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Frequency Domain Sufficient Conditions for Stability Analysis of Linear Neutral Time-Delay Systems

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Abstract— **In this paper, we study the problem of stability of linear neutral time-delay systems. Specifically, using the notion of *structured phase margin*, which characterizes stability margins in terms of a nominal plant transfer function in the presence of unknown structured phase perturbations, we derive several new frequency-domain sufficient conditions for stability of linear neutral time-delay systems. We provide both delay-independent as well as delay-dependent sufficient conditions for stability.**

I. Introduction

In control systems literature, mathematical models of physical/engineering systems and ordinary differential equations are practically synonymous. However, for many physical systems, ordinary differential equations may be inadequate for capturing the dynamic behavior. Generalizations to ordinary differential equations such as hybrid system models [1] and functional differential equations [2] are necessary in order to capture the complex behavior of some systems. Specifically, in many complex systems such as communication networks involving power transfers between interconnecting system components that are not instantaneous, realistic dynamic models should account for information in transit [3]. Such models lead to time-delay dynamical systems. Time-delay dynamical systems and more generally functional differential equations have been extensively studied in the literature (see [3–15] and the numerous references therein). Functional differential equations have been classified into two categories, namely, *retarded-type* and *neutral-type*. A retarded time-delay differential equation is a differential equation where the time-derivative of the state depends on current state as well as past (delayed) state and a neutral time-delay differential equation is a differential equation where the time-derivative of the state not only depends on the current and delayed stated but also the past (delayed) derivative [2], [10–13]. Neutral time-delay systems arise in many engineering systems and have been studied extensively in the literature (see [2], [12], [13], [16], [17] and references within). In this paper, we study the stability problem for neutral time-delay systems. Specifically, we focus on deriving frequency-domain conditions for linear neutral time-delay systems. The basic idea relies on the fact that the stability characteristics of a linear neutral time-delay system can be studied in terms of a feedback interconnection of a matrix transfer function and a *phase* uncertainty block [12], [13]. Since phase uncertainties have unit gain, many delay-independent stability criteria were derived in the literature using the classical small gain theorem or, more generally, the scaled small gain theorem [12–14]. Furthermore, many delay-dependent stability criteria were also derived by applying the (scaled) small gain approach on a *transformed* time-delayed system [12–14].

In a recent paper [18], we introduced the notion of *structured phase margin* for characterizing stability margins in terms of a nominal plant transfer function perturbed by an unknown structured phase uncertainty. In the special case where the uncertainty has no internal structure, the structured phase margin specializes to the multivariable phase margin given in [19], [20]. Furthermore, the concept of the structured phase margin is identical to that of the structured singular value (see for example [21–27], and the numerous references therein) except for the fact that uncertainties are phase bounded as opposed to being gain bounded as in the case of structured singular value. As in the case of structured singular value, computation of the structured phase margin is very difficult. Hence, in [18] we derived many computable lower bounds to the structured phase margin.

In this paper, using the results on structured phase margin [18], we derive several new frequency-domain sufficient conditions for stability of linear neutral time-delay systems. We provide both delay-independent as well as delay-dependent sufficient conditions for stability. Since the lower bounds derived in [18] are given in terms of a minimization problem involving linear matrix inequalities all the sufficient conditions presented in this paper can be solved as generalized eigenvalue problems [28].

II. Mathematical Preliminaries

In this section, we introduce notation and a key result necessary for developing the main results of this paper. Let $\mathbb{R}$ (resp., $\mathbb{C}$) denote the set of real (resp., complex) numbers, let $\mathbb{R}^{n\times m}$ (resp., $\mathbb{C}^{n\times m}$) denote the set of real (resp., complex) $n \times m$ matrices, and let A^{T} and A^* denote the transpose and complex conjugate transpose of A, respectively. Let $\mathbb{S}^{n\times n}$ (resp., $\mathbb{H}^{n\times n}$) denote the set of $n \times n$ symmetric (resp.,

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Hermitian) matrices and let $\mathbb{P}^{n\times n}$ (resp., $\mathbb{N}^{n\times n}$) denote the set of $n \times n$ positive-definite (resp., nonnegative-definite) Hermitian matrices. We write $0_{n\times m}$ to denote the $n \times m$ zero matrix, I_n to denote the $n \times n$ identity matrix, $M > 0$ (resp., $M \geq 0$) to denote the fact that the Hermitian matrix M is positive-definite (resp., nonnegative-definite), and $\mathrm{spec}(M)$ and $\rho(M)$ to denote the spectrum and the spectral radius, respectively, of a square complex matrix M. Furthermore, we write $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ to denote the state space realization of the transfer function $G(s) = C(sI - A)^{-1}B + D$. Finally, $A \otimes B$ denotes the Kronecker product of matrices A and B.

The following result is a generalization of the Kalman-Yakubovich-Popov (KYP) lemma, and establishes the equivalence between a generalized frequency domain inequality and a linear matrix inequality (LMI).

Proposition 2.1 ([29]): Let $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times m}$, and $D \in \mathbb{R}^{p\times m}$. Furthermore, let $Q \in \mathbb{S}^{p\times p}$, $S \in \mathbb{R}^{p\times m}$, and $R \in \mathbb{S}^{m\times m}$. Then,

$$G^*(\jmath\omega)QG(\jmath\omega)+G^*(\jmath\omega)S+S^{\mathrm{T}}G(\jmath\omega)+R < 0,\ \omega \in [0,\infty), \tag{1}$$

if and only if there exists $P \in \mathbb{S}^{n\times n}$ such that

$$\begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Z \end{bmatrix} < 0, \tag{2}$$

where $X = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}QC$, $Y = PB + C^{\mathrm{T}}(QD + S)$, and $Z = R + S^{\mathrm{T}}C + C^{\mathrm{T}}S + D^{\mathrm{T}}QD$.

