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Minimum time optimality for a bang-singular arc: second order sufficient conditions

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Abstract— **We consider a bang–singular normal extremal for the minimum time problem associated to a single–input control system affine in the control. We use Hamiltonian methods to prove that the coercivity of a suitable *strongly-extended second variation* associated to the singular arc is sufficient to prove the local optimality of the extremal in the strong topology. As a consequence we give a sufficient condition easily checkable and we apply it to three examples in $\mathbb{R}^3$.**

I. Introduction

We consider the minimum-time problem for a single-input affine system:

$$\text{minimize} \quad T$$

subject to

$$\dot{\xi}(t) = f_0(\xi(t)) + u f_1(\xi(t)) \quad t \in [0, T] \tag{1}$$

$$\xi(0) = \widehat{x}_0 , \quad \xi(T) = \widehat{x}_f \tag{2}$$

$$u \in [-1, 1]$$

The state space is a smooth $n-$dimensional manifold M, and $f_0, f_1 : M \to TM$ are smooth vector fields, by smooth we mean C^∞.

The aim of the authors is to give second order sufficient conditions for a reference extremal $\widehat{\xi}$ to be a *local optimizer in the strong topology,* i.e. with respect to the admissible trajectories whose graph belongs to a neighborhood of the graph of $\widehat{\xi}$ in $\mathbb{R} \times M$, independently on the values of the associated control.

Sufficient second order conditions for an extremal of this problem are given in [1], when the extremal is bang-bang, and in [2], when the extremal is totally singular, see also [3], [4] (where the optimality in the weak topology is considered) and the references therein.

Here we consider a normal "bang-singular" Pontryagin extremal as a first attempt to give second order sufficient conditions for the optimality of an extremal containing any finite number of bang and singular arcs. Namely we consider a triplet $(\widehat{T}, \widehat{\xi}, \widehat{u})$, with associated adjoint covector

$$\widehat{\lambda} : t \in [0, \widehat{T}] \to \widehat{\lambda}(t) \in T^*M,$$

satisfying Pontryagin Maximum Principle (PMP) and such that there exists a *switching time* $\widehat{T}_s \in (0, \widehat{T})$, for which

$$\widehat{u}(t) \equiv 1 \qquad \forall\, t \in (0, \widehat{T}_s) \tag{3}$$

$$\widehat{u}(t) \in (-1, 1) \qquad \forall\, t \in (\widehat{T}_s, \widehat{T}) . \tag{4}$$

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Nothing changes if the bang control is identically equal to -1.

We assume that the vector fields $f_1,\ [f_0, f_1]$ are linearly independent at

$$\widehat{x}_s \equiv \widehat{\xi}(\widehat{T}_s)$$

and that the bang arc is *regular* in the interval $[0, \widehat{T}_s)$, i.e. we assume

$$\langle \widehat{\lambda}(t) \,,\ f_1(\widehat{\xi}(t)) \rangle > 0 \quad \forall\, t \in [0, \widehat{T}_s) . \tag{5}$$

A consequence of PMP on the bang arc is that

$$\langle \widehat{\lambda}(\widehat{T}_s) \,,\ ([f_0, [f_0, f_1]] + [f_1, [f_0, f_1]])\,(\widehat{x}_s) \rangle \geq 0 , \tag{6}$$

while on the singular arc we get

$$\langle \widehat{\lambda}(t) \,,\ f_1(\widehat{\xi}(t)) \rangle \equiv 0 \tag{7}$$

$$\langle \widehat{\lambda}(t) \,,\ [f_0, f_1](\widehat{\xi}(t)) \rangle \equiv 0 \tag{8}$$

$$\langle \widehat{\lambda}(t) \,,\ ([f_0, [f_0, f_1]] + \widehat{u}(t)[f_1, [f_0, f_1]])\,(\widehat{\xi}(t)) \rangle \equiv 0 . \tag{9}$$

A natural necessary condition is that $\widehat{\xi}$ is a minimum-time trajectory joining $\widehat{x}_s$ with $\widehat{x}_f$, so that further necessary conditions can be obtained analyzing the *extended second variation* J'' on the singular arc as defined in [2], see also [5] and [4]. In particular we obtain the *generalized Legendre condition* (GLC)

$$\langle \widehat{\lambda}(t), [f_1, [f_0, f_1]](\widehat{\xi}(t)) \rangle \geq 0, \quad \forall\, t \in (\widehat{T}_s, \widehat{T}) , \tag{10}$$

and that J'' has to be non–negative if the normalized adjoint covector is unique.

