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Construction of Optimal Norms for Semi-Groups of Matrices

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Abstract— **The notion of spectral radius of a set of matrices is a natural extension of spectral radius of a single matrix. The Finiteness Conjecture (FC) claims that among the infinite products made from the elements of a given finite set of matrices, there is a certain periodic product, made from the repetition of a finite product (the optimal product), whose rate of growth is maximal. FC has been disproved. In this paper it is conjectured that FC is almost always true, and an algorithm is presented to verify the optimality of a given product. The algorithm uses optimal norms, as a special subset of extremal extremal norms. The algorithm has successfully calculated the spectral radius of the pair of matrices associated with compactly supported multi-resolution analyses and wavelets.**

I. INTRODUCTION

Iteration, as a tool or a concept, is central to many branches of mathematics. While most classical applications of iteration, such as fractal generation [3], complex dynamics [4], and iterative functional equations [32], use a single function throughout the process, there is a wide spectrum of emerging important cases where there is a choice of functions at each stage of iteration. Linear multi-function iteration occurs in refinement algorithms for computer aided design [13], [41], image analysis techniques [3], Markov Chains [2], [45], asynchronous processes in control theory [47], the analysis of magnetic recording systems [40], the construction of scaling functions or pre-wavelets of compact support using the cascade algorithm and the Hölder regularity analysis of the resulting wavelets [16], [19]–[21], [28], hybrid systems as they occur in intelligent transport systems or industrial process control [8], the stability analysis of autonomous differential equations [1], [10], [11], and the asymptotic behavior of solutions of linear difference equations with variable coefficients [24]–[26].

Each of these applications requires detailed analysis of the convergence rate of long products of a given set of (at least two) matrices. This rate dictates either the global degree of stability [27] or smoothness of an associated system. (The corresponding local degree is also determined by analyzing the product of the matrices as a function of the ordering of the elements of the product.) There have been many different approaches to quantify this rate. We give an overview in the next section.

A. A Host of Definitions for the Radius of a Set of Matrices

Let $\mathcal{M}$ be a finite collection of square matrices of the same dimension. Assume $\mathcal{L}_n = \mathcal{L}_n(\mathcal{M})$ indicates the set of products of length n of elements of $\mathcal{M}$. The semi-group generated by $\mathcal{M}$ is then $\mathcal{L} = \mathcal{L}(\mathcal{M}) = \bigcup_{n=1}^{\infty} \mathcal{L}_n$.

There are two distinct views toward defining a radius for a set of matrices. The first one focuses on finding a rate of growth for the size of the elements of the semi-group.

Definition 1 *A matrix size function s is one of (an arbitrary fixed) norm, spectral radius, or the absolute value of the trace. For a finite collection of matrices $\mathcal{A}$ and $1 \le p < \infty$ the induced s_p-size is an averaging function defined as*

$$s_p(\mathcal{A}) = (\sum_{A\in\mathcal{A}} s^p(A)/|\mathcal{A}|)^{1/p}, \qquad (1)$$

where $|\mathcal{A}|$ is the cardinality of $\mathcal{A}$. For $p = \infty$ we have the simplified induced sup*-size*

$$s_\infty(\mathcal{A}) = \sup_{A\in\mathcal{A}} s(A). \qquad (2)$$

For the semi-group generated by $\mathcal{M}$ the induced size is defined as

$$S_p(\mathcal{M}) = \sup_n s_p(\mathcal{L}_n), \qquad (3)$$

and the semi-group is called s_p-bounded if $S_p(\mathcal{M})$ is finite. The induced spectral radius of $\mathcal{M}$ is defined as

$$\rho_{s_p}(\mathcal{M}) = \limsup_{n\to\infty} [s_p(\mathcal{L}_n)]^{1/n}. \qquad (4)$$

Some authors do not include division by $|\mathcal{A}|$ in the definition of $s_p(\mathcal{A})$. Also note that the spectral radius of $\mathcal{M}$, with respect to s_p, can be defined as the infimum of positive numbers r such that $\mathcal{M}/r$ generates an s_p-bounded semi-group.

