^*T_1) \le T_1^*XT_1 + T_2^*X^{-1}T_2.$$

*Proof:* Follows from $(KT_1 \mp K^{-T}T_2)^*(KT_1 \mp K^{-T}T_2) \ge 0$, where $X = K^TK$ with $K, K^{-1} \in \mathbb{R}^{m\times m}$. ■

*Lemma 2 (Schur-Complement Lemma):* Let $T_{11} = T_{11}^* \in \mathscr{L}^{m_1\times m_1}$, $T_{22} = T_{22}^* \in \mathscr{L}^{m_2\times m_2}$ and $T_{12} \in \mathscr{L}^{m_1\times m_2}$, with $T_{22}$ being invertible. Then the following are equivalent:

(i) $\begin{bmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{bmatrix} < 0$

(ii) $T_{22} < 0$ and $T_{11} - T_{12}T_{22}^{-1}T_{12}^* < 0$

For LTI operators, $H \in \mathscr{RL}_\infty^{m\times n}$, with realizations of the form $H = \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right] \triangleq C_H(Z - A_H)^{-1}B_H + D_H$, the positive and negative-definiteness conditions can be expressed as frequency domain inequalities (by replacing the operators with their corresponding frequency functions, $H(e^{j\omega})$) and then transformed into LMI's via the KYP lemma, which we provide below in a discrete-time generalized form from [20]:

*Lemma 3 (Generalized Kalman-Yakubovich-Popov Lemma):* Let $\Phi = \left[\begin{array}{c|c} A_\Phi & B_\Phi \\ \hline C_\Phi & D_\Phi \end{array}\right] \in \mathscr{RH}_\infty^{m_\Phi\times n_\Phi}$ be a stable and proper real-rational transfer function, where $A_\Phi \in \mathbb{R}^{k_\Phi\times k_\Phi}, B_\Phi \in \mathbb{R}^{k_\Phi\times n_\Phi}, C_\Phi \in \mathbb{R}^{m_\Phi\times k_\Phi}, D_\Phi \in \mathbb{R}^{m_\Phi\times n_\Phi}$, with $A_\Phi$ having all its eigenvalues strictly inside the unit disk, and $Q \in \mathbb{H}^{m_\Phi}$ be a Hermitian matrix. With $d\varpi \in [0, 2\pi]$ and $\varpi \in [0.5d\varpi, 2\pi - 0.5d\varpi]$, the following conditions are equivalent:

(i) For all $\omega \in [\varpi - 0.5d\varpi, \varpi + 0.5d\varpi]$,

$$\left[\, \Phi(e^{j\omega}) \,\right]^* Q \left[\, \Phi(e^{j\omega}) \,\right] < 0. \tag{1}$$

(ii) There exists $L \in \mathbb{L}_{\varpi,d\varpi}$ such that $R_\Phi^T L R_\Phi < 0$, where

$$R_\Phi \triangleq \begin{bmatrix} I & 0 \\ A_\Phi & B_\Phi \\ C_\Phi & D_\Phi \end{bmatrix}, \tag{2}$$

$$\mathbb{L}_{\varpi,d\varpi} \triangleq \left\{ L = \begin{bmatrix} -P_0 & e^{-j\varpi}P_1 & 0 \\ e^{j\varpi}P_1 & P_0 - 2\cos(0.5d\varpi)P_1 & 0 \\ 0 & 0 & Q \end{bmatrix} : P_0 \in \mathbb{H}^{k_\Phi},\, P_1 \in \mathbb{H}_+^{k_\Phi} \right\}. \tag{3}$$

*Proof:* See [18] for a proof of the standard version and [19], [6] for the underlying proofs of various generalized versions. ■

*Remark 1:* For $d\varpi = 2\pi$, we have $\varpi = \pi$ and thus $2\cos(0.5d\varpi) = 2$, $e^{j\varpi} = -1$. In this case, we will have $R_\Phi^T L(P_0, P_1)R_\Phi < 0$ for some $P_1 \in \mathbb{H}_+^{k_\Phi}$ if and only if $R_\Phi^T L(P_0, 0)R_\Phi < 0$ (see [20]). If we moreover have $Q \in \mathbb{S}^{m_\Phi}$, then we can define the set $\mathbb{L}_{\varpi,d\varpi}$ with $P_1 = 0_{k_\Phi\times k_\Phi}, P_0 \in \mathbb{S}^{k_\Phi}$, and thus recover the standard version of the KYP lemma. We can define $\mathbb{L}_{\varpi,d\varpi}$ with $P_1 = 0_{k_\Phi\times k_\Phi}, P_0 \in \mathbb{S}^{k_\Phi}$ also for $d\varpi = \pi$, $\varpi = \pi/2$ and $Q \in \mathbb{S}^{m_\Phi}$, since $\Phi$ is real-rational.