In this paper we give second order sufficient conditions for the reference trajectory $\widehat{\xi}$ to be a *local optimizer in the strong topology*. The sufficient conditions include the strict version of inequality (6) and the coercivity of J''.

Remark that the coercivity of J'' implies the *strict generalized Legendre condition* (SGLC)

$$\langle \widehat{\lambda}(t), [f_1, [f_0, f_1]](\widehat{\xi}(t)) \rangle > 0, \quad \forall\, t \in (\widehat{T}_s, \widehat{T}) \tag{11}$$

and the optimality of the singular arc between its end–points, see [2].

On the other hand it is possible to prove that the strict inequality in (6) is sufficient to ensure optimality of the bang arc between its end–points.

In order to obtain the minimum–time local optimality in the strong topology for the entire trajectory, we require the coercivity of a *strongly extended second variation*, $J''_{\rm st}$, which is an extension of J'' to a larger space of initial conditions, see Section II and III for the definitions. The main result of the paper is

Theorem 1.1: Assume $\widehat{\xi}\colon [0,\widehat{T}] \to M$ is a bang–singular normal extremal of the minimum–time problem for system (1) – (2) such that $f_1,\ f_{01}$ are linearly independent at $\widehat{x}_s$, the bang arc is regular on $[0,\widehat{T}_s)$ and the strict inequality holds in (6). If the strongly extended second variation on the singular arc J''_{st} defined in (25) – (26) is coercive, then $\widehat{\xi}$ is a local optimizer in the strong topology.

□

Generically, the sufficient condition proposed here is not very close to the necessary conditions known to the authors, however we emphasize that we get the local optimality in the strong topology and that we obtain, as a consequence of Theorem 1.1, a sufficient condition in dimension 3 which is easy to check, see Corollary 4.3.

In the last Section we apply Corollary 4.3 to three examples where it is possible to prove that every normal bang–singular extremal for the proposed problem is optimal.

II. Notations and preliminary results

In this Section we give some notations and preliminary results concerning the extended second variation J'' defined in [2].

For $j = 1,\ldots,k$, $i_j \in \{0,1\}$ we denote by

$$f_{i_1 i_2 \ldots i_k} \equiv [f_{i_1}, [\ldots [f_{i_{k-1}}, f_{i_k}] \ldots]$$

the iterated Lie brackets of the vector fields f_0 and f_1 and the associated Hamiltonian functions by

$$H_{i_1 i_2 \ldots i_k}\colon \ell \mapsto \langle \ell\,,\ f_{i_1 i_2 \ldots i_k}(\pi\ell)\rangle$$

where

$$\pi\colon T^*M \to M$$

is the canonical projection; in coordinates

$$\pi\colon (p,x) \mapsto x\,.$$

For each Hamiltonian function H,

$$\overrightarrow{H}\colon T^*M \to TT^*M$$

is the associated Hamiltonian vector field and, for each vector field V on any manifold X, we denote by

$$(t,x) \mapsto \exp tV(x)$$

the flow of V at time t starting at x, while

$$\exp tV_*\colon T_xX \to T_{\exp tV(x)}X$$

denotes its derivative.

With these notations SGLC (11) can be written as

$$H_{101}(\widehat{\lambda}(t)) > 0\,, \quad t \in [\widehat{T}_s, \widehat{T}] \tag{12}$$

while equations (7) – (8) can be summarized by saying that, for all $t \in [\widehat{T}_s, \widehat{T}]$, $\widehat{\lambda}(t)$ belongs to the symplectic sub–manifold $\mathcal{S} \subset T^*M$ defined by

$$\mathcal{S} \equiv \{\ell \in T^*M\colon H_1(\ell) = H_{01}(\ell) = 0\}\,. \tag{13}$$

To simplify the notation we assume

$$\widehat{u}(t) \equiv 0 \quad \forall\, t \in (\widehat{T}_s, \widehat{T}),$$

hence, from (9), we obtain that inequality (6) is a consequence of GLC (10). We remark that Theorem 1.1 holds also if the control is not identically zero on the singular arc.