III. Structured Phase Margin of a Complex Matrix

In this section we introduce the notion of structured phase margin of a complex matrix which proves essential in deriving delay-dependent stability criteria for time-delay dynamical systems.

Definition 3.1 ([18]): Let $M \in \mathbb{C}^{n\times n}$. The *structured phase margin* $\phi(M)$ is defined by

$$\phi(M) \triangleq \begin{cases} \infty,\ \text{if } \det(I_n + Me^{\jmath\Theta}) \neq 0,\ \Theta \in \boldsymbol{\Theta}, \\ \min\{\rho(\Theta) : \det(I_n + Me^{\jmath\Theta}) = 0,\ \Theta \in \boldsymbol{\Theta}\}, \\ \quad \text{otherwise,} \end{cases}$$

where $\boldsymbol{\Theta} \subseteq \mathbb{C}^{n\times n}$ is a set of block-diagonal phase uncertainty matrices defined by

$$\begin{aligned} \boldsymbol{\Theta} \ \triangleq\ & \{\Theta \in \mathbb{H}^{n\times n} : -\pi I_n \leq \Theta \leq \pi I_n, \\ & \Theta = \text{block-diag}(I_{l_1} \otimes \Theta_1, I_{l_2} \otimes \Theta_2, \ldots, I_{l_r} \otimes \Theta_r), \\ & \Theta_i \in \mathbb{H}^{n_i\times n_i}, i = 1,\ldots,r\}, \end{aligned} \tag{3}$$

where the dimension n_i and the number of repetitions l_i of each block are such that $\sum_{i=1}^{r} l_i n_i = n$ and $r \geq 1$.

In the case where $r = 1$, $l_1 = 1$, and $n_1 = n$, $\phi(M)$ is referred to as the *multivariable phase margin* $\underline{\phi}(M)$. Furthermore, in the case where $r = 1$, $n_1 = 1$, and $l_1 = n$, $\phi(M)$ is referred to as the (*scalar*) *phase margin* of M and is denoted by $\overline{\phi}(M)$.

Remark 3.1: In the case where $r = 1$, $n_1 = n = 1$, and $l_1 = 1$, $\phi(M)$ corresponds to the smallest angle by which the complex number M can be rotated (clockwise or counterclockwise) in the complex plane before intersecting the $-1+\jmath 0$ point. Specifically, if $|M| \neq 1$, then $\phi(M) = \infty$ since no amount of rotation of M in the complex plane will intersect $-1 + \jmath 0$. Alternatively, if $M = e^{\jmath\alpha}$, where $\alpha \in [-\pi,\pi]$, then the angle of rotation of M in the complex plane needed to intersect $-1 + \jmath 0$ is simply $|\pi - \alpha|$, that is, $\phi(M) = |\pi - \alpha|$. More generally, let $G(s)$ denote a single-input, single-output transfer function. In this case, $\inf_{\omega\in\mathbb{R}} \phi(G(\jmath\omega))$ is the phase margin of $G(s)$.

Next, in order to account for the phase uncertainty structure we introduce the following scaling matrix set $\mathcal{T}$ defined by

$$\mathcal{T} \ \triangleq\ \{T \in \mathbb{H}^{n\times n} :\ T\Theta = \Theta T,\quad \Theta \in \boldsymbol{\Theta}\}. \tag{4}$$

Note that in light of the definition of $\boldsymbol{\Theta}$, $\mathcal{T}$ is the set of Hermitian matrices given by

$$\begin{aligned} \mathcal{T} \ =\ & \{T \in \mathbb{H}^{n\times n} :\ T = \text{block-diag}(T_1 \otimes I_{n_1}, T_2 \otimes I_{n_2}, \\ & \ldots, T_r \otimes I_{n_r}), T_i \in \mathbb{H}^{l_i\times l_i},\ i = 1,\ldots,r\}. \end{aligned} \tag{5}$$

Proposition 3.1: Let $M \in \mathbb{C}^{n\times n}$. Then the following statements hold:

i) Let $\alpha > 0$. Then $\alpha < \phi(M)$ (resp., $\alpha \leq \phi(M)$) if and only if $\det(I_n + Me^{\jmath\Theta}) \neq 0$, $\Theta \in \boldsymbol{\Theta}$, $\rho(\Theta) \leq \alpha$ (resp., $\rho(\Theta) < \alpha$).
ii) Let $T \in \mathcal{T}$ be nonsingular. Then $\phi(M) = \phi(T^{-1}MT)$.
iii) $\underline{\phi}(M) \leq \phi(M) \leq \overline{\phi}(M)$.
iv) $\overline{\phi}(M) \in [0,\pi] \cup \{\infty\}$.
v) $\overline{\phi}(M) \leq \pi$ if and only if there exists $\lambda \in \mathrm{spec}(M)$ such that $|\lambda| = 1$.

