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Geometrical description of quasi-hemispherical and calotte-like surfaces using discretised argument-transformed Chebyshev-polynomials

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Abstract— **The even Chebyshev-polynomials of the second kind – modified with appropriate argument-transforms – were used to describe quasi-hemispherical and calotte-like natural surfaces over a set of discrete points – and via interpolation – between these points. The point-set used was selected in a manner that promotes the proper approximation of such surfaces and the numerical implementation of the surface representation algorithm. The representations of such optical surfaces (e.g., the outer surface of the human cornea) in the basis formed by these Chebyshev-polynomials – and by their transformed versions – provide computational alternatives to the Zernike-based description of the optical aberrations caused by such surfaces.**

I. INTRODUCTION

The concise geometrical description of quasi-spherical, quasi-hemispherical and calotte-like surfaces is of high importance in fields ranging from geography to adaptive optics, and from ophthalmologic measurements to inspection of spherical mechanical parts.

The optical features and aberrations (e.g., astigmatism, coma) induced by such optically refractive or reflective surfaces are often characterized by Zernike-coefficients. This characterisation is frequently used in ophthalmic practice for characterising the optical aberrations of the living cornea [1]. It is of importance from diagnostic point of view as the outer calotte-like corneal surface is responsible for about 60% of the optical power of the eye and much of the aberrations of the eye are caused by irregularities of the corneal surface.

In case of surgical operations aiming to correct the sight of patients by reshaping their cornea, the precise knowledge of the corneal surface is of utmost importance, a surface oriented approach was suggested by [2]. The authors of that paper used argument-transformed Chebyshev-polynomials of the second kind for describing the outer surface of human cornea. In the present paper, an advantageous discretisation scheme of the mentioned continuous function-basis is described in detail.

In Section II, continuous 2D orthogonal transformations based on the even Chebyshev-polynomials and their argument-transformed versions are shown as these transformations have the potential of producing concise representations of quasi-hemispherical and calotte-like surfaces. In Section III, the discrete versions of the transformations covered in II are presented. The discretisation presented promotes the approximation of the quasi-hemispherical and calotte-like surfaces and the numerical implementation of the surface representation algorithm. In Section IV conclusions are drawn, and further work directions are mentioned.

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II. CONTINOUOS MODELLING OF QUASI-HEMISPHERICAL AND CALOTTE-LIKE SURFACES

A. Modelling of quasi-hemispherical surfaces

A surface defined on the unit disk $\mathbb{D} := \{(x, y) : x^2+y^2 \leqq 1\}$ can be described by a two-variable function $f(x, y)$. The application of the polar transform to variables x and y results in

$$x = r\cos\varphi, \quad y = r\sin\varphi, \tag{1}$$

where r and φ are the radial and the azimuthal variables, respectively, and where

$$0 \leq r \leq 1, \quad 0 \leq \varphi \leq 2\pi.$$

Using variables r and φ, f(x,y) can be transcribed into the following form:

$$F(r, \varphi) := f(r\cos\varphi, r\sin\varphi) \tag{2}$$

Function F – being a function over the unit disk – can be expressed in Zernike-basis [3], [4]. This representation is often used in the ophthalmological practice for optical power maps of the cornea. The system of Zernike-functions is orthogonal over the unit-circle with respect to the measure $r\,dr\,d\varphi$, that is

$$\int_0^1\int_0^{2\pi} Z_n^m(r,\varphi)\overline{Z}_{n'}^{m'}(r,\varphi) r\,dr d\varphi = \frac{\pi}{n+1}\delta_{nn'}\delta_{mm'},$$

where $\delta_{nn'}$ is the Kronecker-symbol. A discretised form of the Zernike-system that provides the means of a fast computation has been recently obtained [5].

F, however, can be expressed in other function-bases, as well; for instance, in ones which are more "friendly" towards quasi-hemispherical, or calotte-like surfaces. In such a function-basis, the hemisphere (shown in Fig. 1), or a spherical calotte – the one shown in Fig. 2 is tuned to the "standard" shape of the human cornea – over the unit-disk can be represented in a very concise way [2].