The s_∞ is the most commonly used induced size function and the corresponding quantities are well-defined when $\mathcal{M}$ is not finite. If s is a norm and $p = \infty$ then we have Rota and Strang's definition of joint spectral radius (jsr) which was originally given in [44] as

$$\mathrm{jsr}(\mathcal{M}) = \limsup_{n\to\infty} \sup_{Q\in\mathcal{L}_n} ||Q||^{1/n}. \qquad (5)$$

If s is the usual spectral radius and $p = \infty$ then we have Daubechies and Lagarias' definition of generalized spectral radius (gsr) which was originally given in [20] as

$$\mathrm{gsr}(\mathcal{M}) = \limsup_{n\to\infty} \sup_{Q\in\mathcal{L}_n} [\rho(Q)]^{1/n}. \qquad (6)$$

They used gsr and jsr to obtain regularity estimates for certain wavelets. If s is the absolute value of the trace and $p = \infty$ then we have the definition of Chen and Zhou [14]. We refer to it as the mutual spectral radius (msr)

$$\mathrm{msr}(\mathcal{M}) = \limsup_{n\to\infty} \sup_{Q\in\mathcal{L}_n} |\mathrm{tr}(Q)|^{1/n}. \qquad (7)$$

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The L_p-type definitions have also played a role. Jia [31] used a p-norm joint spectral radius similar to jsr_p

$$\mathrm{jsr}_p(\mathcal{M}) = \limsup_{n\to\infty}(\sum_{Q\in\mathcal{L}_n} ||Q^p||/|\mathcal{L}_n|)^{1/pn}, \qquad (8)$$

for the study of L_p-regularity of the solutions of the refinement equations. Muller [42] used a similar concept for Banach Algebras. Also, Wang [48] used jsr_1 for L_1-regularity analysis of wavelets.

A second view of spectral radius of a set of matrices searches for an extremal norm [49]. This definition was also given by Rota and Strang. We refer to it as the common spectral radius (csr).

Definition 2 *The common spectral radius of $\mathcal{M}$ is*

$$\mathrm{csr}(\mathcal{M}) = \inf_{||\cdot||} \sup_{M\in\mathcal{M}} ||M||, \qquad (9)$$

where the infimum is over all sub-multiplicative norms (these norms satisfy $||AB|| \leq ||A||||B||$, see [5], [30]). More generally we may define

$$\mathrm{csr}_{n,p}(\mathcal{M}) = \inf_{||\cdot||}(\sum_{Q\in\mathcal{L}_n} ||Q^p||/|\mathcal{L}_n|)^{1/pn}. \qquad (10)$$

Hence csr *corresponds to* $\mathrm{csr}_{1,\infty}$.

In the next section we review some attempts at simplifying the definitions of radius.

B. Simplifications and Calculation Issues:

Fortunately, the definitions that have been advanced for the spectral radius point to the same quantity.

Theorem 3 *We have*

$$\mathrm{csr}(\mathcal{M}) = \mathrm{jsr}(\mathcal{M}) = \mathrm{gsr}(\mathcal{M}) = \mathrm{msr}(\mathcal{M}).$$

The common value is denoted by $\rho(\mathcal{M})$.

The equality of csr and jsr was proven by Rota and Strang. The equality of jsr and gsr was conjectured by Daubechies and Lagarias. It was proven by Berger and Wang [6], Elsner [22], and Chen and Zhou [14]. The latter also proved that gsr and msr are equal. The above theorem is still valid for infinite but norm-bounded $\mathcal{M}$. Heil and Strang [29] establish the continuity of radius.