Proof: The proof is a direct consequence of the definitions of $\phi(M)$, $\underline{\phi}(M)$, and $\overline{\phi}(M)$. ■

Since the computation of the structured phase margin $\phi(M)$ is difficult in general, we next derive a lower bound for the structured phase margin. This lower bound will be presented in the form of a generalized eigenvalue problem and hence can be computed using linear matrix inequalities [30]. Specifically, let $M \in \mathbb{C}^{n\times n}$ and define

$$\begin{aligned} \gamma_{\mathrm{lb}}(M) \triangleq \inf\{\gamma \in \mathbb{R} : \text{there exist } R \in \mathcal{T} \text{ and } S \in \mathcal{T} \text{ such} \\ \text{that } S \geq 0,\ \text{and } M^*RM - R - M^*S - SM < 2\gamma S\}. \end{aligned} \tag{6}$$

Furthermore, define

$$\phi_{\mathrm{lb}}(M) \triangleq \begin{cases} \alpha, & \text{if } \gamma_{\mathrm{lb}}(M) \in [-1, 1], \\ \infty, & \text{if } \gamma_{\mathrm{lb}}(M) < -1, \end{cases} \tag{7}$$

where $\alpha \in [0,\pi]$ is such that $\cos(\alpha) = \gamma_{\mathrm{lb}}(M)$.

Theorem 3.1 ([18]): For every $M \in \mathbb{C}^{n\times n}$, $\phi_{\mathrm{lb}}(\cdot)$ is well-defined and $\phi_{\mathrm{lb}}(M) \leq \phi(M)$.

Corollary 3.1: Let $M \in \mathbb{C}^{n\times n}$ and let $\beta = \phi_{\rm lb}(M) - \pi$. If $\phi_{\rm lb}(Me^{-\jmath\beta}) = \pi$, then $\det(I_n + Me^{\jmath\Theta}) \neq 0$, $\Theta \in \mathbf{\Theta}$, and $\phi_{\rm lb}(M)I_n < \Theta < (2\pi - \phi_{\rm lb}(M))I_n$.

Proof: It follows from Theorem 3.1 that $\phi_{\rm lb}(Me^{-\jmath\beta}) \leq \phi(Me^{-\jmath\beta})$, and hence, by $i)$ of Proposition 3.1 $\det(I_n + Me^{-\jmath\beta}e^{\jmath\Theta}) \neq 0$, $\Theta \in \mathbf{\Theta}$, $-\pi I_n \leq \Theta \leq \pi I_n$. Hence, $\det(I_n + Me^{\jmath\Theta}) \neq 0$, $\Theta \in \mathbf{\Theta}$, and $\phi_{\rm lb}(M)I_n < \Theta < (2\pi - \phi_{\rm lb}(M))I_n$. ■

Remark 3.2: It follows from Theorem 3.1 and Proposition 3.1 that $\det(I_n + Me^{\jmath\Theta}) \neq 0$, $\Theta \in \mathbf{\Theta}$, $-\phi_{\rm lb}(M)I_n < \Theta < \phi_{\rm lb}(M)I_n$. If, in addition, $\phi_{\rm lb}(Me^{-\jmath(\pi-\phi_{\rm lb}(M))}) = \pi$, then it follows from Corollary 3.1 that $\det(I_n + Me^{\jmath\Theta}) \neq 0$, $\Theta \in \mathbf{\Theta}$, and $\phi_{\rm lb}(M)I_n < \Theta < (2\pi - \phi_{\rm lb}(M))I_n$. Hence, it follows from Theorem 3.1 and Corollary 3.1 that if $\phi_{\rm lb}(Me^{-\jmath\beta}) = \pi$, then $\det(I_n + Me^{\jmath\Theta}) \neq 0$, $\theta \in \mathbf{\Theta}$, and $-\phi_{\rm lb}(M)I_n < \theta < (2\pi - \phi_{\rm lb}(M))I_n$.

The following two results present special cases when the lower bound $\phi_{\rm lb}(\cdot)$ is equal to $\phi(\cdot)$.

Theorem 3.2 ([18]): Let $\mathcal{T} = \mathbb{C}^{n\times n}$ and let $M \in \mathbb{C}^{n\times n}$. Then $\overline{\phi}(M) = \phi_{\rm lb}(M)$.

Theorem 3.3 ([18]): Let $\mathcal{T} = \{T \in \mathbb{R}^{n\times n} : T = tI_n,\ t \in \mathbb{R}\}$ and let $M \in \mathbb{C}^{n\times n}$. Then $\underline{\phi}(M) = \phi_{\rm lb}(M)$.

IV. Stability of Linear Dynamical Systems with Structured Phase Uncertainties

In this section we state and prove a stability criterion for multivariable systems with generalized positive real frequency-dependent multipliers [31]. This criterion involves a square nominal transfer function $G(s)$ in a negative feedback interconnection with a complex, square, uncertain matrix Δ as shown in Figure 1. For this result, define the set $\mathbf{\Delta}_\alpha$ consisting

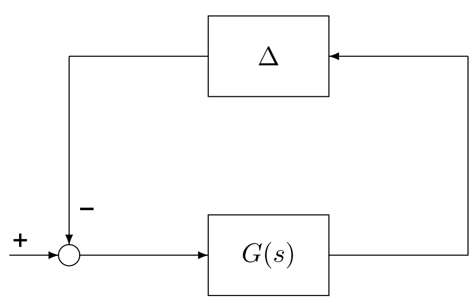

Fig. 1. Interconnection of transfer function $G(s)$ with uncertain matrix Δ

of unitary matrices given by

$$\mathbf{\Delta}_\alpha \triangleq \{\Delta \in \mathbb{C}^{n\times n} : \Delta = e^{\jmath\Theta},\ \Theta \in \mathbf{\Theta}, \rho(\Theta) < \alpha\},$$

where $\alpha \in (-\pi, \pi] \cup \{\infty\}$.

The following result is a direct consequence of multivariable Nyquist criterion.