## III. Problem Formulation

The standard setup for robust stability and performance analysis is given in Figure 1, where $M \in \mathscr{RH}_\infty^{\mu\times\eta}$ denotes a known causal and stable LTI operator (representing the -controlled- plant) and $\Delta \in \mathscr{L}_{\mathrm{c}}^{\eta_{\mathrm{u}}\times\mu_{\mathrm{u}}}$ denotes some causal and bounded LTV operator coming out of the set $\boldsymbol{\Delta}$ (representing the set of unknown perturbations). The feedback interconnection is **well-posed** (or in short $(M_{\mathrm{uu}},\Delta)$ **is well-posed**), if the map $(I - M_{\mathrm{uu}}\Delta)$ has a causal inverse, $(I - M_{\mathrm{uu}}\Delta)^{-1} \in \mathscr{L}_{\mathrm{c}}^{\mu_{\mathrm{u}}\times\mu_{\mathrm{u}}}$. We say that the system in Figure 1 is **uniformly robustly ($\ell_2$-)stable against $\boldsymbol{\Delta}$**, if, moreover, $\left\|(I - M_{\mathrm{uu}}\Delta)^{-1}\right\|_{\mathrm{i2}}$ is bounded in $\boldsymbol{\Delta}$ (i.e. $\exists\beta \in \mathbb{R}_+$ such that $\|(I - M_{\mathrm{uu}}\Delta)^{-1}\|_{\mathrm{i2}} < \beta$, $\forall\Delta \in \boldsymbol{\Delta}$). In this case, the map from $w_{\mathrm{p}}$ to $z_{\mathrm{p}}$ will be well-defined and given by $T_{z_{\mathrm{p}}w_{\mathrm{p}}}(M,\Delta) = M_{\mathrm{pp}} + M_{\mathrm{pu}}\Delta(I - M_{\mathrm{uu}}\Delta)^{-1}M_{\mathrm{up}}$, facilitating the assessment of the performance of the (controlled) plant against the specified set of perturbations. Among the various possible choices as performance indicators, we consider in this paper the induced $\ell_2$-gain of $T_{z_{\mathrm{p}}w_{\mathrm{p}}}$, which is well-known as a natural extension of $\mathscr{H}_\infty$ performance to LTV maps. Hence, we say that $M$ has **uniform robust ($\ell_2$-gain) performance of level $\gamma$ against $\boldsymbol{\Delta}$**, if it is uniformly robustly stable against $\boldsymbol{\Delta}$ and $\sup_{\Delta\in\boldsymbol{\Delta}}\|T_{z_{\mathrm{p}}w_{\mathrm{p}}}(M,\Delta)\|_{\mathrm{i2}} < \gamma$.

Based on these notions, the problem that we study in this paper is formulated as follows:

*Problem 1:* Consider the feedback system in Figure 1 with a given causal and stable LTI plant, $M \in \mathscr{RH}_\infty^{(\mu_{\mathrm{u}}+\mu_{\mathrm{p}})\times(\eta_{\mathrm{u}}+\eta_{\mathrm{p}})}$, together with causal and stable, structured LTV perturbations $\Delta$ that are contained in the set $\boldsymbol{\Delta} = \boldsymbol{\Delta}_\nu^{\mathrm{s}} \subset \mathscr{BL}_{\mathrm{c}}^{\eta_{\mathrm{u}}\times\mu_{\mathrm{u}}}$, defined as

$$\boldsymbol{\Delta}_\nu^{\mathrm{s}} \triangleq \left\{ \begin{array}{l} [I_{l_i}\otimes\Delta_i]_{i=1}^{\kappa} : \Delta_i \in \mathscr{BL}_{\mathrm{c}}^{m_i\times n_i}; \\ \|\mathscr{V}_{\Delta_i}\|_{\mathrm{i2}} \le \nu_i, i\in\mathscr{I}; \Delta_i^* = \Delta_i, i \in \mathscr{I}_{\mathrm{p}} \end{array} \right\}, \quad (4)$$

where $\mathrm{s} = \{l_i, m_i, n_i, a_i\}_{i=1}^{\kappa}$ (with $a_i = -1$, for $i \in \mathscr{I}_{\mathrm{p}} \subseteq \mathscr{I} = \{1,\dots,\kappa\}$; $a_i = 0$, for $i \in \mathscr{I}_{\mathrm{d}} \triangleq \mathscr{I}\setminus\mathscr{I}_{\mathrm{p}}$, and $\eta_{\mathrm{u}} = \sum_{i=1}^{\kappa} l_i m_i$, $\mu_{\mathrm{u}} = \sum_{i=1}^{\kappa} l_i n_i$) and $\nu = \{\nu_i\}_{i=1}^{\kappa}$, $\nu_i \in [0,2]$ describe, respectively, the structure and the maximum rate-of-variation of the uncertainty. For later use define $\mathscr{I}_{\mathrm{v}} \triangleq \{i \in \mathscr{I} : \nu_i \neq 0\}$, $\mathscr{I}_{\mathrm{pv}} \triangleq \mathscr{I}_{\mathrm{p}}\cap\mathscr{I}_{\mathrm{v}}$, $\mathscr{I}_{\mathrm{dv}} \triangleq \mathscr{I}_{\mathrm{d}}\cap\mathscr{I}_{\mathrm{v}}$ and $\kappa_{\mathrm{p}} \triangleq \sum_{i\in\mathscr{I}_{\mathrm{p}}} 1$, $\kappa_{\mathrm{v}} \triangleq \sum_{i\in\mathscr{I}_{\mathrm{v}}} 1$. Assume that the feedback system is uniformly robustly stable and define

$$\gamma^{\mathrm{opt}}(M,\boldsymbol{\Delta}) \triangleq \inf_{\|T_{z_{\mathrm{p}}w_{\mathrm{p}}}(M,\Delta)\|_{\mathrm{i2}}<\gamma, \forall\Delta\in\boldsymbol{\Delta}} \gamma, \quad (5)$$

as the optimum level of uniform robust performance. Within this setting;

1.1 Determine an upper bound, $\gamma_{\mathrm{ub}}^{\mathrm{opt}} \ge \gamma^{\mathrm{opt}}$.
1.2 Determine a lower bound, $\gamma_{\mathrm{lb}}^{\mathrm{opt}} \le \gamma^{\mathrm{opt}}$.

## IV. An IQC Approach to Robust Performance Analysis Against Structured LTV Perturbations

In this section, we concentrate on the first part of Problem 1. We first summarize the IQC approach to robust performance analysis and then provide the extension to robust performance analysis, of the swapping-lemma-based robust stability analysis test developed by [13], in its generalized form due to [17].