The authors of the mentioned paper used Chebyshev-polynomials of 2nd kind (in the radial direction) and functions of the widely used trigonometrical system (in the asimuthal direction) for representing quasi-hemispherical

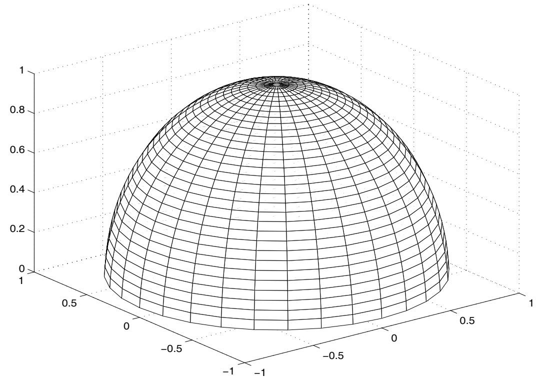

Fig. 1. The hemisphere over the unit circle

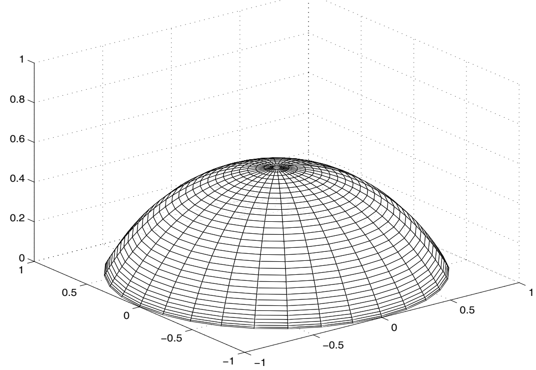

Fig. 2. A cornea-like "calotte" surface

surfaces, and the argument-transformed versions of these for representing calotte-like surfaces.

There are good reasons for using Chebyshev-polynomials for representing quasi-hemispherical and calotte-like surfaces. For example, the Chebyshev-polynomials are closely connected to the trigonometric system and hence many of the methods and algorithms readily available for the trigonometric system are expected to work well also with the new system; the application of the summation methods used for the trigonometric systems result in approximations that are uniformly convergent over the unit circle.

Furthermore, the polynomials of the Chebyshev-system are orthogonal with respect to the weight function $\sqrt{1-r^2}$ $(0 \leq r \leq 1)$ [6]. As a consequence, the function $F(r,\varphi) = \sqrt{1-r^2}$, which describes a hemisphere, can be represented – using the function of the proposed system and its weight function – as a single component.

The Chebyshev-polynomials $\{U_n(x)\}$ of 2nd kind can be introduced as follows:

$$U_n(\cos t) := \frac{\sin(n+1)t}{\sin t} = \quad (t \in \mathbb{R}, n \in \mathbb{N}). \tag{3}$$

The Chebyshev-polynomials of second kind satisfy the following second order recursion

$$\begin{aligned} U_0(x) = 1, \quad U_1(x) = 2x, \\ U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \\ (x \in \mathbb{R}, n = 1, 2, \cdots). \end{aligned} \tag{4}$$

Given $V_n := U_{2n}$ and $W_n := U_{2n+1}$, the orthogonality of both systems – i.e., $\{V_n\}$ and $\{W_n\}$ – over the interval $[0,1]$ can be easily demonstrated. In the rest of the paper, only the V_n system will be considered. For this system the following orthogonality relation holds:

$$\int_0^1 V_n(r)V_m(r)\sqrt{1-r^2}\,dr = \frac{\pi}{4}\delta_{mn}. \tag{5}$$

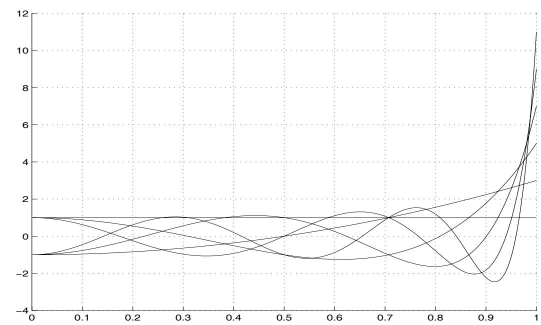

Fig. 3. The first few functions of the $\{V_n\}$ system

Building upon this system, let the following system of complex valued functions of variables r and φ be introduced:

$$\begin{aligned} \mathcal{V}_{nm}(r,\varphi) := V_n(r)e^{im\varphi} \\ ((r,\varphi) \in I,\ n \in \mathbb{N}, m \in \mathbb{Z}) \end{aligned} \tag{6}$$

This system – as the trigonometrical system is also an orthogonal system – forms a complete orthogonal system over the unit disk $\mathbb{D}$ with respect to the weight-function $\rho(r,\varphi) = \sqrt{1-r^2}$, where $((r,\varphi) \in I := [0,1] \times [0,2\pi])$, i.e.

$$\begin{aligned} \int_0^1\int_0^{2\pi} \mathcal{V}_{nm}(r,\varphi)\overline{\mathcal{V}_{n'm'}(r,\varphi)}\rho(r,\varphi)d\varphi dr = \frac{\pi^2}{2}\delta_{mm'}\delta_{nn'} \\ ((m,n),(m',n') \in \mathcal{N} := \mathbb{N} \times \mathbb{Z}). \end{aligned} \tag{7}$$

The systems $\{\mathcal{V}_{nm}\}$ and $\{\mathcal{V}_{nm}\rho\}$ $((n,m) \in \mathcal{N})$ can be interpreted as a bi-orthogonal system in $L^2(I)$, with respect to the usual inner product

$$\langle f,g \rangle := \int_0^1 \int_0^{2\pi} f\overline{g} d\varphi\, dr.$$

Hence any function $F \in L^2(I)$ can be realized by

$$F \sim \sum_{(n,m)\in\mathcal{N}} \langle F, \mathcal{V}_{nm}\rangle \mathcal{V}_{nm}\rho \tag{8}$$

bi-orthogonal representation.

Specifically, if $F(r,\varphi) := \sqrt{1-r^2}$ – i.e. a hemisphere surface, as shown in Fig. 1 – the representation will be reduced to a single component. The coefficient of the series expansion in (8) can be expressed with the trigonometrical

Fourier-coefficients. It is advantageous to express function F in the following form: $F(r,\varphi) = G(r,\varphi)\sqrt{1-r^2}$ ($(r,\varphi) \in I$). (Note that in case of $G(r,\varphi) = 1$, F is the function describing the hemisphere over the unit disk.)

After the substitution $r = \cos t$ and considering that

$$V_n(r) = \frac{\sin\left((2n+1)\arccos r\right)}{\sqrt{1-r^2}}\ , \tag{9}$$

the coefficients of the series expansion in (8) can be given as follows.

$$\begin{aligned} &\langle F, \mathcal{V}_{nm}\rangle = \\ &= \int_0^1 \int_0^{2\pi} F(r,\varphi)\frac{\sin\left((2n+1)\arccos r\right)}{\sqrt{1-r^2}}\cdot \qquad (10)\\ &\cdot \exp\left(-im\varphi\right)d\varphi\, dr = \\ &= \int_0^{\pi/2}\int_0^{2\pi} G(\cos t,\varphi)\sin t \sin\left((2n+1)t\right)\cdot \qquad (11)\\ &\cdot \exp\left(-im\varphi\right)d\varphi\, dt = \qquad (12)\\ &= \frac{1}{4}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} G(\cos t,\varphi)\cdot \\ &\cdot(\cos\left(2nt\right) - \cos\left((2n+2)t\right)\exp\left(-im\varphi\right)d\varphi\, dt. \qquad (13)\end{aligned}$$