Lemma 4.1: Let $\alpha \in (-\pi, \pi]$. Assume the negative feedback interconnection of $G(s)$ and $\Delta = I_n$ is asymptotically stable. Then the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\alpha$ if and only if $\det(I_n + G(\jmath\omega)\Delta) \neq 0$, $\Delta \in \mathbf{\Delta}_\alpha$, $\omega \in \mathbb{R}$.

The following result is a direct consequence of Proposition 3.1.

Theorem 4.1: Let $\alpha \in (-\pi, \pi]$. Assume the negative feedback interconnection of $G(s)$ and $\Delta = I_n$ is asymptotically stable. Then the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\alpha$ if and only if $\alpha \leq \phi(G(\jmath\omega))$, $\omega \in \mathbb{R}$.

Proof: Let $\omega \in \mathbb{R}$. It follows from Proposition 3.1 that $\alpha \leq \phi(G(\jmath\omega))$ if and only if $\det(I_n + G(\jmath\omega)\Delta) \neq 0$, $\Delta \in \mathbf{\Delta}_\alpha$. Now, the result is a direct consequence of Lemma 4.1. ■

Since it is difficult to compute $\phi(G(\jmath\omega))$ in general we provide the following sufficient condition for robust stability of the negative feedback interconnection of $G(s)$ and Δ.

Corollary 4.1: Let $\alpha \in (-\pi, \pi]$. Assume the negative feedback interconnection of $G(s)$ and $\Delta = I_n$ is asymptotically stable. Furthermore, assume there exist $R, S : \jmath\mathbb{R} \to \mathcal{T}$ such that for every $\omega \in \mathbb{R}$, $S(\jmath\omega) \geq 0$, and

$$\begin{aligned} 2\cos\alpha S(\jmath\omega) \quad > \quad & G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) - G^*(\jmath\omega)S(\jmath\omega) \\ & -S(\jmath\omega)G(\jmath\omega) - R(\jmath\omega). \end{aligned} \tag{8}$$

Then the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\alpha$.

Proof: Note that $\gamma_{\rm lb}(G(\jmath\omega)) \leq \cos\alpha$, $\omega \in [0, \infty)$, and hence $\phi(G(\jmath\omega)) \geq \phi_{\rm lb}(G(\jmath\omega)) \geq \alpha$, $\omega \in [0, \infty)$. Now, the result is a direct consequence of Theorem 4.1. ■

Corollary 4.2: Assume that the negative feedback interconnection of $G(s)$ and $\Delta = I_n$ is asymptotically stable. Furthermore, assume there exists $R : \jmath\mathbb{R} \to \mathcal{T}$ such that

$$R(\jmath\omega) - G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) > 0, \quad \omega \in [0, \infty). \tag{9}$$

Then the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\infty$.

Proof: The proof is a direct consequence of Corollary 4.1. Specifically, it follows from (9) that $\gamma_{\rm lb}(G(\jmath\omega)) < -1$, $\omega \in \mathbb{R}$ which implies that $\phi(G(\jmath\omega)) = \phi_{\rm lb}(G(\jmath\omega)) = \infty$. ■

Remark 4.1: Corollaries 4.1 and 4.2 provide sufficient conditions for robust stability of linear dynamical systems with block-structured phase uncertainties in terms of generalized frequency domain inequalities involving frequency-dependent multipliers. Hence, using Proposition 2.1, one can obtain sufficient conditions for robust stability using linear matrix inequalities involving the state space realizations of $G(s)$, $R(s)$, and $S(s)$.

V. Sufficient Conditions for Stability of Neutral Time-Delay Systems

In this section, we consider the problem of stability analysis of linear neutral time-delay dynamical systems. Specifically, we will transform the neutral time-delay stability analysis problem to a robust stability analysis problem with phase perturbations as discussed in Section IV. Then using the results developed in Section III we present several sufficient conditions for stability analysis of dynamical systems with neutral time-delay. Although the results presented below are restricted to the case single time-delay it should be noted that all the sufficient conditions can be trivially generalized to the case of multiple time-delays (see [18] for results on multiple time-delays in the case of retarded time-delay systems).

Consider the linear neutral time-delay dynamical system $\mathcal{G}$ given by

$$\begin{aligned}\dot{x}(t) + A_{\rm n}\dot{x}(t-\tau) &= Ax(t) + A_{\rm d}x(t-\tau),\\ x(\theta) &= \eta(\theta),\ -\tau \le \theta \le 0,\ t \ge 0, \quad (10)\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $t \ge 0$, $A, A_{\rm d}, A_{\rm n} \in \mathbb{R}^{n\times n}$, $\tau \ge 0$, $\eta(\cdot) \in \mathcal{C} = \mathcal{C}([-\tau,0],\mathbb{R}^n)$ is a continuously differentiable vector valued function specifying the initial state of the system, and $\mathcal{C}([-\tau,0],\mathbb{R}^n)$ denotes a Banach space of continuously differentiable functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Note that the state of (10) at time t is the *piece of trajectories* x between $t-\tau$ and t, or, equivalently, the *element* x_t in the space of continuously differentiable functions defined on the interval $[-\tau,0]$ and taking values in $\mathbb{R}^n$; that is, $x_t \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$, where $x_t(\theta) \triangleq x(t+\theta)$, $\theta \in [-\tau,0]$. Here, we assume that $\dot{\eta}(0) + A_{\rm n}\dot{\eta}(-\tau) = A\eta(0) + A_{\rm d}\eta(-\tau)$, so that $x_t \in \mathcal{C}$ for all $t \ge 0$ [2]. Furthermore, since for a given time t the piece of the trajectories x_t is defined on $[-\tau,0]$, the uniform norm $|||x_t||| = \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\|$ is used for the definition of asymptotic stability of (10) where $\|\cdot\|$ is a vector norm defined on $\mathbb{R}^n$. For further details see [2], [6].