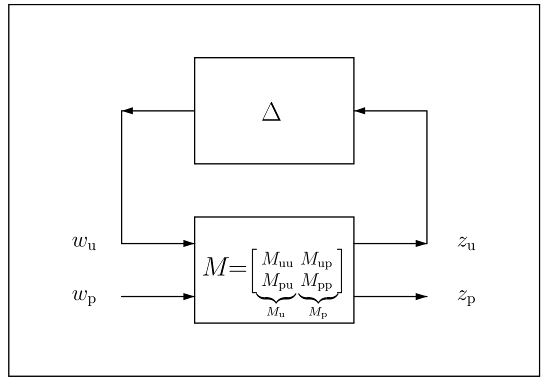


Fig. 1. Standard setup for robust performance analysis.

### A. IQC-Based Robust Stability and Performance Analysis

The IQC approach of [3] for robust stability analysis can be extended as follows to perform robust performance analysis:

*Theorem 1: (IQC-Based Robust Stability and Performance Analysis [3])* Let the set $\boldsymbol{\Delta}_\Pi$ be defined as

$$\boldsymbol{\Delta}_\Pi \triangleq \left\{\Delta \in \mathscr{L}_{\mathrm{c}}^{\eta_{\mathrm{u}}\times\mu_{\mathrm{u}}} : \begin{bmatrix} I \\ \Delta \end{bmatrix}^* \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} \begin{bmatrix} I \\ \Delta \end{bmatrix} \ge 0 \right\}, \quad (6)$$

where $\Pi_{11} = \Pi_{11}^* \in \mathscr{RL}_\infty^{\mu_{\mathrm{u}}\times\mu_{\mathrm{u}}}, \Pi_{22} = \Pi_{22}^* \in \mathscr{RL}_\infty^{\eta_{\mathrm{u}}\times\eta_{\mathrm{u}}}, \Pi_{12} \in \mathscr{RL}_\infty^{\mu_{\mathrm{u}}\times\eta_{\mathrm{u}}}$. Assume that the following two conditions are satisfied:

(i) $(M_{\mathrm{uu}},\tau\Delta)$ is well-posed $\forall\tau\in[0,1]$ and $\forall\Delta\in\boldsymbol{\Delta}$.
(ii) $\tau\Delta \in \boldsymbol{\Delta}_\Pi$, $\forall\tau\in[0,1]$ and $\forall\Delta\in\boldsymbol{\Delta}$.

$M$ will be uniformly robustly $\ell_2$-stable against $\boldsymbol{\Delta}$ and will admit uniform robust $\ell_2$-gain performance of level $\gamma$, if

$$\begin{bmatrix} M_{\mathrm{uu}} & M_{\mathrm{up}} \\ I & 0 \\ M_{\mathrm{pu}} & M_{\mathrm{pp}} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \Pi_{11} & \Pi_{12} & 0 & 0 \\ \Pi_{12}^* & \Pi_{22} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} M_{\mathrm{uu}} & M_{\mathrm{up}} \\ I & 0 \\ M_{\mathrm{pu}} & M_{\mathrm{pp}} \\ 0 & I \end{bmatrix} < 0. \quad (7)$$

### B. A Swapping-Lemma-Based IQC Method for Computing Upper Bounds for the Optimum Level of Robust Performance

Before we state the main result of this section, we cite the generalized version of the swapping lemma from [17]:

*Lemma 4 (Generalized Swapping Lemma):* Consider, in the setup of Problem 1, a $\Delta_{\mathrm{r}} = [I_{l_i}\otimes\Delta_i]_{i\in\mathscr{I}} \in \boldsymbol{\Delta}_\nu^{\mathrm{s}}$, together with causal and stable transfer functions $H_i \in \mathscr{RH}_\infty^{l_i\times q_i}, i = 1,\dots,\kappa$ that admit minimal realizations of the form $H_i = \left[\begin{array}{c|c} A_{H_i} & B_{H_i} \\ \hline C_{H_i} & D_{H_i} \end{array}\right]$, where $A_{H_i} \in \mathbb{R}^{k_i\times k_i}$, $B_{H_i} \in \mathbb{R}^{k_i\times l_i}$, $C_{H_i} \in \mathbb{R}^{q_i\times k_i}$, $D_{H_i} \in \mathbb{R}^{q_i\times l_i}$, with $A_i$'s having all their eigenvalues strictly inside the unit disk. With $H_{iB}$, $H_{iC}$ being defined as

$$H_{iB} \triangleq Z(Z - A_{H_i})^{-1}B_{H_i}, \quad (8)$$
$$H_{iC} \triangleq C_{H_i}(Z - A_{H_i})^{-1}Z, \quad (9)$$

and with $\mathscr{V}_{\Delta_i}$ denoting the variation of $\Delta_i$, we have

$$\underbrace{[I_{q_i}\otimes\Delta_i]_{i=1}^{\kappa}}_{\Delta_{\mathrm{l}}}\underbrace{[H_i\otimes I_{n_i}]_{i=1}^{\kappa}}_{H_{\mathrm{r}}} - \underbrace{[H_i\otimes I_{m_i}]_{i=1}^{\kappa}}_{H_{\mathrm{l}}}\underbrace{[I_{l_i}\otimes\Delta_i]_{i=1}^{\kappa}}_{\Delta_{\mathrm{r}}}$$
$$= \underbrace{[H_{iC}\otimes I_{m_i}]_{i=1}^{\kappa}}_{H_C}\underbrace{\left[I_{k_i}\otimes\mathscr{V}_{\Delta_i}\right]_{i=1}^{\kappa}}_{V_{\mathrm{l}}\mathscr{V}_{\Delta_{\mathrm{c}}}V_{\mathrm{r}}^T}\underbrace{[H_{iB}\otimes I_{n_i}]_{i=1}^{\kappa}}_{H_B}. \quad (10)$$

*Proof:* The proof is based on sequential application of $[I_{q_i} \otimes \Delta_i][X_i \otimes I_{n_i}] = [X_i \otimes I_{m_i}][I_{l_i} \otimes \Delta_i]$, $X_i \in \mathbb{R}^{q_i \times l_i}$, $\Delta_i \in \mathscr{L}_{\rm c}^{m_i \times n_i}$, after replacing $H_i$'s with their realizations (see [17]). ■