After the substitution $t = s/2$, the coefficients can be expressed as the complex trigonometrical Fourier-coefficients of the following function:

$$H(s,\varphi) := G(\cos\frac{s}{2}) \quad (-\pi \le s \le \pi, 0 \le \varphi \le 2\pi),$$

as H is even with respect to s.

$$\hat{H}(k,l) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^{2\pi} H(s,\varphi)\exp\left(-iks - il\varphi\right)d\varphi\, ds. \tag{14}$$

$$\langle F, \mathcal{V}_{nm}\rangle = \frac{\pi^2}{2}(\hat{H}(n+1,m) - \hat{H}(n,m)) \qquad (n \in \mathbb{N}, m \in \mathbb{Z}). \tag{15}$$

B. The argument transform

Using the approach of argument transform described below, various useful orthogonal systems can be built from the $V_n(r)$ functions. The functions generated in this manner are orthogonal with respect to a non-negative weight function $\rho(t)$ $(0 \le t \le 1)$. The actual weight function can be chosen to match the requirements of the given application. E.g., when dealing with cornea surfaces - that is, surfaces assumed to be spherical calotte-like surfaces – the proper choice for the weight function could be a circular segment rather than a semi-circle. In this case, the weight function $\sqrt{1-r^2}$ – which is used together with the original $V_n(r)$ functions – is replaced by

$$\rho_a(r) := \sqrt{a^2 - r^2} - \sqrt{a^2 - 1} \qquad (0 \le r \le 1, a \ge 1). \tag{16}$$

For the standard cornea dimensions the value of parameter a is around 1.4.

Applying the continuously differentiable bijection $R : [0,1] \to [0,1]$ (with $R(0) = 0$ and $R(1) = 1$) on function $V_n(r)$ and using the argument transform $r = R(t)$ for (5), results in

$$\begin{aligned}\frac{\pi}{4}\delta_{mn} &= \int_0^1 V_m(r)V_n(r)\sqrt{1-r^2}\, dr = \\ &= \int_0^1 V_m(R(t))V_n(R(t))R'(t)\sqrt{1-R(t)^2}\, dt\end{aligned}$$

Let the function R together with an appropriate constant c - be chosen in the following manner:

$$R'(t)\sqrt{1-R^2(t)} = c\rho(t) \quad (0 \le t \le 1) \tag{17}$$

where $\rho(t)$ is the required weight function for a particular application. This choice of function R results in a non-linear differential equation. In order to solve this differential equation, consider that the left-hand-side of this differential equation can be re-written as $d\Phi(R(t))/dt$, where

$$\Phi'(x) = \sqrt{1-x^2} \quad (-1 \le x \le 1) \tag{18}$$

$$\Phi(x) = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x.$$

With the mentioned transcription, the above differential equation takes the following form:

$$\frac{d}{dt}\Phi(R(t)) = c\rho(t) \tag{19}$$

Integrating both sides of the equation according to variable t, and taking into consideration that $R(0) = 0$ and $R(1) = 1$,

$$\Phi(R(r)) = c\int_0^r \rho(t)\, dt \quad (0 \le r \le 1) \tag{20}$$

is obtained, while for constant c

$$c = \frac{\Phi(1)}{\int_0^1 \rho(t)\, dt} = \frac{\pi}{4\int_0^1 \rho(t)\, dt}. \tag{21}$$

Using these results, the sought argument transform is as follows.

$$R(r) = \Phi^{-1}\Big(c\int_0^r \rho(t)\, dt\Big) \ \ (0 \le r \le 1), \tag{22}$$

where Φ^{-1} is the inverse function of Φ. Practically, by this train of thought the following theorem has been proved:

Theorem 2.1: Let $\rho(r)$ $(r \in [0,1])$ be a continuous non-negative function, starting from the Chebyshev-polynomials of 2nd kind $V_n := U_{2n}$ by applying the argument-transform defined by (21) and (22) let the function

$$Y_n^\rho(r) := V_n(R(r)) \quad (n \in \mathbb{N},\ 0 \le r \le 1), \tag{23}$$

be introduced. In this case, the following orthogonality relation holds:

$$\int_0^1 Y_n^\rho(r)Y_m^\rho(r)\,\rho(r)\, dr = c_\rho\delta_{mn} \tag{24}$$

where $\quad c_\rho := \int_0^1 \rho(t)\, dt, \quad (m,n \in \mathbb{N}).$

The concept of the argument transform has successfully been applied by the authors also in other fields, see e.g. [7].

C. Modelling of calotte-like surfaces

Simple eye-models describe the corneal surface as a spherical calotte. One can use the argument transform approach – described in in Section II-B for any arbitrary non-negative weight function - for the weight function $\rho_a(r)$ (defined in (16). The argument transform shifts the Chebyshev-polynomials and their roots in a non-linear fashion determined by the argument transform function.

The functions of the proposed system, and also those of the modified system behave rather "awkwardly" over the origin: they take on a wide range of values near and over the origin. This behaviour is indicated by the crisp star-like artifacts in the centre of the circle, as it has been detailed in [2]. Such behaviour clearly should be avoided if real cornea surfaces are to be modelled.

In order to save the useability of the Chebyshev-based system proposed in this paper, elimination of this artifact is necessary e.g. by choosing weighting functions that take 0 at the origin and are smooth and flat enough. One can use, for example, a weighting function ρ_b that resembles a circular segment turned upside down, that is

$$\rho_b(r) := b - \sqrt{b^2 - r^2} \quad (0 \leq r \leq 1, b \geq 1). \tag{25}$$

The value for parameter b is selected as stated for parameter a in Section II-B.

The use of this weight function in (20) and (21) radically flattens the Chebyshev-polynomials, see [2] for details. It should be noted it is the properly selected weight function (25) that smoothes the elements in the representation series.

III. Discrete modelling of quasi-hemispherical and calotte-like surfaces

A. Discrete modelling of quasi-hemispherical surfaces

The representation (8) forms the basis of computation algorithms, but an appropriate discretization process is needed to approximate the involved coefficients. In order to achieve this goal the following discrete set is being introduced:

$$I_{NM} := \{(r_k^N, \varphi_\ell^M) : 1 \leq k \leq N, 0 \leq \ell < M\}$$

where

$$r_k^N := \cos\frac{k\pi}{2N+1} \quad (k = 1, 2, \cdots, N) \tag{26}$$

$$\varphi_\ell^M := 2\pi\ell/M \quad (\ell = 0, 1, \cdots, M-1), \tag{27}$$

r_k^N represent the positive roots of the polynomial V_N. It can easily be shown, that the system V_n is orthogonal with respect to the discrete inner-product

$$[f, g]_N := \frac{4}{2N+1}\sum_{k=1}^{N} f(r_k^N)\overline{g(r_k^N)}(1 - |r_k^N|^2). \tag{28}$$

Applying the inner-product (28) to (9) for indices $0 \leq n, n' < N$, and $x = r_k^N$ $(k = 1, 2, \ldots, N)$ we get

$$\begin{aligned} &[V_n, V_{n'}]_N = \\ &= \frac{4}{2N+1}\sum_{k=1}^{N} \sin((2n+1)\arccos r_k^N)\cdot \\ &\quad \cdot \sin((2n'+1)\arccos r_k^N) = \\ &= \frac{2}{2N+1}\sum_{k=1}^{N} \sin\frac{(2n+1)k\pi}{2N+1}\sin\frac{(2n'+1)k\pi}{2N+1}. \end{aligned} \tag{29}$$