The reference vector field is given by

$$\widehat{f}_t = \begin{cases} f^+ \equiv f_0 + f_1 & \text{if } t \in [0, T_s] \\ f_0 & \text{if } t \in [T_s, \widehat{T}] \end{cases} \tag{14}$$

and the pull–back of the controlled vector field f_1 at the switching time $\widehat{T}_s$ is defined by

$$g_t(x) \equiv \begin{cases} \exp(\widehat{T}_s - t)f^+_* f_1(\exp(t - \widehat{T}_s)f^+(x)) \\ \qquad\qquad t \in [0, \widehat{T}_s] \\ \exp(\widehat{T}_s - t)f_{0*} f_1(\exp(t - \widehat{T}_s)f_0(x)) \\ \qquad\qquad t \in [\widehat{T}_s, \widehat{T}]\,. \end{cases} \tag{15}$$

Remark that

$$\dot{g}_t(x) = \begin{cases} \exp(\widehat{T}_s - t)f^+_* f_{01}(\exp(t - \widehat{T}_s)f^+(x)) \\ \qquad\qquad t \in [0, \widehat{T}_s] \\ \exp(\widehat{T}_s - t)f_{0*} f_{01}(\exp(t - \widehat{T}_s)f_0(x)) \\ \qquad\qquad t \in [\widehat{T}_s, \widehat{T}] \end{cases} \tag{16}$$

$$\langle \widehat{\lambda}(\widehat{T}_s)\,,\ [g_t, \dot{g}_t](\widehat{x}_s)\rangle = H_{101}(\widehat{\lambda}(t)) \quad t \in [0, \widehat{T}]. \tag{17}$$

In order to obtain the needed second variation on the singular arc, we consider the minimum–time problem between the fixed points $\widehat{x}_s$ and $\widehat{x}_f$ and the singular arc as reference trajectory. For this problem, in [2], the extended second variation is obtained by writing the intrinsic second variation as defined in [6], and embedding each control variation $v \in L^2([\widehat{T}_s, \widehat{T}], \mathbb{R})$ into $\mathbb{R} \times L^2([\widehat{T}_s, \widehat{T}], \mathbb{R})$ by

$$v \mapsto \left(\int_{\widehat{T}_s}^{\widehat{T}} v(r)\ \mathrm{d}r\ ,\ t \mapsto \int_t^{\widehat{T}} v(r)\ \mathrm{d}r \right).$$

Notice that the resulting quadratic form on $\mathbb{R} \times L^2([\widehat{T}_s, \widehat{T}], \mathbb{R})$ is a different formulation of the one used in [5] to prove necessary conditions.

In order to introduce the extended second variation of [2], we choose a function $\widehat{\beta}\colon M \to \mathbb{R}$ such that

$$\mathrm{d}\widehat{\beta}(\widehat{x}_s) = -\widehat{\lambda}(\widehat{T}_s)\,.$$

The extended second variation J'' of the singular arc is a linear quadratic form defined on any $\delta e \equiv (w_1, w(\cdot)) \in \mathbb{R} \times L^2([\widehat{T}_s, \widehat{T}], \mathbb{R})$, such that the system

$$\begin{aligned} &\dot{\zeta}(t) = w(t)\dot{g}_t(\widehat{x}_s) \\ &\zeta(\widehat{T}_s) = w_1 f_1(\widehat{x}_s)\,, \quad \zeta(\widehat{T}) = 0 \end{aligned}$$

admits a solution on $[\widehat{T}_s, \widehat{T}]$ and it is given by

$$\begin{aligned} J''(\delta e) = {} & \frac{w_1^2}{2} L_{f_1} L_{f_1} \widehat{\beta}(\widehat{x}_s) \\ & + \frac{1}{2} \int_{\widehat{T}_s}^{\widehat{T}} w(t)^2 H_{101}(\widehat{\lambda}(t))\, \mathrm{d}t \\ & + \int_{\widehat{T}_s}^{\widehat{T}} w(t) L_{\zeta(t)} L_{\dot{g}_t} \widehat{\beta}(\widehat{x}_s)\, \mathrm{d}t\,. \end{aligned} \tag{18}$$