*Remark 2:* In the compact representation of (10), $\Delta_{\rm c} \triangleq [I_{k_i} \otimes \nu_i^{-1}\Delta_i]_{i\in\mathscr{I}_{\rm v}}$, and $V_{\rm l}$, $V_{\rm r}$ are matrices that satisfy $V_{\rm l}\mathscr{V}_{\Delta_{\rm c}}V_{\rm r} = [I_{k_i} \otimes \mathscr{V}_{\Delta_i}]_{i=1}^{\kappa}$. A possible way to define $V_{\rm l}$ and $V_{\rm r}$ is as follows: introduce the matrices $V_{ij} = \sqrt{\nu_i} \otimes I_{k_i}$, if $\mathscr{I}_{\rm v}(j) = i$ and $V_{ij} = 0_{k_i \times k_{\mathscr{I}_{\rm v}(j)}}$, otherwise, for $i = 1, \ldots, \kappa$, $j = 1, \ldots, \kappa_{\rm v}$, and let $V_{\rm r} = [V_{{\rm r}ij}]$, $V_{{\rm r}ij} \triangleq V_{ij} \otimes I_{n_i \times n_{\mathscr{I}_{\rm v}(j)}}$ and $V_{\rm l} = [V_{{\rm l}ij}]$, $V_{{\rm l}ij} \triangleq V_{ij} \otimes I_{m_i \times m_{\mathscr{I}_{\rm v}(j)}}$.

The following theorem extends the robust stability analysis method of [17] to robust performance analysis and thus provides a possible approach to tackle Problem 1.1:

*Theorem 2:* Consider the setup of Problem 1 and assume that $(M_{\rm uu}, \tau\Delta)$ is well-posed for all $\Delta \in \mathbf{\Delta}_\nu^{\rm s}$ and $\tau \in [0,1]$. With $H_i \in \mathscr{RH}_\infty^{q_i \times l_i}$, $i = 1, \ldots, \kappa$ being given stable (basis) transfer functions, let $H \triangleq [H_i]_{i=1}^{\kappa}$ and define $H_{\rm r}$, $H_{\rm l}$, $H_B$, $H_C$, $V_{\rm r}$ and $V_{\rm l}$ as in Lemma 4. For extension (or picking) purposes, introduce the matrices $E_{\rm r}$, $E_{\rm l}$ such that $E_{\rm l}\Delta_{\rm pc}E_{\rm r}^T = [I_{k_i} \otimes a_i\nu_i^{-1}\Delta_i]_{i\in\mathscr{I}_{\rm v}}$, where $\Delta_{\rm pc} \triangleq [I_{k_i} \otimes \Delta_i]_{i\in\mathscr{I}_{\rm pv}}$ (cf. Remark 2). Obtain the transfer function $\Phi$ as $\Phi = \Psi \begin{bmatrix} M_{\rm e} \\ I \end{bmatrix}$, where $M_{\rm e} \triangleq \begin{bmatrix} M_{\rm u} & 0_{\mu\times(\eta_{\rm e}+\eta_{\rm v}+\eta_{\rm pv})} & M_{\rm p} \end{bmatrix}$, $\eta_{\rm e} \triangleq \sum_{i\in\mathscr{I}} q_i m_i$, $\eta_{\rm v} \triangleq \sum_{i\in\mathscr{I}_{\rm v}} k_i m_i$, $\eta_{\rm pv} \triangleq \sum_{i\in\mathscr{I}_{\rm pv}} k_i m_i$, and

$$\Psi \triangleq \begin{bmatrix} H_{\rm r} & & & & & & \\ V_{\rm r}^T Z^{-1} H_B & & & & & & \\ & I_{\mu_{\rm p}} & & & & & \\ & & H_{\rm l} & 0 & 0 & H_C Z^{-1} V_{\rm l} E_{\rm l} & \\ & & H_{\rm l} & 0 & 0 & 0 & \\ & & H_{\rm l} & I_{\eta_{\rm e}} & H_C Z^{-1} V_{\rm l} & 0 & \\ & & 0 & I_{\eta_{\rm e}} & 0 & 0 & \\ & & 0 & 0 & I_{\eta_{\rm v}} & 0 & \\ & & 0 & 0 & 0 & I_{\eta_{\rm pv}} & \\ & & & & & & I_{\eta_{\rm p}} \end{bmatrix}. \quad (11)$$

Construct the matrix $R_\Phi$ according to (2) with a (preferably) minimal realization for $\Phi$ and define the set $\mathbb{L}$ as

$$\mathbb{L} \triangleq \left\{ \begin{bmatrix} -P & & & & & & & & & & & \\ & P & & & & & & & & & & \\ & & X_{\rm r} & 0 & 0 & Y & & & & & & \\ & & 0 & X_{\rm vr}+E_{\rm r}Y_{\rm v}E_{\rm r}^T & 0 & 0 & & & & & & \\ & & 0 & 0 & I_{\mu_{\rm p}} & 0 & & & & & & \\ & & Y^T & 0 & 0 & 0 & & & & & & \\ & & & & & & 0 & U & & & & \\ & & & & & & U^T & 0 & & & & \\ & & & & & & & & -X_{\rm l} & & & \\ & & & & & & & & & -X_{\rm vl} & & \\ & & & & & & & & & & -Y_{\rm v} & \\ & & & & & & & & & & & -\alpha I_{\eta_{\rm p}} \end{bmatrix} : \begin{array}{l} X_{\rm r} = [X_i \otimes I_{n_i}]_{i\in\mathscr{I}}, X_{\rm l} = [X_i \otimes I_{m_i}]_{i\in\mathscr{I}}, \\ X_{\rm vr} = [X_{{\rm v}_i} \otimes I_{n_i}]_{i\in\mathscr{I}_{\rm v}}, X_{\rm vl} = [X_{{\rm v}_i} \otimes I_{m_i}]_{i\in\mathscr{I}_{\rm v}}, \\ Y = [Y_i \otimes I_{n_i\times m_i}]_{i\in\mathscr{I}}, Y_{\rm v} = [Y_{{\rm v}_i} \otimes I_{n_i}]_{i\in\mathscr{I}_{\rm pv}}; \\ \alpha \in \mathbb{R}, U \in \mathbb{R}^{\eta_{\rm e}\times\eta_{\rm e}}, P \in \mathbb{S}^{k_\Phi}, \\ X_i \in \mathbb{S}_+^{q_i}, X_{{\rm v}_i} \in \mathbb{S}_+^{k_i}, Y_i = a_i Y_i^T \in \mathbb{R}^{q_i\times q_i}, Y_{{\rm v}_i} \in \mathbb{S}_+^{k_i} \end{array} \right\}. \quad (12)$$