Applying the trigonometrical identity

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)),$$

considering the evenness of the cos function, and considering that the expression for index $k = 0$ equals to zero, the sum can be extended to index values ranging from $-N$ to N, hence we get

$$\begin{aligned} &[V_n, V_{n'}]_N = \frac{2}{2N+1}\cdot \\ &\cdot \sum_{k=-N}^{N}\left[\cos\frac{(n-n')2k\pi}{2N+1} - \cos\frac{(n+n'+1)2k\pi}{2N+1}\right]. \end{aligned} \tag{30}$$

A well-known discrete orthogonality relation of the trigonometric system is

$$\frac{1}{M}\sum_{\ell=0}^{M-1} e^{i(m-m')\varphi_\ell^M} = \delta_{mm'} \quad 0 \leq m, m' < M). \tag{31}$$

The orthogonality of V_N with respect to the inner-product (28) can easily be verified, since according to (31) the second sum in (30) equals to zero, as $n + n' + 1 \neq 0$, the first one is equal to 1 for $n = n'$ and 0 otherwise, hence

$$\begin{aligned} &[V_n, V_{n'}]_N = \\ &= \frac{2}{2N+1}\sum_{k=-N}^{N}\cos\frac{(n-n')2k\pi}{2N+1} = \delta_{nn'}. \end{aligned} \tag{32}$$

This implies that the system

$$T_n(x) = V_n(x)\sqrt{1-x^2} \quad (0 \leq n < N) \tag{33}$$

is orthonormal with respect the inner product

$$\langle f, g\rangle_N := \frac{4}{2N+1}\sum_{k=1}^{N} f(r_k^N)\overline{g(r_k^N)} \tag{34}$$

This orthogonality relation forms the basis for applying fast computational algorithms, e.g. FFT, for the approximate computing the coefficients of the surface representation.

The discrete Fourier-coefficients with respect to the system T_n are

$$\begin{aligned} &c_n := \langle f, T_n\rangle_N = \\ &= \frac{4}{2N+1}\sum_{k=1}^{N} f\Big(\cos\big(\frac{\pi k}{2N+1}\big)\Big)\sin\big(\frac{(2n+1)\pi k}{2N+1}\big) \end{aligned} \tag{35}$$

Introducing function G in the following manner

$$
\begin{aligned}
G(s) &:= \quad f(\cos(s/2))\exp(is/2) \quad \big(s \in (0,\pi]\,\big) \\
G(0) &:= \quad 0 \\
G(s) &:= -f(\cos(s/2))\exp(is/2) \quad \big(s \in [-\pi,0)\,\big) \qquad (36)
\end{aligned}
$$

the discrete Fourier-coefficients c_n can be written as follows.

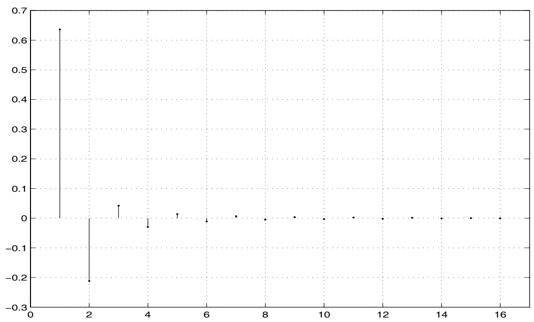

Fig. 4. The coefficients c_n for the cone

$$
\begin{aligned}
c_n \;:=\;& \frac{2}{(2N+1)i}\sum_{k=1}^{N} f\Big(\cos\big(\frac{\pi k}{2N+1}\big)\Big)\,\cdot \\
&\cdot\;\Big(\exp\big(\frac{(2n+1)i\pi k}{2N+1}\big)-\exp\big(\frac{-(2n+1)i\pi k}{2N+1}\big)\Big)= \\
&= \frac{2}{(2N+1)i}\,\cdot \\
&\cdot\sum_{k=1}^{N}\Big(G(t_k^N)\exp(it_k^N n)-\overline{G(t_k^N)\exp(-it_k^N n)}\Big)= \\
&= \frac{2}{(2N+1)i}\sum_{k=-N}^{N} G(t_k^N)\exp(it_k^N n), \qquad (37)
\end{aligned}
$$