Remark that J'' is independent on the choice of $\widehat{\beta}$ with the stated property, hence we can choose $\widehat{\beta}$ such that $\widehat{\beta}|_\Gamma = 0$, where Γ is the integral line of f_1 through $\widehat{x}_s$. With such a choice it is easy to see that J'' is the standard second variation of the problem

$$\text{minimize} \quad T$$

subject to

$$\begin{aligned}
&\dot{\xi}(t) = f_0(\xi(t)) + w(t) f_{01}(\xi(t)) + \frac{w(t)^2}{2} f_{101}(\xi(t)) \\
&\xi(\widehat{T}_s) \in \Gamma\,, \ \xi(T) = \widehat{x}_f
\end{aligned} \tag{19}$$

with reference triple $(\widehat{\xi}, \widehat{w} \equiv 0, \widehat{T})$ and adjoint covector $\widehat{\lambda}$.

If $M = \mathbb{R}^n$, then J'' can be written along $\widehat{\lambda}$ by

$$\begin{aligned}
&\frac{1}{2}\int_{\widehat{T}_s}^{\widehat{T}} \langle \widehat{\lambda}(t)\,,\ D^2 f_0(\widehat{\xi}(t))[\zeta(t)]^2 \rangle \,\mathrm{d}t \\
&+ \int_{\widehat{T}_s}^{\widehat{T}} w(t) \langle \widehat{\lambda}(t)\,,\ D f_{01}(\widehat{\xi}(t))\zeta(t) \rangle \,\mathrm{d}t \\
&+ \frac{1}{2}\int_{\widehat{T}_s}^{\widehat{T}} w(t)^2 \langle \widehat{\lambda}(t)\,,\ f_{101}(\widehat{\xi}(t)) \rangle \,\mathrm{d}t
\end{aligned} \tag{20}$$

where

$$\dot{\zeta}(t) = D f_0(\widehat{\xi}(t))\zeta(t) + w(t) f_{01}(\widehat{\xi}(t)) \tag{21}$$

$$\zeta(\widehat{T}_s) \in \mathbb{R} f_1(\widehat{x}_s)\,,\ \zeta(\widehat{T}) = 0\,. \tag{22}$$

III. The strongly extended second variation

Locally around $0 \in \mathbb{R}^2$ define

$$\Gamma_{\mathrm{st}} \equiv \{\exp s f_1 \circ \exp \tau f_{01}(\widehat{x}_s) \colon (s, \tau) \in \mathbb{R}^2\}\,.$$

We define the *strongly extended second variation* J''_{st} as the standard second variation of the minimum time problem for system (19) with end–conditions

$$\xi(\widehat{T}_s) \in \Gamma_{\mathrm{st}}\,,\ \xi(T) = \widehat{x}_f \tag{23}$$

and reference triple $(\widehat{\xi}, \widehat{w} \equiv 0, \widehat{T})$ with adjoint covector $\widehat{\lambda}$. Remark that the definition is consistent since the reference triple satisfies PMP also for the above problem with the same adjoint covector.

If $M = \mathbb{R}^n$, then J''_{st} can be written as equations (20) – (21) with initial conditions (22) replaced by

$$\zeta(\widehat{T}_s) \in \mathrm{span}\,\{f_1(\widehat{x}_s)\,,\ f_{01}(\widehat{x}_s)\}\ ,\ \zeta(\widehat{T}) = 0.$$