Next, we derive sufficient conditions for stability of neutral time-delay systems which are classified into two categories, namely *delay-dependent* and *delay-independent*. A sufficient condition is delay-independent if it guarantees stability for all $\tau \in [0,\infty)$ and it is delay-dependent if it guarantees stability for all $\tau \in [0,\bar{\tau})$ for some $\bar{\tau} > 0$. In both cases, two necessary conditions are that $\rho(A_{\rm n}) < 1$ and the linear system given by (10) is stable with $\tau = 0$, that is $(I_n + A_{\rm n})^{-1}(A + A_{\rm d})$ is Hurwitz. Hence, in what follows, we assume that $\rho(A_{\rm n}) < 1$ and $(I_n + A_{\rm n})^{-1}(A + A_{\rm d})$ is Hurwitz. The following lemma is a direct consequence of Theorems 3.19 and 3.20 of [13, p. 109].

Lemma 5.1: Let $\bar{\tau} \in [0,\infty]$ Assume that $\rho(A_{\rm n}) < 1$ and $(I_n + A_{\rm n})^{-1}(A + A_{\rm d})$ is Hurwitz. Then, the neutral time-delay system $\mathcal{G}$ given by (10) is asymptotically stable for all $\tau \in [0,\bar{\tau})$ if and only if

$$\det[I_n + G(j\omega)\Delta(j\omega)] \ne 0,\ \omega \in (0,\infty),\ \tau \in [0,\bar{\tau}), \quad (11)$$

where $G(s) \sim \left[\begin{array}{c|c} A & A_{\rm d} - AA_{\rm n} \\ \hline I_n & -A_{\rm n}\end{array}\right]$ and $\Delta(s) = e^{-\tau s}I_n$.

In this section, we consider a special structure of $\boldsymbol{\Theta}$ given by

$$\boldsymbol{\Theta} = \{\Theta \in \mathbb{R}^{n\times n}:\ \Theta = \theta I_n,\ \theta \ge 0,\ \}, \quad (12)$$

so that $\mathcal{T} = \mathbb{C}^{n\times n}$. Note that if $\boldsymbol{\Theta}$ is given by (12) then it follows from Theorem 3.2 that $\phi(M) = \phi_{\rm lb}(M) = \overline{\phi}(M)$, $M \in \mathbb{C}^{n\times n}$.

For the following result define $\alpha_{\rm lb} : \mathbb{C}^{n\times n} \to [0,2\pi)$ by

$$\alpha_{\rm lb}(M) \triangleq \begin{cases} \infty, & \text{if } \phi_{\rm lb}(M) = \infty,\\ \phi_{\rm lb}(M), & \text{if } \phi_{\rm lb}(M) < \infty \\ & \text{and } \phi_{\rm lb}(Me^{j(\phi_{\rm lb}(M)-\pi)}) < \pi,\\ 2\pi - \phi_{\rm lb}(M), & \text{otherwise.}\end{cases} \quad (13)$$

Theorem 5.1: Assume $\rho(A_{\rm n}) < 1$ and $(I_n+A_{\rm n})^{-1}(A+A_{\rm d})$ is Hurwitz and let $\tau_{\rm lb} \triangleq \inf_{\omega>0} \frac{\alpha_{\rm lb}(G(j\omega))}{\omega}$. Then the neutral time-delay dynamical system $\mathcal{G}$ is asymptotically stable for $\tau \in [0,\tau_{\rm lb})$.

Proof: If $\tau_{\rm lb} = \infty$, then for all $\omega \in (0,\infty)$, $\alpha_{\rm lb}(G(j\omega)) = \phi_{\rm lb}(G(j\omega)) = \infty$. Now, it follows from Theorem 3.1 that $\phi(G(j\omega)) = \infty$, $\omega \in (0,\infty)$. Hence, by definition, $\det(I_n + G(j\omega)e^{j\Theta}) \ne 0$, $\Theta \in \boldsymbol{\Theta}$, which implies that $\det(I_n + G(j\omega)\Delta(j\omega)) \ne 0$, $\omega \in (0,\infty)$, $\tau \in [0,\tau_{\rm lb})$.

Next, assume $\tau_{\rm lb} < \infty$ and let $\omega \in (0,\infty)$ be such that $\alpha_{\rm lb}(G(j\omega)) = \phi_{\rm lb}(G(j\omega))$. Now, since $\phi_{\rm lb}(G(j\omega)) \le \phi(G(j\omega))$, it follows from *i)* of Proposition 3.1 that $\det(I_n + G(j\omega)e^{j\Theta}) \ne 0$, $\Theta \in \boldsymbol{\Theta}$, $-\alpha_{\rm lb}(G(j\omega))I_n < \Theta < \alpha_{\rm lb}(G(j\omega))I_n$. Now, for all $\tau \in [0,\tau_{\rm lb})$, it follows that $-\alpha_{\rm lb}(G(j\omega)) < -\omega\tau$ which implies that $\det(I_n + G(j\omega)\Delta(j\omega)) \ne 0$, $\tau \in [0,\tau_{\rm lb})$. Next, let $\omega \in (0,\infty)$ be such that $\phi_{\rm lb}(G(j\omega)) < \infty$, $\alpha_{\rm lb}(G(j\omega)) = 2\pi - \phi_{\rm lb}(G(j\omega))$ or, equivalently, $\alpha_{\rm lb}(G(j\omega)e^{-j\beta}) = \pi$, where $\beta = \alpha_{\rm lb}(G(j\omega)) - \pi$. Now, it follows from Corollary 3.1 that $\det(I_n + G(j\omega)e^{j\Theta}) \ne 0$, $\Theta \in \boldsymbol{\Theta}$, $\phi_{\rm lb}(G(j\omega))I_n < \Theta < (2\pi - \phi_{\rm lb}(G(j\omega)))I_n$. Once again, for all $\tau \in [0,\tau_{\rm lb})$, it follows that $-\alpha_{\rm lb}(G(j\omega)) < -\omega\tau$ which implies that $\det(I_n + G(j\omega)\Delta(j\omega)) \ne 0$, $\tau \in [0,\tau_{\rm lb})$. Thus, $\det(I_n + G(j\omega)\Delta(j\omega)) \ne 0$, $\omega \in (0,\infty)$, $\tau \in [0,\tau_{\rm lb})$, and hence, it follows from Lemma 5.1 that $\mathcal{G}$ is asymptotically stable for $\tau \in [0,\tau_{\rm lb})$. ■