Question 4 *To the author's knowledge gsr_p, msr_p, and $\mathrm{csr}_{n,p}$, or their relationship with jsr_p have not been studied. Also a general notion of size of a matrix, beyond the three concepts of norm, radius, and trace, has not been advanced. Moreover, the relationship between different notions of s_p-boundedness has not been investigated. In particular, if the spectral radii of all elements of a semi-group are less than 1 is the semi-group norm-bounded?*

A critical question is the degree to which the last limit operation in the definition of radius can be simplified. Rota and Strang showed that lim sup in the definition of jsr can be replaced by lim, and if a sub-multiplicative norm is used it can be replaced by inf. Jia [31] and Protasov [43] have a similar result for jsr_p. Daubechies and Lagarias showed that lim sup in the definition of gsr can be replaced by sup. They conjectured that for finite $\mathcal{M}$ it can be replaced by max, that is, a finite product will attain the limit radius. This is known as the Finiteness Conjecture (FC):

Conjecture 5 *For finite $\mathcal{M}$ there exists a finite n and $P \in \mathcal{L}_n$ such that $\rho(P)^{1/n} = \rho(\mathcal{M})$. A product P that satisfies FC is called an optimal product.*

Bousch and Mairess [9] have disproved this conjecture.

In a similar manner finding necessary and sufficient conditions under which inf in the definition of csr can be replaced by min, that is a particular norm achieves the radius, have been investigated.

Definition 6 *A sub-multiplicative norm $||\cdot||_e$ is called an extremal norm for $\mathcal{M}$ if $\rho(\mathcal{M}) = \sup_{M\in\mathcal{M}} ||M||_e$. A set $\mathcal{M}$ is called*

- *product bounded if it generates a norm-bounded semi-group.*
- *regular if it has an extremal norm.*
- *asymptotically non-defective if either $\mathcal{M} = 0$ or $\rho(\mathcal{M}) > 0$ and $\mathcal{M}/\rho(\mathcal{M})$ is product bounded.*
- *irreducible or non-decomposable if $\mathcal{M}$ has two or more matrices which do not have a common invariant subspace other than 0 and the entire space.*

Rota and Strang show that $\mathcal{M}$ is regular iff it is asymptotically non-defective. Protasov [43] and Elsner [22] show that if $\mathcal{M}$ is irreducible then it is regular. Brayton and Tong [10] give a sufficient condition for non-defectiveness in terms of "uniform linear independence" of the columns of each of the similarity transformations which reduce the elements of the semi-group generated by $\mathcal{M}$ to their Jordan form. Blondel and Tsitsiklis [7] show that the problem of determining whether or not $\mathcal{M}$ generates a bounded semi-group or that $\rho(\mathcal{M}) \leq 1$ is undecidable. A detailed analysis of defective sets of matrices appears in [25].

The most widely used method for calculating the radius is the Branch-and-Bound Method. It was introduced by Daubechies and Lagarias to provide upper estimates. It was utilized by Colella and Heil [16]. Gripenberg [23] refined it to provide lower estimates as well. This method identifies a base of finite products out of which near-optimal products can be built. A problem with this method is the extremely slow rate of convergence. For example in the benchmark experiment involving a specific pair of 2×2 matrices and using products of lengths 50, 150, and 250, Gripenberg's method produced a relative error of 1.5×10^{-4}, 3×10^{-5}, and 2×10^{-5} respectively. In contrast, with the optimal norm construction, as explained below, one obtains the exact answer (to machine precision) using products of nearly same length as the optimal product. In the benchmark calculation the optimal product is of length 13.

A central question is the complexity of algorithms aimed at measuring the radius. Tsitsiklis and Blondel [46] show that such algorithms are NP-hard. The point of view advanced in this paper is that the NP-hardness is due to certain rare and extreme cases and the "average" case, while computationally intensive, is still feasible.

Definition 7 *Exceptional matrix sets are finite sets of matrices for which the Finiteness Conjecture is not true.*

We propose:

Conjecture 8 *The Finiteness Conjecture is almost always true. The matrix sets which are exceptional form a set of measure zero in the space of matrices.*

If this conjectures is true, then it suggests that one should seek out candidates for optimal product and validate them in order to find the radius. In the next section we explain how to perform the validation step. This step is based on using extremal norms for the given set. The next conjecture states that instances where such norms may fail to exist are rare.

Conjecture 9 *Decomposable matrix sets form a set of measure zero in the corresponding space of matrices. Asymptotically defective matrix sets form a set of measure zero within the set of decomposable matrices.*

II. Constructing Optimal Norms for Semi-Groups

Here we propose an "Optimal Norm Conjecture" (ONC) and a companion algorithm aimed at deciding if a product is optimal, determining the exact value of radius, and mapping points in the space of sets matrices to their particular optimal products.