For a general manifold we write an *explicit coordinate free formula* for J''_{st} by means of a function $\widehat{\beta}$ such that

$$\mathrm{d}\widehat{\beta}(\widehat{x}_s) = -\widehat{\lambda}(\widehat{T}_s) \text{ and } \widehat{\beta}_{|\Gamma_{\mathrm{st}}} = 0\,, \tag{24}$$

see for example [6]. With such a $\widehat{\beta}$,

$$J''_{\mathrm{st}} \colon T_{\widehat{x}_s}\Gamma_{\mathrm{st}} \times L^2([\widehat{T}_s, \widehat{T}], \mathbb{R}) \to \mathbb{R}$$

is defined by

$$\begin{aligned}
J''_{\mathrm{st}}(\delta x, w) = {} & \frac{1}{2}\int_{\widehat{T}_s}^{\widehat{T}} w(t)^2 H_{101}(\widehat{\lambda}(t))\,\mathrm{d}t \\
& + \int_{\widehat{T}_s}^{\widehat{T}} w(t) L_{\zeta(t)} L_{\dot{g}_t} \widehat{\beta}(\widehat{x}_s)\,\mathrm{d}t
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
&\dot{\zeta}(t) = w(t)\dot{g}_t(\widehat{x}_s) \quad t \in [\widehat{T}_s, \widehat{T}] \\
&\zeta(\widehat{T}_s) = \delta x \in \mathrm{span}\,\{f_1(\widehat{x}_s),\ f_{01}(\widehat{x}_s)\}\,,\ \zeta(\widehat{T}) = 0.
\end{aligned} \tag{26}$$

It is well known that the coercivity of J'' (and hence of J''_{st}) implies the SGLC (12).

With the methods developed in [6] the coercivity of J''_{st} can be checked starting from the Hamiltonian

$$H = H_0 + \frac{1}{2}\frac{H_{01}^2}{H_{101}} \tag{27}$$

associated to system (19) and the tangent space L_0 to the Lagrangean sub–manifold of the initial transversality condition associated to the constraints (23)

$$L_0 = \mathbb{R}\overrightarrow{H}_1(\widehat{\lambda}(\widehat{T}_s)) \oplus \mathbb{R}\,\mathrm{d}\widehat{\beta}_* f_{01} \oplus \Pi$$

where

$$\Pi \equiv \{p \in T_{\widehat{x}_s}M \colon \langle p\,,\ f_1(\widehat{x}_s)\rangle = \langle p\,,\ f_{01}(\widehat{x}_s)\rangle = 0\}$$

is embedded in $T_{\widehat{\lambda}(\widehat{T}_s)}\mathcal{S}$ as a space of tangent vectors to vertical curves.

Lemma 3.1: J''_{st} is coercive if and only if

$$\begin{gathered}
\pi_* \exp(t - \widehat{T}_s)\overrightarrow{H}_* \delta\ell = 0, \quad t \in [\widehat{T}_s, \widehat{T}]\,,\ \delta\ell \in L_0 \\
\Downarrow \\
\pi_* \exp(s - \widehat{T}_s)\overrightarrow{H}_* \delta\ell = 0, \quad \forall\, s \in [\widehat{T}_s, t]\,.
\end{gathered}$$

□

The proof can be found in [6].

IV. A model case

In this section we consider the case when

$$H_{001|\mathcal{S}} = 0\,. \tag{28}$$

Since $\overrightarrow{H}_{|\mathcal{S}} = \overrightarrow{H}_0$, and equation (28) implies that $\overrightarrow{H}_0$ is tangent to $\mathcal{S}$, then, for any $\delta\ell \in T_{\widehat{\lambda}(\widehat{T}_s)}\mathcal{S}$,

$$\exp(t - \widehat{T}_s)\overrightarrow{H}_* \delta\ell = \exp(t - \widehat{T}_s)\overrightarrow{H}_{0*} \delta\ell\,.$$

By the choice of $\widehat{\beta}$, $\mathrm{d}\widehat{\beta}_* f_{01}$ belongs to $T_{\widehat{\lambda}(\widehat{T}_s)}\mathcal{S}$, therefore we have, for all $t \in [\widehat{T}_s, \widehat{T}]$

$$\begin{aligned}
&\pi_* \exp(t - \widehat{T}_s)\overrightarrow{H}_*\, \mathrm{d}\widehat{\beta}_* f_{01}(\widehat{x}_s) = \exp(t - \widehat{T}_s) f_{0*} f_{01}(\widehat{x}_s) \\
&\pi_* \exp(t - \widehat{T}_s)\overrightarrow{H}_* p = p\, \exp(t - \widehat{T}_s) f_{0*} = 0\,, \quad p \in \Pi.
\end{aligned}$$

