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On the design of stable state dependent switching laws for single-input single-output systems

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Abstract— **We provide a constructive method for finding stabilising state-dependent switching laws to orchestrate switching between a set of linear vector fields. The method requires that some convex combination of the vector fields should be Hurwitz; no assumptions are made about stability of the individual vector fields.**

I. Introduction

Research on switched linear systems, and more generally, on time-varying linear systems, has been progressing steadily over the past four decades. In particular, given a set of vector fields, the problem of determining a state dependent switching law that orchestrates switching between the vector fields without causing instability has emerged as a problem of considerable merit [6], [5], [1], [4], [9], [8], [3], [2], [12]. While this problem is at the heart of many classical results (the Passivity theorem, the Popov criterion) [7], and has been more recently visited by the Hybrid Systems community, many questions related to the design of state dependent switching systems remain open. In particular, the following problem has yet to be adequately resolved.

Consider a set $\mathcal{A} = \{(A_1, \Omega_1), ..., (A_m, \Omega_m)\}$, where $A_i \in \mathbb{R}^{n\times n}$, and where $\Omega_i \subset \mathbb{R}^n$, $\bigcup_{i=1}^m \Omega_i = \mathbb{R}^n$, and $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$. Determine all $\mathcal{A}$ such that the differential equation

$$\dot{x} = A(x)x,\ A(x) = A_i \ \forall\, x \in\ \Omega_i, \tag{1}$$

is exponentially stable.

Our objective in this paper is to develop tools that can be used to address this problem. Our strategy is to seek conditions for the existence of a quadratic Lyapunov function for the system (1) and use these conditions to construct the (A_i, Ω_i). A central theme in this work involves the notion of Passivity [11]. In particular, we present an extension of the classical passivity theorem, and use the notion of positive realness to provide a partial solution to the above problem.

II. Partitions of the state space and state dependent switching

The principal tool that we shall use in deriving our results is the non-linear system

$$\dot{x} = Ax(t) - z(x(t), t) \tag{2}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, and $z : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$. The vector $z(\cdot)$ is allowed to depend on the state $x(t)$ as well as explicitly on t. We will assume throughout that the matrix A is Hurwitz. A basic question in the analysis of the above system relates to the manner in which $z(x(t), t)$ can vary in order to guarantee stability. The following theorem, which is a generalisation of the Passivity theorem, provides a partial answer to this question.

Theorem 1: Let P, R be positive definite $n \times n$ matrices such that

$$PA + A^T P = -R \tag{3}$$

and suppose that for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$,

$$x^T Pz(x, t)) \geq 0 \tag{4}$$

Then the system (2) is exponentially stable.

Proof: The function $x^T Px$ is a Lyapunov function for the system (2), as the following calculation shows:

$$\begin{aligned} \frac{d}{dt}(x^T Px) &= x^T P(Ax - z(x, t)) + (Ax - z(x, t))^T Px && (5) \\ &= x^T(PA + A^T P)x - 2x^T Pz(x, t) && (6) \\ &= -x^T Rx - 2x^T Pz(x, t) && (7) \\ &\leq -x^T Rx && (8) \end{aligned}$$

which proves the result since R is positive definite. **QED**

Theorem 1 gives a condition on the vector field $z(x, t)$ which guarantees exponential stability of the system (2). The condition states that at each point x in state space, the vector field $z(x, t)$ must have a non-negative inner product with the vector field Px. We shall use this theorem to design stable switched linear systems with state dependent switching laws.

Let $\{A_1, ..., A_m\}$ be a collection of real matrices such that the convex combination $\tilde{A} = \sum_i^m \alpha_i A_i$ is Hurwitz for some $\alpha_1, ..., \alpha_m, \sum_{i=1}^m \alpha_i = 1$ and all $\alpha_i \geq 0$. Choose $P = P^T > 0$ such that $\tilde{A}^T P + P\tilde{A} < 0$ and define $K_i = \{x : x^T P(\tilde{A} - A_i)x \geq 0\}$. Note that for all $x \in \mathbb{R}^n$, $\sum_{i=1}^m \alpha_i x^T P(\tilde{A} - A_i)x = 0$; hence

$$\bigcup_{i=1}^m K_i = \mathbb{R}^n. \tag{9}$$

Corollary : *For $i = 1, ..., m$, let $\Omega_i \subset K_i$ such that the Ω_i are disjoint and their union is $\mathbb{R}^n$. Then the system (1) is exponentially stable with $\mathcal{A} = \{(A_1, \Omega_1), ..., (A_m, \Omega_m)\}$.*

Comment 1 : The Corollary to Theorem 1 provides a recipe for constructing exponentially stable switched linear systems with state dependent switching laws for a given set of matrices $A_i, ..., A_m$. Note that the requirement that $\tilde{A}$ is Hurwitz does not imply that the A_i are also Hurwitz.

Comment 2 : It known that the existence of a convex combination $\tilde{A}$ that is Hurwitz is sufficient for the system (1) to be quadratically stabilizable [2]. In the case when $m = 2$, this condition is also necessary [2].