To describe ONC first we define optimal norms essentially as the "tightest" possible extremal norms.

Definition 10 *Let a bounded set of points S that contains at least one point other than origin be given. Suppose $\mathcal{M}$ is real and has an extremal norm. Let $\mathcal{U} = \mathcal{U}(S)$ be the intersection of the unit balls of all extremal norms of $\mathcal{M}$ that contain S. Suppose $\mathcal{U}$ has a non-empty interior then there is a norm whose unit ball is $\mathcal{U}$. We refer to this norm as an optimal norm of $\mathcal{M}$ and $\mathcal{U}$ will be called an optimal unit ball of $\mathcal{M}$. If $\mathcal{U}(S)$ has an empty interior then we refer to it as a reduced optimal ball. In particular if $\mathcal{M}$ is the single matrix M and $\mathcal{U}$ is a ball in the eigenspace associated with eigenvalues λ where $|\lambda| = \rho(M)$ then we refer to $\mathcal{U}$ as a spectral ball of M.*

Conjecture 11 *Suppose $\mathcal{M}$ is non-decomposable then the optimal ball of $\mathcal{M}$ is unique up to a multiple.*

A uniqueness theorem for the case where $\mathcal{M}$ is only non-defective appears to hold for most $\mathcal{M}$ but counterexamples involving special rotation matrices are easy to build.

Let a real matrix A also represent a set of points indicated by its column vectors. Denote by $\mathrm{cvx}(S)$ the convex hull of the set S.

Conjecture 12 The Optimal Norm Conjecture (ONC) *Assume $\mathcal{M}$ is finite, real, product-bounded and of unit radius $\rho(\mathcal{M}) = 1$. Let $\mathcal{L}$ be the semi-group generated by $\mathcal{M}$. Then a product $P \in \mathcal{L}$ is an optimal product of $\mathcal{M}$ only if there exists $\mathcal{G}$, a finite subset of $\mathcal{L}$, such that $\mathrm{cvx}(\mathcal{L}V) = \mathrm{cvx}(\mathcal{G}V)$, where V is a spectral ball of P.*

In other words the optimal unit ball can be finitely generated provided that we have the optimal product. The following algorithm formalizes the process of construction. Recall that x is called an extreme point of a set S if whenever y and z belong to S and x is on the line segment connecting y to z then $x = y = z$.

Algorithm 13 The ONC-Based Algorithm *An algorithm to verify the optimality of a product P of elements of a set $\mathcal{M}$: Suppose P is of length n, then $\rho(\mathcal{M}) = \rho(P)^{1/n}$ if P is indeed optimal.*

1) *Scale all matrices so that the radius of the set is 1, i.e., define $\mathcal{M}^* = \mathcal{M}/\rho(\mathcal{M})$. Then $P^* = P/\rho(P)$ and $\rho(\mathcal{M}^*) = \rho(P^*) = 1$. Define $\mathcal{M}^*_+$ as $\mathcal{M}^*$ augmented with identity.*
2) *Find Ω_0, a spectral ball of P^*.*
3) *For $q \geq 1$ compute $\Omega_q = \mathrm{cvx}(\mathcal{M}^*_+\Omega_{q-1})$.*
4) *Positive exit: If at a certain stage q_c the convex hull does not grow, $\Omega_{q_c} = \Omega_{q_c-1}$, then P is an optimal product.*
5) *Negative exit: If an extreme point of P^* becomes an interior point of the convex hull of its own iterates, then P is not an optimal product.*

In [34] we prove two theorems that establish the sufficiency of the two exit criteria. Optimal Norm Conjecture states that these exit criteria are also necessary.

At the positive termination of the algorithm, Ω_{q_c} can be considered as the unit ball of an optimal norm $||\cdot||_c$ with respect to which $\mathcal{M}$ attains its radius $\rho(\mathcal{M}) = ||\mathcal{M}||_c = \sup_{M\in\mathcal{M}} ||M||_c$. The value of q_c is defined as the *critical index* of the optimal product P. In experiments q_c exceeds the length of the optimal product by a small integer.