Corollary 5.1: Let $\bar{\tau} > 0$ and assume $\rho(A_{\rm n}) < 1$ and $(I_n + A_{\rm n})^{-1}(A + A_{\rm d})$ is Hurwitz. Furthermore, assume there exist functions $M_{\bar{\tau}} : j\mathbb{R} \to \mathbb{C}^{n\times n}$, $R : j\mathbb{R} \to \mathcal{T}$, and $S : j\mathbb{R} \to \mathcal{T}$, such that $M^*_{\bar{\tau}}(j\omega)M_{\bar{\tau}}(j\omega) \in \mathcal{T}$, $S(j\omega) \ge 0$, $\omega \in (0,\infty)$,

$$\sigma_{\min}(M_{\bar{\tau}}(j\omega)) > \begin{cases} 2\sin(\frac{\omega\bar{\tau}}{2}), & \omega \in (0, \frac{\pi}{\bar{\tau}}] \\ 2, & \omega > \frac{\pi}{\bar{\tau}},\end{cases} \quad (14)$$

and

$$\begin{aligned}&2S(j\omega) - M^*_{\bar{\tau}}(j\omega)S(j\omega)M_{\bar{\tau}}(j\omega) + G^*(j\omega)R(j\omega)G(j\omega)\\ &+ G^*(j\omega)S(j\omega) + S(j\omega)G(j\omega)\\ &- R(j\omega) > 0,\ \omega \in (0,\infty). \quad (15)\end{aligned}$$

Then the linear neutral time-delay dynamical system $\mathcal{G}$ given by (10) is asymptotically stable for all $\tau \in [0, \bar{\tau})$.

Proof: It follows from (15) that

$$\gamma_{\rm lb}(G(\jmath\omega)) \leq 1 - \frac{1}{2}\lambda_{\min}(M_{\bar{\tau}}^*(\jmath\omega)M_{\bar{\tau}}(\jmath\omega)).$$

Now, it follows from (14) that

$$\begin{array}{rcl}\gamma_{\rm lb}(G(\jmath\omega)) & \leq & 1 - \frac{1}{2}\sigma_{\min}^2(M_{\bar{\tau}}(\jmath\omega)) \\ & < & 1 - 2\sin^2(\frac{\omega\bar{\tau}}{2}) \\ & = & \cos(\omega\bar{\tau}),\ \omega \in (0, \frac{\pi}{\bar{\tau}}],\end{array}$$

and $\gamma_{\rm lb}(G(\jmath\omega)) < -1$, $\omega > \frac{\pi}{\bar{\tau}}$, or, equivalently, $\phi_{\rm lb}(G(\jmath\omega)) > \omega\bar{\tau}$, $\omega \in (0, \frac{\pi}{\bar{\tau}}]$, and $\phi_{\rm lb}(G(\jmath\omega)) = \infty$, $\omega > \frac{\pi}{\bar{\tau}}$. Hence, it follows that $\inf_{\omega>0} \frac{\phi_{\rm lb}(G(\jmath\omega))}{\omega} \geq \bar{\tau}$ which further implies that $\bar{\tau} \leq \tau_{\rm lb}$. Now, the result is a direct consequence of Theorem 5.1. ■

Remark 5.1: An obvious choice for $M_{\bar{\tau}}(\jmath\omega)$ is $M_{\bar{\tau}}(\jmath\omega) = \jmath\omega\bar{\tau}I_n$, since, for $\omega\bar{\tau} \in (0, \pi]$, we have $\omega\bar{\tau} > 2\sin(\frac{\omega\bar{\tau}}{2})$, and $\omega\bar{\tau} > 2$ otherwise.

Remark 5.2: If (15) holds with $M_{\bar{\tau}}(\jmath\omega) = (2+\varepsilon)I_n$ for some $\varepsilon > 0$, then it follows from Corollary 5.1 that $\mathcal{G}$ is asymptotically stable for all $\tau \in [0, \infty)$. In addition, it can be shown that if (15) holds with $M_{\bar{\tau}}(\jmath\omega) = 2I_n$, then $\mathcal{G}$ is asymptotically stable for all $\tau \in [0, \infty)$.