With $\gamma_{\rm ub}^{\rm opt}(M, \triangle, H)$ obtained as

$$\gamma_{\rm ub}^{\rm opt}(M, \triangle, H) = \left( \inf_{L\in\mathbb{L},\, R_\Phi^T L R_\Phi < 0} \alpha \right)^{1/2}, \quad (13)$$

$M$ has guaranteed uniform robust $\ell_2$-gain performance of level less than or equal to $\gamma_{\rm ub}^{\rm opt}$.

*Proof:* The proof is based on finding a set of IQC's satisfied by all $\Delta \in \mathbf{\Delta}_\nu^{\rm s}$. To construct the underlying IQC multiplier, we first consider the operator $-\mathfrak{He}\left\{\Delta_{\rm r}^* H_{\rm l}^* U H_{\rm l} \Delta_{\rm r}\right\}$ for an arbitrary $\Delta_{\rm r} \in \mathbf{\Delta}_\nu^{\rm s}$, with $U \in \mathbb{R}^{\eta_{\rm e}\times\eta_{\rm e}}$. By employing the swapping and the bounding lemmas, we obtain

$$\begin{aligned} -\Delta_{\rm r}^* H_{\rm l}^*(U+U^T)H_{\rm l}\Delta_{\rm r} \le{}& H_{\rm r}^*\Delta_{\rm l}^* X_{\rm l}\Delta_{\rm l} H_{\rm r} + \Delta_{\rm r}^* H_{\rm l}^* U X_{\rm l}^{-1} U^T H_{\rm l}\Delta_{\rm r} \\ &+ H_B^* V_{\rm r}\mathscr{V}_{\Delta_{\rm c}}^* X_{\rm vl}\mathscr{V}_{\Delta_{\rm c}} V_{\rm r}^T H_B + \Delta_{\rm r}^* H_{\rm l}^* U H_C V_{\rm l} X_{\rm vl}^{-1} V_{\rm l}^T H_C^* U^T H_{\rm l}\Delta_{\rm r}, \end{aligned}$$

for positive-definite $X_{\rm l}$ and $X_{\rm vl}$. If $X_{\rm l}$ and $X_{\rm vl}$ are chosen as in (12), we will have $\Delta_{\rm l}^* X_{\rm l}\Delta_{\rm l} \le X_{\rm r}$, $\mathscr{V}_{\Delta_{\rm c}}^* X_{\rm vl}\mathscr{V}_{\Delta_{\rm c}} \le X_{\rm vr}$, and thus $\Pi_{11}^{(1)} + \Delta^*\Pi_{22}^{(1)}\Delta \ge 0$, $\forall\Delta \in \mathbf{\Delta}_\nu^{\rm s}$, with

$$\begin{aligned} \Pi_{11}^{(1)} &\triangleq H_{\rm r}^* X_{\rm r} H_{\rm r} + H_B^* V_{\rm r} X_{\rm vr} V_{\rm r}^T H_B, \\ \Pi_{22}^{(1)} &\triangleq H_{\rm l}^*\left[U + U^T + U\left(X_{\rm l}^{-1} + H_C V_{\rm l} X_{\rm vl}^{-1} V_{\rm l}^T H_C^*\right)U^T\right]H_{\rm l}. \end{aligned}$$

Next, let us consider the operator $-\mathfrak{He}\{H_{\rm r}^* Y H_{\rm l}\Delta_{\rm r}\}$ for an arbitrary $\Delta_{\rm r} \in \mathbf{\Delta}_\nu^{\rm s}$, with $Y$ being a structured matrix as in (12). Proceeding similarly as above and noting that $Y\Delta_{\rm l} + \Delta_{\rm l}^* Y^T = 0$ thanks to the structure of $Y$, we arrive at

$$\begin{aligned} -H_{\rm r}^* Y H_{\rm l}\Delta_{\rm r} - \Delta_{\rm r}^* H_{\rm l}^* Y^T H_{\rm r} \le{}& H_B^* V_{\rm r} E_{\rm r} Y_{\rm v} E_{\rm r}^T V_{\rm r}^T H_B \\ &+ H_{\rm r}^* Y H_C V_{\rm l} E_{\rm l}\mathscr{V}_{\Delta_{\rm pc}} Y_{\rm v}^{-1}\mathscr{V}_{\Delta_{\rm pc}}^* E_{\rm l}^T V_{\rm l}^T H_C^* Y^T H_{\rm r}, \end{aligned}$$

for positive-definite $Y_{\rm v}$. With $Y_{\rm v}$ chosen as in (12), we will have $\mathscr{V}_{\Delta_{\rm pc}} Y_{\rm v}^{-1}\mathscr{V}_{\Delta_{\rm pc}}^* \le Y_{\rm v}^{-1}$. Thus we conclude that $\Pi_{11}^{(2)} + \Pi_{12}^{(2)}\Delta + \Delta^*\Pi_{12}^{(2)*} \ge 0$, $\forall\Delta \in \mathbf{\Delta}_\nu^{\rm s}$, with

$$\begin{aligned} \Pi_{11}^{(2)} &\triangleq H_B^* V_{\rm r} E_{\rm r} Y_{\rm v} E_{\rm r}^T V_{\rm r}^T H_B + H_{\rm r}^* Y H_C V_{\rm l} E_{\rm l} Y_{\rm v}^{-1} E_{\rm l}^T V_{\rm l}^T H_C^* Y^T H_{\rm r}, \\ \Pi_{12}^{(2)} &\triangleq H_{\rm r}^* Y H_{\rm l}. \end{aligned}$$