Constructing special unit balls, through a convex hull of the action of semi-group matrices on an arbitrary starting ball, is a recurrent theme in the papers on this topic. It appears in Rota and Strang's paper as the "alternative construction of the norm" and in Brayton and Tong's papers as the "constructive algorithm." What is new about our approach is the special choice of the starting ball. We use the optimal ball of the optimal product of Lagarias and Daubechies' Finiteness Conjecture as our starting ball. Then we observe that Rota and Strang's procedure terminates in a finite number of steps.

The calculation of convex hulls, especially in high dimensions, is of course expensive. A brute-force approach to the calculation of spectral radius of a set of matrices, by subjecting every possible product to Algorithm 13, will have a prohibitive cost. However, there are well established branch-and-bound methods [19], [20], [23] for selecting products which are the only likely ones to be a prefix of an optimal product.

III. Numerical tests for the ONC-based algorithm

The proposed ONC-based algorithm is both exact and faster than branch-and-bound type algorithms. Among successful applications of the algorithm is the numerical discovery [39] of the Hölder-smoothest four-coefficient orthogonal scaling functions and the associated multiresolution analysis (MRA) or wavelet, predating the theoretical discovery of the same by Bröker and Zhou [12]. We also describe the smoothest six-coefficient orthogonal scaling function and point out an error of Daubechies in the approximation of the same. (For a description of MRA see [15], [34].)

1) The Hölder exponent of four-coefficient MRA: Consider the 4-coefficient dilation equation

$$\phi(x) = c_0\phi(2x) + c_1\phi(2x-1) + c_2\phi(2x-2) + c_3\phi(2x-3), \quad (11)$$

subject to sum and orthogonality rules:

$$c_0 + c_2 = 1, \qquad c_1 + c_3 = 1, \quad (12a)$$
$$(c_0 - 1/2)^2 + (c_3 - 1/2)^2 = 1/2. \quad (12b)$$

Then the Hölder exponent of the associated wavelet is $h = -\log_2(\rho(\mathcal{M}))$ (if $\rho(\mathcal{M}) < 1$) where $\mathcal{M} = \{M_0, M_1\}$ and

$$M_0 = \begin{pmatrix} c_0 & 0 \\ -c_3 & 1 - c_0 - c_3 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 - c_0 - c_3 & -c_0 \\ 0 & c_3 \end{pmatrix}. \quad (13)$$

Colella and Heil's conjectured [16], [18], [17], that at $(c_0, c_3) = (0.6, -0.2)$ the radius of $\mathcal{M}$ attains its smallest value and the optimal product is $P = M_1 M_0^{12}$. We disproved the first statement and confirmed the second one. Our numerical experiments [37], [39], showed that the optimal product at any point (c_0, c_3) of (12b) is one of

$$M_0 M_1^n \quad \text{or} \quad M_1 M_0^n. \quad (14)$$

Furthermore, we obtained a very detailed picture of the structure of the optimal balls, dependence of n on (c_0, c_3), dependence of q_c on n, the smallest value of the radius, the resulting smoothest wavelet, and the critical arcs on which $n > 0$, etc. Here we give a brief report on such findings.

To determine h for each wavelet we will travel on the half-circle below $c_0 = c_3$, from $(0, 0)$ toward $(1, 1)$ in the counter-clockwise direction on the orthogonality circle (12b). (The properties on the upper half can be described similarly.) First the optimal product is simply M_0 and the optimal ball is a quadrilateral. Then, starting at $(1/2, (1 - \sqrt{2})/2)$, there is a critical strip on which the optimal product is of the form $M_1 M_0^n$ where n starts at infinity, descends to 11, and goes back to infinity. On the second stretch of the critical strip (where n goes from 11 to infinity) we pass through Heil-Colella point $(c_0, c_3) = (0.6, -0.2)$, which is on a subinterval where $n = 12$. The spectral radius decreases throughout that interval and no minimum occurs. Next, there is a point on the border between $n = 22$ and $n = 23$ at which the smallest joint spectral radius and the smoothest multiresolution is realized. At this point the ball has 54 sides,

$$c_0 = +0.64319821225683, \qquad c_1 = +1.19245524910022,$$
$$c_2 = +0.35680178774317, \qquad c_3 = -0.19245524910022,$$