Corollary 5.2: Let $\bar{\tau} > 0$ and assume $\rho(A_{\rm n}) < 1$ and $(I_n + A_{\rm n})^{-1}(A + A_{\rm d})$ is Hurwitz. Furthermore, assume there exist functions $M_{\bar{\tau}} : \jmath\mathbb{R} \to \mathbb{C}^{n\times n}$, $R : \jmath\mathbb{R} \to \mathcal{T}$, and $S : \jmath\mathbb{R} \to \mathcal{T}$ such that $M_{\bar{\tau}}^*(\jmath\omega)M_{\bar{\tau}}(\jmath\omega) \in \mathcal{T}$, $S(\jmath\omega) \geq 0$, $\omega \in (0, \infty)$,

$$\sigma_{\min}(M_{\bar{\tau}}(\jmath\omega)) > \left\{\begin{array}{ll} 2\sin(\frac{\omega\bar{\tau}}{2}), & \omega \in (0, \frac{\pi}{\bar{\tau}}] \\ 2, & \omega > \frac{\pi}{\bar{\tau}}, \end{array}\right. \quad (16)$$

and

$$\begin{array}{rl} & [I_n + G^*(\jmath\omega)]S(\jmath\omega)[I_n + G(\jmath\omega)] \\ - & G^*(\jmath\omega)M_{\bar{\tau}}^*(\jmath\omega)S(\jmath\omega)M_{\bar{\tau}}(\jmath\omega)G(\jmath\omega) + R(\jmath\omega) \\ - & G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) > 0,\ \omega \in (0, \infty). \end{array} \quad (17)$$

Then the linear neutral time-delay dynamical system $\mathcal{G}$ given by (10) is asymptotically stable for all $\tau \in [0, \bar{\tau})$.

Proof: The proof is a direct consequence of Corollary 5.1 by replacing $R(\jmath\omega)$ with $S(\jmath\omega) - M_{\bar{\tau}}^*(\jmath\omega)S(\jmath\omega)M_{\bar{\tau}}(\jmath\omega) - R(\jmath\omega)$. ■

Corollary 5.3: Let $\bar{\tau} > 0$ and assume $\rho(A_{\rm n}) < 1$ and $(I_n + A_{\rm n})^{-1}(A + A_{\rm d})$ is Hurwitz. Furthermore, let $M_{\bar{\tau}} : \jmath\mathbb{R} \to \mathbb{C}^{n\times n}$ be such that

$$\sigma_{\min}(M_{\bar{\tau}}(\jmath\omega)) > \left\{\begin{array}{ll} 2\sin(\frac{\omega\bar{\tau}}{2}), & \omega \in (0, \frac{\pi}{\bar{\tau}}] \\ 2, & \omega > \frac{\pi}{\bar{\tau}}, \end{array}\right. \quad (18)$$

and $\rho(H(\jmath\omega)) < 1$, $\omega \in (0, \infty)$, where $H(\jmath\omega) \triangleq M_{\bar{\tau}}(\jmath\omega)G(\jmath\omega)(I_n + G(\jmath\omega))^{-1}$. Then the linear neutral time-delay dynamical system $\mathcal{G}$ given by (10) is asymptotically stable for all $\tau \in [0, \bar{\tau})$.

Proof: The proof is a direct consequence of Corollary 5.2 with $R(\jmath\omega) = 0$. ■

Remark 5.3: If $M_{\bar{\tau}}(\jmath\omega) = \jmath\omega\bar{\tau}I_n$, then it follows from Corollary 5.3 that $\mathcal{G}$ is asymptotically stable for all $\tau \in [0, \bar{\tau})$, where $\bar{\tau} = \inf_{\omega\in(0,\infty)} \frac{1}{\rho(H(\jmath\omega))}$ and $H(s) \triangleq sG(s)(I_n + G(s))^{-1}$.

Corollary 5.4: Let $\bar{\tau} > 0$ and assume $\rho(A_{\rm n}) < 1$ and $(I_n + A_{\rm n})^{-1}(A + A_{\rm d})$ is Hurwitz. Let $\hat{\gamma} \in \mathbb{R}$ and $\hat{\omega} \in [0, \infty)$ be defined by

$$\begin{array}{rl}\hat{\gamma} \triangleq \inf\limits_{\omega>0}\{\gamma \in \mathbb{R} : & \text{there exist } R : \jmath\mathbb{R} \to \mathcal{T} \text{ and } S : \jmath\mathbb{R} \to \mathcal{T} \\ & \text{such that for every } \omega \in (0, \infty), \\ & S(\jmath\omega) \geq 0 \text{ and } 2\gamma S(\jmath\omega) + G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) \\ & +G^*(\jmath\omega)S(\jmath\omega) + S(\jmath\omega)G(\jmath\omega) - R(\jmath\omega) > 0\},\end{array} \quad (19)$$

$$\begin{array}{rl}\hat{\omega} \triangleq & \inf\{\bar{\omega} \in (0, \infty) : \text{there exists } R : \jmath\mathbb{R} \to \mathcal{T} \text{ such that} \\ & \text{for every } \omega \geq \bar{\omega},\ R(\jmath\omega) - G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) > 0\}.\end{array} \quad (20)$$

Then the neutral time-delay dynamical system $\mathcal{G}$ is asymptotically stable for $\tau \in [0, \hat{\tau})$, where $\hat{\tau} = \frac{\cos^{-1}(\hat{\gamma})}{\hat{\omega}}$.