Theorem 1 provides a procedure to find state-space conditions that guarantee stability of the system (2) and thereby a solution to the problem posed in Section (1); namely find P, R which satisfy (3), then use them to define the constraint (4). For a given $\tilde{A}$, this raises the question of determining all $P = P^T > 0$ such that $\tilde{A}^T P + P\tilde{A} < 0$. In general, it is difficult to find a compact parameterisation of these matrices. However, a convenient subset of these matrices can be obtained using a special version of the Kalman-Yacubovich-Popov lemma. For given matrices $B, C \in \mathbb{R}^{n\times k}$ the transfer functions $H(s)$ and $H_{inv}(s)$ are defined as

$$H(s) = C^T (sI - A)^{-1} B, \quad (10)$$
$$H_{inv}(s) = C^T (sI - A^{-1})^{-1} B \quad (11)$$

where $s \in \mathbb{C}$.

Lemma 2: Suppose there exist $B, C \in \mathbb{R}^{n\times k}$ and $\epsilon > 0$ such that $A + \epsilon I$ is Hurwitz, and for all $\omega \in \mathbb{R}$ either

$$H(j\omega - \epsilon) + H(j\omega - \epsilon)^* \geq 0 \quad (12)$$
$$H_{inv}(j\omega - \epsilon) + H_{inv}(j\omega - \epsilon)^* \geq 0. \quad (13)$$

Then there is a $P = P^T > 0$ such that

$$PA + A^T P < 0, \quad PB = C. \quad (14)$$

Proof : The proof follows from the standard proof of the Kalman-Yacubovic-Popov lemma [11] and from the fact that any $P = P^T > 0$ that satisfies $A^T P + PA < 0$ also satisfies $A^{-1}P + PA^{-1} < 0$.

Comment 3: For a Hurwitz matrix A the condition (12) is equivalent to the statement that the transfer function $H(s)$ is strictly positive real (SPR) [**?**].

Comment 4: The above lemma is useful in the case when the matrices A_i can be written in the form $A_i = \tilde{A} - BD_i^T$ with $\tilde{A}$ Hurwitz and B some fixed matrix. In this case the cones become $K_i : x^T CD_i^T x \geq 0$ where the matrix C is determined using Lemma 2. This bypasses the need to determine P in order to construct stabilising switching laws. In the next Section we will provide constructive procedures to determine all such laws that can be found in this manner.

Comment 5: Note that Theorem 1 and Lemma 2 together consitute an extension of the Passivity theorem [11]. There is a special case of (2) where exponential stability is well-understood, namely the case when

$$z(x(t), t) = B\phi(y(t), t), \quad y(t) = C^T x(t) \quad (15)$$

where $\phi(y(t), t)$ is a vector field that maps $\mathbb{R}^k \times \mathbb{R}$ to $\mathbb{R}^k$. The stability of (2) in this case is called the Passivity Theorem [11] and is known to follow as a simple corollary of the circle criterion. Theorem 1 goes further by allowing the support of ϕ to depend directly on the state variable x rather than on the output variable y.

III. Design of stable switching systems

We are principally interested in designing stable switching systems. It follows from the Corollary to Theorem 1 that the generalised Passivity theorem provides a mechanism for the design of such systems. For a given $\tilde{A}$ and B, the family of stablising switching laws which can be determined using our procedure is given by all matrices for which Lemma 2 is satisfied.

Definition 3: For a given pair of matrices (A, B), we denote by $\mathcal{C}$ the set of all matrices C for which Lemma 2 is satisfied. From the point of view of using these results to design stable systems, two issues naturally arise: (i) given (A, B, C), can one efficiently check whether $H(s)$ is SPR; and secondly, (ii) given (A, B) can we describe all the matrices C such that $H(s)$ is SPR. Our focus in the remainder of the paper is addressing these questions when B and C are vectors. This case corresponds to the design of single-input single-output (SISO) switching systems.

(i) Triplets (A, b, c) that satisfy Lemma 2

The problem of determining whether or not a transfer function if SPR has been considered in a previous paper [10]. We give here the main result from this paper.

Theorem 4: Consider the transfer function $H(s) = c^T(sI - A)^{-1}b$. $H(s)$ is SPR if and only if: (i) $c^T Ab < 0$; (ii) $c^T A^{-1}b < 0$; (iii) A is Hurwitz; and (iv) $A(I - \frac{1}{c^T Ab} Abc^T)A$ has no eigenvalues on the open negative real axis $(-\infty, 0)$.

(ii) Finding $\mathcal{C}$

The main objective of this paper is to provide a mechanism for designing switching systems with state dependent switching laws. The problem of determining $\mathcal{C}$ lies at the heart of this problem. In this section we provide a simple characterisation of $\mathcal{C}$ for SISO systems.