where

$$
t_k^N := \frac{2\pi k}{(2N+1)} \quad (k \in \mathbb{Z}).
$$

In order to obtain a discrete two-dimensional extension of (33), we introduce the following system:

$$
\mathcal{T}_{n\ell}(r,\varphi) := T_k(r)e^{i\ell\varphi} \quad (0 \le n < N, 0 \le \ell < M). \qquad (38)
$$

This system is orthonormal with respect to the following two-dimensional discrete inner product:

$$
[F,G]_{NM} := \frac{4}{(2N+1)M}\sum_{z=(r,\varphi)\in I_{NM}} F(z)\overline{G(z)}. \qquad (39)
$$

According to (37), the Fourier-coefficients with respect to $\mathcal{T}_{k\ell}$ can be expressed by 2D discrete Fourier transform. As a consequence these coefficients can be computed by FFT algorithm.

In Fig. 4 the coefficients – computed according to (37) for a cone over the unit-disk – are shown. The coefficient c_1, for example, correspond to the "hemisphericity" of the conic surface. The stages of the reconstruction leading to the conic surface are shown in Fig. 5. Similar series of diagrams are shown for a perturbed asymmetric cone-like surface in Figs. 7 and 6. The last diagrams in Figs. 4 and 7 are fairly closed approximations of the original surfaces. These surfaces has been selected in an *ad hoc* fashion, realistic corneal surfaces can be obtained from patient data. On the other hand, several corneal test-surfaces have been suggested in [8].

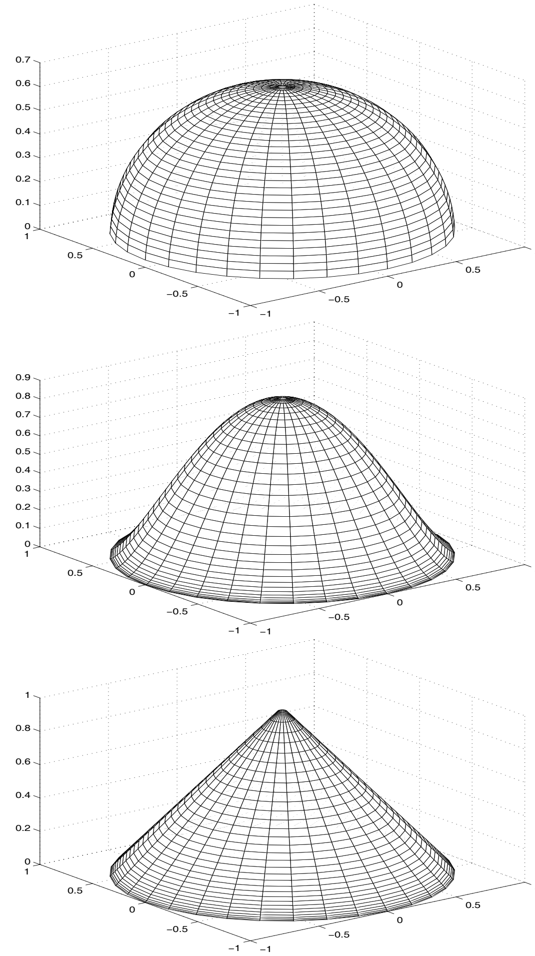

Fig. 5. The stages of reconstruction of a cone over the unit disc (including 1,2 and 25 components

B. Discrete modelling of calotte-like surfaces

The discrete system of (38) – which is advantageous for the representation of hemispherical surfaces – can be converted into a system, which is very suitable for representing calotte-like surfaces. This conversion can be done in a similar fashion as for the continuous case.