We can also observe that $\tau\Delta \in \mathbf{\Delta}_\Pi$ for all $\tau \in [0,1]$ and all $\Delta \in \mathbf{\Delta}_\nu^{\rm s}$, with $\Pi = \Pi^{(1)} + \Pi^{(2)}$, where $\Pi^{(1)}$ and $\Pi^{(2)}$ are the multipliers identified by the IQC's derived above. A close analysis of this IQC multiplier reveals that it can be expressed as $\Pi^{(1)} + \Pi^{(2)} = \Pi_{\rm e11} - \Pi_{\rm e12}\Pi_{\rm e22}^{-1}\Pi_{\rm e12}^*$, where

$$\begin{aligned} \Pi_{\rm e11} &\triangleq \begin{bmatrix} H_{\rm r}^* X_{\rm r} H_{\rm r} + H_B^* V_{\rm r}\left(X_{\rm vr} + E_{\rm r} Y_{\rm v} E_{\rm r}^T\right)V_{\rm r}^T H_B & H_{\rm r}^* Y H_{\rm l} \\ H_{\rm l}^* Y^T H_{\rm r} & H_{\rm l}^*\left(U + U^T\right)H_{\rm l} \end{bmatrix}, \\ \Pi_{\rm e12} &\triangleq \begin{bmatrix} 0 & 0 & H_{\rm r}^* Y H_C V_{\rm l} E_{\rm l} \\ H_{\rm l}^* U & H_{\rm l}^* U H_C V_{\rm l} & 0 \end{bmatrix}, \\ \Pi_{\rm e22} &\triangleq \begin{bmatrix} -X_{\rm l} & 0 & 0 \\ 0 & -X_{\rm vl} & 0 \\ 0 & 0 & -Y_{\rm v} \end{bmatrix}. \end{aligned}$$

Now applying the Schur-complement lemma, we can linearize the inequality of (7) in all the matrix variables and then factorize it into the form $\Phi^* Q \Phi < 0$, which we can re-express, by favor of the standard KYP lemma, as $R_\Phi^T L R_\Phi < 0$. Clearly, we can then obtain upper bounds by solving the minimization problem described by (13). ■

*Remark 3:* Referring also to the proof, we can realize that the constant D-G scaling-test-based upper bounds can be obtained with $H_i = I, \forall i$ and $U = -X_1$. On the other hand, for $\nu_i = 0, \forall i$ and $U = -X_1$, we obtain the frequency-dependent D-G-scaling-test-based upper bounds. Recalling the results of [8], [9] and [10], we note that $U$ can be set to $U = -X_1$, for $\nu_i = 0, \forall i$ as well as for $H_i = I, \forall i$, without introducing any conservatism.

*Remark 4:* The IQC multiplier on which the analysis method of Theorem 2 is based has some minor differences than the one in [17], where $U$ was required to be symmetric negative-definite and structured similarly to $X_1$. Moreover, the variable corresponding to $Y$ in the IQC multiplier of [17] has a different structure and again (usually) less number of variables.

*Remark 5:* The optimization problem of (13) is basically a linear objective minimization under LMI constraints and can be solved efficiently by present day LMI solvers depending on the number of variables, which is especially affected by the McMillan degree of $\Psi$ and thus the choice of the basis functions. For instance, let us consider bases of the form $H_i = \left[ I_{l_i}\, Z^{-1} \cdots Z^{-p_i} \right]^T$. Recalling the standard controllable canonical form, we can easily realize that (for $p_i \geq 1$) such bases admit minimal realizations of the form

$$\left[\begin{array}{c|c} A_{H_i} & B_{H_i} \\ \hline C_{H_i} & D_{H_i} \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ I_{(p_i-1)} & 0 & 0 \\ \hline 0 & 0 & 1 \\ I_{(p_i-1)} & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \otimes I_{l_i}, \qquad (14)$$

where we clearly have $q_i = (p_i + 1) l_i$ and $k_i = p_i l_i$. With $A_{H_r} \triangleq [A_{H_i} \otimes I_{n_i}]_{i=1}^{\kappa}$, $B_{H_r} \triangleq [B_{H_i} \otimes I_{n_i}]_{i=1}^{\kappa}$, $C_{H_r} \triangleq [C_{H_i} \otimes I_{n_i}]_{i=1}^{\kappa}$, $D_{H_r} \triangleq [D_{H_i} \otimes I_{n_i}]_{i=1}^{\kappa}$, $A_{H_l} \triangleq [A_{H_i} \otimes I_{m_i}]_{i=1}^{\kappa}$, $B_{H_l} \triangleq [B_{H_i} \otimes I_{m_i}]_{i=1}^{\kappa}$, $C_{H_l} \triangleq [C_{H_i} \otimes I_{m_i}]_{i=1}^{\kappa}$ and $D_{H_l} \triangleq [D_{H_i} \otimes I_{m_i}]_{i=1}^{\kappa}$, we can form a (not necessarily minimal) realization for $\Psi$ as

$$\Psi = \left[\begin{array}{cccc|ccccccc} A_{H_r} & 0 & 0 & 0 & B_{H_r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{H_l} & 0 & 0 & 0 & 0 & B_{H_l} & 0 & 0 & V_l E_l & 0 \\ 0 & 0 & A_{H_l} & 0 & 0 & 0 & B_{H_l} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{H_l} & 0 & 0 & B_{H_l} & 0 & V_l & 0 & 0 \\ \hline C_{H_r} & 0 & 0 & 0 & D_{H_r} & 0 & 0 & 0 & 0 & 0 & 0 \\ V_r^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{H_l} & 0 & 0 & 0 & 0 & D_{H_l} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{H_l} & 0 & 0 & 0 & D_{H_l} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{H_l} & 0 & 0 & D_{H_l} & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right]. \qquad (15)$$

For this realization, we have $k_\Psi = \sum_{i \in \mathscr{I}} p_i l_i (n_i + 3m_i)$. It can be observed that the number of decision variables for this generic basis is proportional to $p_i^2$.