$\rho(\mathcal{M}) = 0.64705462513820$ and the Hölder exponent of the resulting MRA is $h = 0.62804058345878$. Bröker and Zhou [12] obtain the same result by analytical means. As we leave the critical strip (at $c_3 = 1 - a^{1/3} - 1/3a^{-1/3}$ where $a = 1/4 + 33^{1/2}/36$, *i.e.*, $c_0 = 0.64779887126104$, and $c_3 = -0.19148788395312$), we enter an interval where once again the optimal product is of length one and the optimal ball is first a quadrilateral (Daubechies' D_4 is here) and then a hexagon. Finally we arrive at $(1, 1)$.

Table I records sample values of $\rho(\mathcal{M})$ at different values of c_3 over the critical strip. Between two consecutively recorded values of c_3 the structure of optimal unit ball is determined. The critical exponent q is half of the number of vertices of the ball. The columns of matrix V, together with $-V$, represent the vertices of the ball. The vector v is the eigenvector of the scaled optimal product associated with eigenvalue -1, $BA^n v = -v$, where $(A, B) = (M_0, M_1)/\rho(\mathcal{M})$. We used $A^{[0:m]}v$ to stand for $v, Av, A^2v, \cdots, A^m v$.

A. The Hölder exponent of six-coefficient MRA

Consider the 6-coefficient dilation equation

$$\phi(x) = c_0\phi(2x) + c_1\phi(2x-1) + c_2\phi(2x-2) + c_3\phi(2x-3) + c_4\phi(2x-4) + c_5\phi(2x-5), \quad (15)$$

subject to sum and orthogonality rules:

$$\begin{aligned} c_0 + c_2 + c_4 = c_1 + c_3 + c_5 &= 1, \\ 0c_0 - 1c_1 + 2c_2 - 3c_3 + 4c_4 - 5c_5 &= 0, \\ c_0c_2 + c_1c_3 + c_2c_4 + c_3c_5 &= 0, \\ c_0c_4 + c_1c_5 &= 0, \\ c_0^2 + c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 &= 2. \end{aligned} \quad (16)$$

These rules can be written in terms of the corner coefficients (c_0, c_5) for the main cases as

$$8c_0^4 + 8c_5^4 + 16c_0^2c_5^2 - 4c_0^3 - 4c_5^3 + 12c_0^2c_5 + 12c_0c_5^2 - c_0^2 - c_5^2 + 4c_0c_5 = 0, \quad (17a)$$

$$z = \frac{2c_0 + 2c_5 + 1}{2c_5 - 2c_0} \quad \text{if} \quad c_0 \neq c_5, \quad (17b)$$

$$c_1 = zc_0, \quad c_4 = zc_5, \quad (17c)$$

$$c_2 = 1 - c_0 - zc_5, \quad c_3 = 1 + zc_0 - c_5. \quad (17d)$$

In a special case where $c_4 = c_5 = 0$ or $c_0 = c_5 = 0$ or $c_0 = c_1 = 0$ we get an MRA with less than six coefficients. If $c_0 = c_5$ then their common value is $-1/4$ and in fact $c_0 = c_5 = -1/4$ is an isolated point on the graph of (17a). In this case $c_1 = c_4 = 0$ and $c_2 = c_3 = 5/4$. The graph of (17a) resembles a bent figure-8 or a butterfly with an eye at $(-1/4, -1/4)$, see Fig. 1.