The discrete form of the inner product used in orthogonality relation (24) is as follows:

$$
[f,g]_{\rho,N} := \sum_{t\in T_N^{\rho}} f(t)\overline{g(t)}(1-R^2(t)). \qquad (40)
$$

T_N^{ρ} system of the discrete measurement points is formed by applying R^{-1} to the radial values r_k^N of (26), while

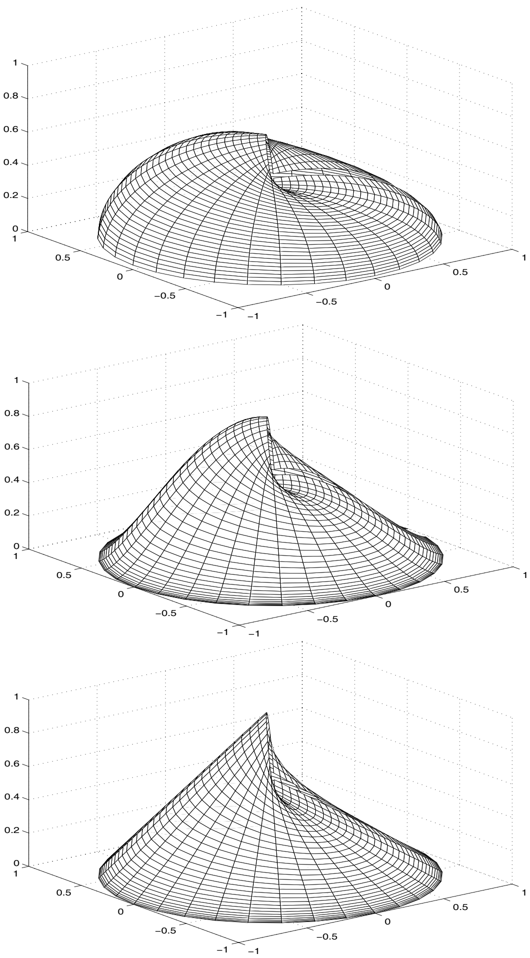

Fig. 6. The stages of reconstruction of a perturbed, asymmetric cone over the unit disc (including 1,2 and 25 components

discretization on the azimuthal variable φ can be the same as in (27).

Substituting (23) for indices m, n into the expression of the discrete inner product in (40), we arrive again at the expression of (29), therefore, the discrete orthogonality relation of (28) remains valid. Hence the c_n coefficients for the new system can be computed as in (37) except for the change in the radial discretisation scheme. In the new system the sample points are moved according the inverse of the argumentum-transform $R(t)$ in radial direction.

IV. Conclusion and future work

In this paper, we have elaborated on the discretisation of the continuous orthogonal system based on the even Chebyshev-polynomials of the second kind, which was suggested to model quasi-hemispherical and calotte-like (e.g. human corneal) surfaces in [2]. It has been shown in the paper that the representation coefficients of the given surface can be computed in an efficient manner via a 2D FFT algorithm. The representation coefficients with respect to the modified system (i.e., the one which is advantageous for representing calotte-like surfaces) can be computed in a very similar manner. This representation method is planned to be used for the reconstruction of natural surfaces based on measured data (e.g., in cornea topography). The proposed surface modelling technique can be incorporated into measurement, control and diagnostic devices in the field of ophthalmology, (e.g., refractive surgery).

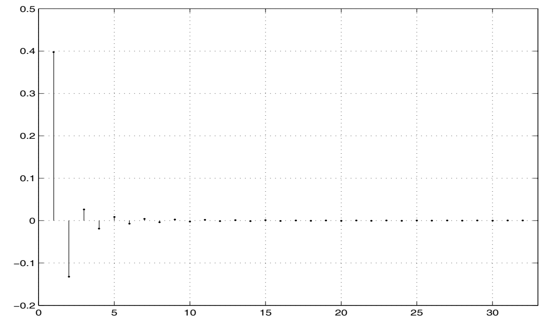

Fig. 7. The coefficients c_n for the perturbed asymmetric cone

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