## V. Estimation of Conservatism in Robust Performance Analysis for Dynamic LTV Perturbations

We know from [8] and [10] that, with $l_i = 1, \forall i$, the upper bounds obtained via (13) will be exact for $\nu_i = 2, \forall i$ and $\nu_i \to 0, \forall i$. The powerful result of [10] that allows to draw the conclusion for the arbitrarily slow LTV perturbations also makes it possible to develop a method for computing lower bounds, as we describe in the following theorem:

*Theorem 3:* Consider the setup of Problem 1 with a strictly proper $M$ and assume that we have $\kappa_p = 0 \,\&\, \nu_i = \nu$, $l_i = 1, \forall i$. Let the matrix $R_\Phi$ be constructed according to (2) with a (preferably) minimal realization for $\Phi = \begin{bmatrix} M \\ I \end{bmatrix}$, and with $d\varpi = 2\arcsin(0.5\nu)$, $\varpi \in [0.5 d\varpi, \pi - 0.5 d\varpi]$, let the sets $\mathbb{Q}$ and $\mathbb{L}_\varpi$ be defined as

$$\mathbb{Q} \triangleq \left\{ \begin{array}{l} \operatorname{diag}\left( [x_i \otimes I_{n_i}]_{i=1}^{\kappa}, I_{\mu_p}, -[x_i \otimes I_{m_i}]_{i=1}^{\kappa}, -\alpha I_{\eta_p} \right) : \\ \alpha, x_i \in \mathbb{R}_+ \end{array} \right\}, \qquad (16)$$

$$\mathbb{L}_\varpi \triangleq \left\{ \begin{array}{l} \begin{bmatrix} -P_0 & e^{-j\varpi} P_1 & \\ e^{j\varpi} P_1 & P_0 - 2\cos(0.5 d\varpi) P_1 & \\ & & Q \end{bmatrix} : \\ P_0 \in \mathbb{H}^{k_\Phi}, P_1 \in \mathbb{H}_+^{k_\Phi}, Q \in \mathbb{Q} \end{array} \right\}. \qquad (17)$$

If $\gamma_{\mathrm{lb}}^{\mathrm{opt}}$, obtained as

$$\gamma_{\mathrm{lb}}^{\mathrm{opt}}(M, \Delta) = \left( \sup_{\varpi \in [0,\pi]} \ \inf_{L \in \mathbb{L}_\varpi,\, R_\Phi^T L R_\Phi < 0} \alpha \right)^{1/2}, \qquad (18)$$

is finite, then $\gamma^{\mathrm{opt}} \geq \gamma_{\mathrm{lb}}^{\mathrm{opt}}$.

*Proof:* The proof proceeds along the lines of the proof of Theorem 3.5 in [10]. We should first note by recalling the generalized KYP lemma that, for $\alpha_\varpi^{\mathrm{opt}}$ defined as $\alpha_\varpi^{\mathrm{opt}} \triangleq \inf_{L \in \mathbb{L}_\varpi,\, R_\Phi^T L R_\Phi < 0} \alpha$, we have

$$\alpha_\varpi^{\mathrm{opt}} = \inf_{Q \in \mathbb{Q},\, \Phi^*(e^{j\omega}) Q \Phi(e^{j\omega}) < 0, \forall \omega \in [\varpi - 0.5 d\varpi, \varpi + 0.5 d\varpi]} \alpha.$$

Now, let us fix an $\varepsilon \in \mathbb{R}_+$. With $\gamma_{\mathrm{lb}}^{\mathrm{opt}}$ being finite, there exists a $\varpi \in [0, \pi]$ such that $\alpha_\varpi^{\mathrm{opt}} \geq (\gamma_{\mathrm{lb}}^{\mathrm{opt}})^2 - \varepsilon$. Clearly, for $\alpha = (\gamma_{\mathrm{lb}}^{\mathrm{opt}})^2 - 2\varepsilon$, we cannot have any $Q \in \mathbb{Q}$ with which $\Phi^*(e^{j\omega}) Q \Phi(e^{j\omega}) < 0, \forall \omega \in [\varpi - 0.5 d\varpi, \varpi + 0.5 d\varpi]$. The essential step is then to conclude (by making use of the Infinite Helly theorem, see [10]) that, there exist $r = \kappa + 1$ frequencies such that, with $\alpha = (\gamma_{\mathrm{lb}}^{\mathrm{opt}})^2 - 2\varepsilon$, $Q \in \mathbb{Q}$, $[\Phi^*(e^{j\omega_i})]_{i=1}^r Q [\Phi(e^{j\omega_i})]_{i=1}^r < 0$ is infeasible. Employing Theorem 3.2 of [10] (which states that, if the constant D-scaling test is not satisfied for a group of frequencies $\{\omega_i\}_{i=1}^r$ in the interval $[\varpi - 0.5 d\varpi, \varpi + 0.5 d\varpi] \subseteq [0, \pi]$, then $M$ cannot admit robust performance against $\Delta_\nu^{\mathrm{s}}$ for the case $\kappa_p = 0 \,\&\, \nu_i = 2\sin(0.5 d\varpi)$, $l_i = 1, \forall i$), we conclude that $\gamma^{\mathrm{opt}} \geq \sqrt{(\gamma_{\mathrm{lb}}^{\mathrm{opt}})^2 - 2\varepsilon}$. Since $\varepsilon$ is arbitrary, $\gamma^{\mathrm{opt}} \geq \gamma_{\mathrm{lb}}^{\mathrm{opt}}$. ■

*Remark 6:* The minimization problem in (18) is a linear objective minimization under complex-valued LMI constraints. The complex-valued LMI constraints can be transformed into real-valued LMI's via the standard route (see e.g. [5]) and an LMI solver can be employed within a line search over $\varpi$ to obtain guaranteed lower bounds for optimum robust performance.