The Hölder exponent of the first derivative of the associated wavelet is $h = -\log_2(2\rho(\mathcal{M}))$ (if $\rho(\mathcal{M}) < 1/2$) where $\mathcal{M} = \{M_0, M_1\}$,

$$
\begin{aligned}
M_0 &= \frac{1}{2}\begin{pmatrix} 1+a & -1-2a & 1+2a \\ 1+b & -1-a-b & 1+2a \\ 0 & 2c_5 & 1+b-2c_5 \end{pmatrix}, \\
a &= 4c_0 - 2c_1 - 2c_5, \\
b &= 2c_0 - 2c_1 - 2c_5, \\
M_1 &= \frac{1}{2}\begin{pmatrix} -a & 1+2a & -1-2a \\ 2c_5 & 1+b-2c_5 & -1-a-b \\ 0 & 0 & 2c_5 \end{pmatrix}.
\end{aligned} \tag{18}
$$

Here the matrices M_0 and M_1 have been obtained by applying the similarity transformations suggested in [19] and the sum-rules (17) to the standard wavelet matrices T_0 and T_1. (For the $m+1$-coefficient dilation equation $(T_d)_{ij} = c_{2i-j+d-1}$, $d = 0$ or 1, $1 \le i, j \le m$.)

We have applied Algorithm 13 to determine $\rho(\mathcal{M})$. As a result we have found that the optimal product is one of

$$M_0 M_1^n \quad \text{or} \quad M_0^2 M_1^n \quad \text{or} \quad M_1 M_0^n \quad \text{or} \quad M_1^2 M_0^n. \tag{19}$$

We report the value of ρ in terms of $m = c_5/c_0$. We start at the origin and move on the loop with $c_0 > c_5$ in the clockwise direction on the graph of (17a). At the beginning either M_0 or M_1 can be considered an optimal product. This occurs on a strip starting at the origin, where $m = 2 - \sqrt{3}$, and continues up to $m = .20091381944779$, where a critical strip starts. On this critical strip the optimal product is $M_0^n M_1$, and n starts from infinity, descends to 4 and increases back to infinity. (The value of n generally, but not always, changes in steps of 2.) This strip ends at $m = .12041694921052$. At $m = .12278337157050$, on the border between two subintervals with optimal products $M_0^8 M_1$ and $M_0^{10} M_1$, we find the smoothest 6-coefficient MRA. Here the coefficients of the dilation equation are