Theorem 5: Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz and let $b \in \mathbb{R}^{n\times 1}$. For each $\omega \in \mathbb{R}$ define the vector $t(\omega) = (A^2 + \omega^2)^{-1}Ab$ and let

$$\mathcal{T} = \Big\{ \frac{t(\omega)}{\| t(\omega) \|} : \omega \in \mathbb{R} \Big\}. \quad (16)$$

Let $\overline{\mathcal{T}}$ denote the closure of $\mathcal{T}$. Then

$$\mathcal{C} = \{c \neq 0 \in \mathbb{R}^n : c^T t < 0 \, \forall \, t \in \overline{\mathcal{T}}\}. \tag{17}$$

Proof : Suppose first that c is such that $c^T t < 0 \; \forall \; t \in \overline{\mathcal{T}}$. Then by continuity, for $\epsilon > 0$ sufficiently small $c^T((A + \epsilon I)^2 + \omega^2)^{-1}(A + \epsilon I)b < 0$ for all ω and hence $c \in \mathcal{C}$. For the converse, suppose that $c \in \mathcal{C}$. Then $c^T(A^2 + \omega^2)^{-1}Ab < 0$ for all ω and $c^T Ab < 0$. Since $\overline{\mathcal{T}} = \mathcal{T} \cup \left\{\frac{Ab}{\|Ab\|}\right\}$, then this implies that $c^T t < 0 \, \forall \, t \in \overline{\mathcal{T}}$.

In the special case of second order systems, the set $\mathcal{C}$ can be conveniently parameterised.

Theorem 6: Let $A \in \mathbb{R}^{2\times 2}$ be Hurwitz and let $b \in \mathbb{R}^{2\times 1}$. Then, $\mathcal{C} = \{c \in \mathbb{R}^{2\times 1} : c^T Ab < 0, c^T A^{-1} b < 0\}$.

Proof : The proof follows by a continuity argument and the fact that $c^T t = c^T A^{-1} b$ at $\omega = 0$ and that $c^T t$ tends toward $c^T Ab$ as ω approaches infinity.

The case of second order systems is of limited practical value. We give a procedure for calculating a subset of $\mathcal{C}$ for higher dimensional systems. Assume without too much loss of generality that A is a companion matrix that is Hurwitz. Let b be a vector whose entries are all zero except for the last entry which is equal to one. Define the vector $u(s) = (1, s, s^2, ..., s^{n-1})$. Then, for all complex s

$$(sI - A)^{-1}b = \frac{u(s)}{s^n - a^T u(s)}, \tag{18}$$

where a^T is the $n'th$ row of the matrix A.
Then, we wish to find c such that $Re\{c^T(j\omega I - A)^{-1}b\} =$

$$Re\Big\{\frac{c^T u(j\omega)[(-j\omega)^n - a^T u(-j\omega)]}{|(j\omega)^n - a^T u(j\omega)|^2}\Big\} > 0. \tag{19}$$

It follows that the numerator of $Re\{c^T(j\omega I - A)^{-1}b\}$ may be written $n(\omega) = \sum_{k=0}^{n-1} \omega^{2k}(c^T y_k)$, where y_k are vectors defined by A and b, for all $k = 0, .., n-1$. We can find a non-empty subset of $\mathcal{C}$ by demanding $c^T y_k > 0$ for all $k = 0, .., n-1$. This gives n linear inequalities that can be solved.

Example : Consider the state dependent switching system

$$\dot{x} = A(x)x, \; A(x) \in \{A_1, A_2\}, \tag{20}$$

with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix}.$$

The matrix A_1 is Hurwitz but the matrix A_2 is not. Further, $A_2 = A_1 + bd^T$ where $b^T = (0, 0, 1)$ and $d^T = (1, 1, 1)$. Since A_1 is Hurwitz one may find stabilising switching laws by finding a vector c such that:

$$\dot{x} = A_2 x, \; \forall \; x : \; (d^T x)(c^T x) > 0 \tag{21}$$

$$\dot{x} = A_1 x \text{ otherwise} \tag{22}$$

As described above, a set of vectors c that satisfy $(d^T x)(c^T x) > 0$ are given by the $c = (c_0, c_1, c_2)^T$ which satisfy:

$$Re(c_0 + j\omega c_1 - \omega^2 c_2)(1 + 2j\omega - 3\omega^2 + j\omega^3) > 0. \tag{23}$$

The vectors c that satisfy the above equation are the set of vectors which have a positive inner product with the polyhedral cone spanned by $y_1^T = (1, 0, 0); y_2^T = (-3, 2, -1); y_3^T = (0, -1, 3)$.

IV. Conclusions

In this paper we have provided constructive procedures for finding stabilising state-dependent switching laws to orchestrate switching between a set of unstable vector fields. To achieve this aim we use a generalised notion of passivity, and the requirement that some convex combination of our system matrices are stable. In this present work we capture a subset of all such state-dependent switching laws; furture work will consider the problem of developing constructive procedures that capture all switching laws from which exponentially stable switching systems can be constructed.

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