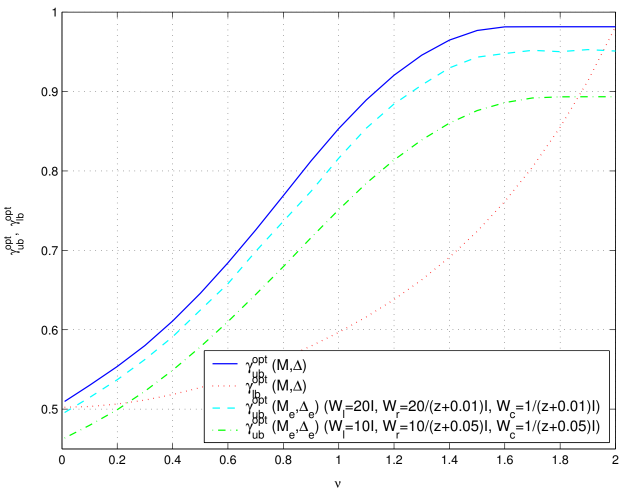


Fig. 2. Upper and lower bounds for optimum robust performance.

## VI. Illustrative Example

In this section, we demonstrate by an academic example how one can simplify the analysis of complex systems subject to LTI and parametric LTV perturbations, by viewing mixed LTI/parametric LTV perturbations as dynamic LTV systems. We consider a plant with realization

$$\left[\begin{array}{c|c} A_M & B_M \\ \hline C_M & D_M \end{array}\right] = \left[\begin{array}{rrr|rrr} -0.28 & -0.37 & 0.44 & -0.21 & -0.14 & 0.10 \\ 0.05 & 0.43 & -0.15 & -0.21 & 0.01 & 0.04 \\ 0.07 & 0.13 & 0.32 & -0.24 & -0.02 & 0.00 \\ \hline 0.35 & 0.31 & -0.84 & 0.00 & 0.00 & 0.00 \\ 1.49 & -0.88 & 0.05 & 0.00 & 0.00 & 0.00 \\ 0.27 & -2.40 & -7.20 & 0.00 & 0.00 & 0.00 \end{array}\right],$$

and perturbations of the form $\Delta = \beta(I + W_l\Delta_l)W_c\Delta_c(I + W_l\Delta_l)$, where $\Delta_l,\ \Delta_r \in \mathscr{BRH}_\infty^{2\times 2}$, $\Delta_c \in \{I_2 \otimes \delta : \delta \in \mathscr{BL}_c^{1\times 1}, \delta^* = \delta, \|\mathscr{V}_\delta\|_{i2} \le \nu\}$; $W_l, W_r, W_c$ are fixed shaping filters and $\beta = 1/(\|W_c\|_\infty(1+\|W_l\|_\infty)(1+\|W_r\|_\infty))$. In Figure 2, we present the computed lower and upper bounds for optimum robust performance versus the maximal rate-of-variation $\nu = \max\|\mathscr{V}_\delta\|_{i2}$. The upper bounds are computed (via implementations in MATLAB-YALMIP toolbox[21] and with bases of the form $H_i = [I\ Z^{-1}\ Z^{-2}]^T$), by considering the $(M,\Delta)$ couple (solid curve), where $\Delta = \{\Delta \in \mathscr{BL}^{2\times 2} : \|\mathscr{V}_\Delta\|_{i2} \le \nu\}$, and the $(M_e,\Delta_e)$ couple (dashed and dash-dotted curves, corresponding to two different group of shaping filters), where $\Delta_e = \{\mathrm{diag}(\Delta_l,\Delta_r,\Delta_c)\}$ and $M_e(M,W_l,W_r,W_c,\beta)$ is the extended plant that results after pulling out the perturbations to obtain the extended uncertainty structure. We note that the simplified analysis provides slightly more conservative results with significant reduction in computational complexity. Moreover, we can also observe that, as the uncertainty (that is expressed by shaping filters) in the LTI perturbations increases, the upper bounds computed by considering the $(M,\Delta)$ couple becomes less conservative. The lower bounds (dotted curve) are obtained as $\gamma_{lb}^{opt} = (\max_{i=1,\ldots,N}\inf_{L\in\mathbb{L}_{\varpi_i},\, R_\Phi^T L R_\Phi<0}\alpha)^{1/2}$ with $\varpi_i = 0.5d\varpi + (i-1)(\pi - d\varpi)/(N-1)$, $N = 30$, by considering the $(M,\Delta)$ couple. We should note that the lower bounds are meaningful (and thus comparable only to the upper bounds) for the $(M,\Delta)$ couple. In fact, the lower and upper bound curves for $(M,\Delta)$ couple approach each other when $\nu \to 2$ and $\nu \to 0$, as expected from [8] and [10] respectively.

## VII. Concluding Remarks

We have developed robust performance analysis methods for structured LTV perturbations. The upper bound computation method is an extension of the robust stability analysis test of [17]. The lower bound computation method is a novel application of the generalized KYP lemma for purposes of estimating conservatism in robust performance analysis. Extension of lower bound computation to parametric perturbations deserves further investigation.

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