TABLE I

OPTIMAL PRODUCTS FOR 4-COEFFICIENT MRAS

$c_3 = -0.20710678118655$	$\rho = 0.70710678118655$
$q = \infty$, $BA^{\infty}v = -v$	$V = [A^{[0:\infty]}v]$
...	...
$c_3 = -0.20685451946438$	$\rho = 0.69618860818864$
$q = 15$, $BA^{12}v = -v$	$V = [A^{[0:13]}v, BA^{13}v]$
interval of shortest products ↓	
$c_3 = -0.20641657740770$	$\rho = 0.69004302279648$
$q = 16$, $BA^{11}v = -v$	$V = [A^{[0:13]}v, BA^{13}v]$
$c_3 = -0.20639313158185$	$\rho = 0.68979383768344$
$q = 15$, $BA^{11}v = -v$	$V = [A^{[0:13]}v, BA^{12}v]$
$c_3 = -0.20634605286404$	$\rho = 0.68930630911961$
$q = 14$, $BA^{11}v = -v$	$V = [A^{[0:12]}v, BA^{12}v]$
$c_3 = -0.20248452406185$	$\rho = 0.66771960222144$
$q = 15$, $BA^{11}v = -v$	$V = [A^{[0:13]}v, BA^{12}v]$
$c_3 = -0.20181564521458$	$\rho = 0.66530105883053$
$q = 16$, $BA^{11}v = -v$	$V = [A^{[0:13]}v, BA^{13}v, BA^{12}v]$
$c_3 = -0.20131323874003$	$\rho = 0.66359100053031$
interval of shortest products ↑	
$q = 15$, $BA^{12}v = -v$	$V = [A^{[0:13]}v, BA^{13}v]$
$c_3 = -0.19994273898044$	$\rho = 0.65951833373125$
$q = 16$, $BA^{12}v = -v$	$V = [A^{[0:14]}v, BA^{13}v]$
$c_3 = -0.19935467077442$	$\rho = 0.65790899824005$
$q = 17$, $BA^{12}v = -v$	$V = [A^{[0:14]}v, BA^{14}v, BA^{13}v]$
$c_3 = -0.19887220524860$	$\rho = 0.65663720229290$
...	...
anomalous interval ↓	
$c_3 = -0.19516075726816$	$\rho = 0.64928146835213$
$q = 19$, $BA^{16}v = -v$	$V = [A^{[0:17]}v, BA^{17}v]$
$c_3 = -0.19512218095930$	$\rho = 0.64923016354484$
$q = 20$, $BA^{16}v = -v$	$V = [A^{[0:18]}v, BA^{17}v]$
$c_3 = -0.19479024238150$	$\rho = 0.64879376983300$
$q = 21$, $BA^{16}v = -v$	$V = [A^{[0:18]}v, BA^{18}v, BA^{17}v]$
$c_3 = -0.19447589464925$	$\rho = 0.64838816938558$
$q = 22$, $BA^{16}v = -v$	$V = [A^{[0:19]}v, BA^{18}v, BA^{17}v]$
$c_3 = -0.19446922675618$	$\rho = 0.64837963968583$
anomalous interval ↑	
...	...
$c_3 = -0.19250565305303$	$\rho = 0.64705734026606$
$q = 28$, $BA^{22}v = -v$	$V = [A^{[0:25]}v, BA^{24}v, BA^{23}v]$
smoothest 4-coef. MRA:	minimum value of ρ:
$c_3 = -0.19245524910022$	$\rho = 0.64705462513820$
$q = 27$, $BA^{23}v = -v$	$V = [A^{[0:25]}v, BA^{24}v]$
$c_3 = -0.19240955523641$	$\rho = 0.64705945464432$
...	...
$q = \infty$, $BA^{\infty}v = -v$	$V = [A^{[0:\infty]}v, BA^{\infty}v]$
$c_3 = -0.19148788395312$	$\rho = 0.64779887126104$

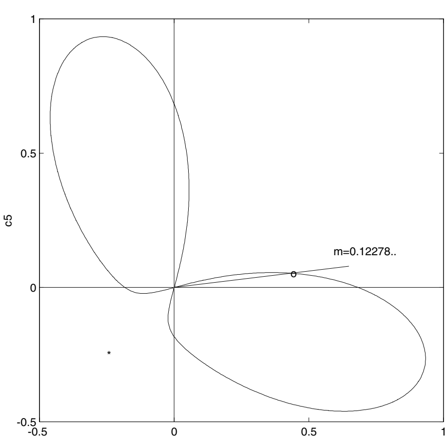

Fig. 1. Butterfly Curve of MRA-6, Smoothest Scaling Function at o

$$
\begin{aligned}
c_0 &= +0.43244669413947, & c_1 &= +1.12348982603632, \\
c_2 &= +0.70549917462643, & c_3 &= -0.17658708916728, \\
c_4 &= -0.13794586876589, & c_5 &= +0.05309726313096,
\end{aligned}
$$

while $\rho = .43707240150802$ and the generalized Hölder exponent is 1.19405581139788. We note a discrepancy between this result and the ones reported in [17, page 510] and [18, page 242] where the smoothest six-coefficient MRA wavelet is said to have a Hölder exponent of at least 1.40198 and at most 1.4176. We calculate an exponent of 1.123543439 for the wavelet reported there.

As we leave the first critical strip either M_0 or M_1 can be considered the optimal product. Then a second critical strip starts at $m = -.26637703880995$. On this strip the optimal product is of the form $M_0^n M_1^2$. The strip terminates at $m = -.58801735569420$. Then once again the optimal product is M_0 or M_1 until we arrive at the origin.

IV. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

We have demonstrated that the concept of optimal ball and the Optimal Norm Conjecture are important tools in investigating the spectral radius of a set of matrices.

B. Future Works

In addition to investigating some of the many conjectures included we intend to use the ONC-based algorithm to find the Hölder smoothest 8-coefficient MRA. This function is most likely the shortest orthogonal wavelet which is twice differentiable.

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