Proof: Note that $\gamma_{\rm lb}(G(\jmath\omega)) \leq \hat{\gamma}$, $\omega \in (0, \infty)$, and hence, $\phi_{\rm lb}(G(\jmath\omega)) \geq \cos^{-1}(\hat{\gamma})$. Now, for all $\omega \in (0, \hat{\omega}]$, $\frac{\phi_{\rm lb}(G(\jmath\omega))}{\omega} \geq \frac{\cos^{-1}(\hat{\gamma})}{\hat{\omega}} = \hat{\tau}$. Next, for every $\omega > \hat{\omega}$, since there exists $R(\jmath\omega)$ such that $G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) - R(\jmath\omega) < 0$, it follows that $\gamma_{\rm lb}(G(\jmath\omega)) = -\infty$ or, equivalently, $\phi_{\rm lb}(G(\jmath\omega)) = \infty$. Hence, $\hat{\tau} \leq \tau_{\rm lb}$. Now, the result is a direct consequence of Theorem 5.1. ■

Remark 5.4: Since $G(j\infty) = A_{\rm n}$ and $\rho(A_{\rm n}) < 1$, it follows that

$$\hat{\omega} = \inf\{\overline{\omega} \in (0, \infty) : \text{for every } \omega \geq \overline{\omega},\ \rho(A_{\rm n}) < 1\}. \quad (21)$$

Corollary 5.5: Assume $\rho(A_{\rm n}) < 1$ and $(I_n + A_{\rm n})^{-1}(A + A_{\rm d})$ is Hurwitz. The neutral time-delay dynamical system $\mathcal{G}$ is asymptotically stable for $\tau \in [0, \infty)$, if and only if there exist $R : \jmath\mathbb{R} \to \mathbb{H}^{n\times n}$ such that

$$R(\jmath\omega) - G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) > 0, \quad \omega \in (0, \infty). \quad (22)$$

Proof: Sufficiency is a direct consequence of Corollary 5.4. Specifically, it follows from (22) that $\hat{\omega} = 0$ which implies that $\hat{\tau} = \infty$. Next, assume that $\mathcal{G}$ is asymptotically stable for $\tau \in [0, \infty)$. Hence, it follows from Lemma 5.1 that $\det(I_n + G(\jmath\omega)e^{-\jmath\omega\tau}) \neq 0$, $\omega \in \mathbb{R}$ and $\tau \in [0, \infty)$ which implies that $\det(I_n + G(\jmath\omega)e^{-\jmath\theta}) \neq 0$, $\theta \in \mathbb{R}$. Thus, it follows that $\overline{\phi}(G(\jmath\omega)) = \phi_{\rm lb}(G(\jmath\omega)) = \infty$ which proves the result. ■

Remark 5.5: As in Remark 5.4, since $G(j\infty) = A_{\rm n}$ and $\rho(A_{\rm n}) < 1$, it follows that (22) holds for all $\omega \in (0, \infty)$ if and only if $\rho(G(j\omega)) < 1$. Hence, the neutral time-delay dynamical system $\mathcal{G}$ is asymptotically stable for all $\tau \in [0, \infty)$ if and only if $\rho(G(j\omega)) < 1$.

Remark 5.6: In the case where $A, A_{\rm n}, A_{\rm d} \in \mathbb{R}$, that is $\mathcal{G}$ is a scalar neutral time-delay system, it follows from Corollary 5.5 that $\mathcal{G}$ is asymptotically stable for all $\tau \in [0, \infty)$ if and only if $|G(\jmath\omega)| < 1$, $\omega \in (0, \infty)$, or, equivalently, $|A_{\rm n}| < 1$ and $|A_{\rm d}| < |A|$.

VI. Illustrative Numerical Examples

In this section we consider two illustrative numerical examples to demonstrate the utility of the proposed theory.

Example 6.1: Consider a two-dimensional, linear neutral time-delay dynamical system of the form (10) given by

$$A = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix},\ A_{\mathrm{n}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \text{and}$$

$$A_{\mathrm{d}} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

For this example, we obtained $\alpha_{\mathrm{lb}}(G(\jmath\omega)) = \infty$, $\omega \neq 0.7063$ and $\alpha_{\mathrm{lb}}(G(\jmath\omega)) = 2.6932$, $\omega = 0.7063$. Hence, $\tau_{\mathrm{lb}} = 3.8133$. Furthermore, using multivariable Nyquist criterion it can be shown that $\tau = \tau_{\mathrm{lb}} = 3.8133$.

Example 6.2: In this example, we adopt the linear neutral time-delay system [17]

$$\dot{x}(t) - \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \dot{x}(t-\tau) = \begin{bmatrix} -0.8 & 0.2 \\ -0.2 & -0.8 \end{bmatrix} x(t-\tau), \quad (23)$$

where $0 < c < 1$.

For this example, we compute τ_{lb} using Theorem 5.1 for several values of $c \in [0, 1)$. We then compare our predictions with those of the results given in [17]. Table I shows that our results provide better predictions than those of [17].

TABLE I
Maximum allowable delay prediction for Example 6.2

c	0.0	0.1	0.2	0.3	0.4
τ_{lb} (Theorem 5.1)	1.6078	1.4468	1.2784	1.1046	0.9273
Theorem 1 [17]	1.176	1.055	0.933	0.812	0.691
c	0.5	0.6	0.7	0.8	0.9
τ_{lb} (Theorem 5.1)	0.7484	0.5704	0.3964	0.2315	0.0863
Theorem 1 [17]	0.570	0.448	0.327	0.206	0.085

VII. Conclusion

In this paper, we studied the problem of stability of linear neutral time-delay systems. Specifically, using the notion of *structured phase margin*, which characterizes stability margins in terms of a nominal plant transfer function in the presence of unknown structured phase perturbations, we derived several new frequency-domain sufficient conditions for stability of linear neutral time-delay systems. We provided both delay-independent as well as delay-dependent sufficient conditions for stability. It should be noted that all the sufficient conditions presented in this paper are in terms of linear matrix inequalities and hence computationally feasible. Finally, we illustrated our results through numerical examples.

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