Moreover, using the variational equation for the derivative of the flow of a vector field, it is not difficult to see that

$$\exp(t - \widehat{T}_s)\overrightarrow{H}_* \overrightarrow{H}_1(\widehat{\lambda}(\widehat{T}_s)) = \overrightarrow{H}_1(\widehat{\lambda}(t)).$$

hence, for all $t \in [\widehat{T}_s, \widehat{T}]$

$$\pi_* \exp(t - \widehat{T}_s)\overrightarrow{H}_* \overrightarrow{H}_1(\widehat{\lambda}(\widehat{T}_s)) = f_1(\widehat{\xi}(t))\,.$$

By the above arguments Lemma 3.1 can be restated in the following way

Lemma 4.1: Let $H_{001|\mathcal{S}} \equiv 0$. If the SGLC (12) holds true, then $J''_{\rm st}$ is coercive if and only if $f_1(\widehat{\xi}(t))$ and $\exp(t - \widehat{T}_s)f_{0*}f_{01}(\widehat{x}_s)$ are linearly independent for any $t \in [\widehat{T}_s, \widehat{T}]$; equivalently:

$g_t(\widehat{x}_s)$, $f_{01}(\widehat{x}_s)$ are linearly independent $\forall\, t \in [\widehat{T}_s, \widehat{T}]$.

□

Remark 4.2: Let M be a 3-dimensional manifold. If f_1, $[f_0, f_1]$ are linearly independent on M, then there is a *singular vector field* f_s, in the sense that there exists a feedback function $\omega\colon M \to \mathbb{R}$ such that any singular extremal trajectory for which SGLC (12) holds is an integral curve of $f_s \equiv f_0 + \omega f_1$. We can substitute f_s to f_0 in the control system obtaining that (28) holds true, therefore we can state the following Corollary to Theorem 1.1.

Corollary 4.3: Assume that M is a 3-dimensional manifold, let f_1, $[f_0, f_1]$ be linearly independent on M and denote by f_s the singular vector field. In this case every bang–singular normal extremal $\widehat{\xi}$ such that

a) the bang arc is regular on $[0, \widehat{T}_s)$,

b) SGLC (12) holds true,

c) the tangent vectors at $\widehat{x}_s$ $\exp(\widehat{T}_s - t)f_{s*}f_1(\widehat{\xi}(t))$ and $[f_s, f_1](\widehat{x}_s)$ are linearly independent for $t \in [\widehat{T}_s, \widehat{T}]$,

is a local optimizer in the strong topology.

□

V. Proof of the Main Theorem

Since f_1, f_{01} are linearly independent at $\widehat{x}_s$, it is possible to choose coordinates in a neighborhood $\mathcal{O}$ of $\widehat{x}_s$ such that

$$\begin{aligned} f_1 &= \frac{\partial}{\partial x_1} \\ f_{01} &= \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + \varphi(x_1)\Phi(x), \quad \varphi(x_1) = O(x_1^2), \\ f_{101} &= \frac{\partial}{\partial x_3} + \psi(x_1, x_2)\Psi(x),\ \psi(0,0) = 0. \end{aligned}$$

As a consequence we obtain

$$\widehat{\lambda}(\widehat{T}_s) = \sum_{i=3}^{n} \lambda_i \,\mathrm{d}x_i,\ \lambda_3 > 0.$$

Choose

$$\widehat{\beta}(x) = -\sum_{i=3}^{n} \lambda_i x_i$$

and define

$$\alpha(x) = -\widehat{\beta}(x) + \frac{\chi}{2}\sum_{i=3}^{n} x_i^2\,.$$

Since $\widehat{\beta}$ satisfies condition (24), then the strongly extended second variation can be considered as the restriction to $\operatorname{span}\left\{\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}\right\}$ of the quadratic form defined by

$$\begin{aligned} J''_\chi[\delta e]^2 = \frac{\chi}{2}\sum_{i=3}^{n} x_i^2 & \\ &+ \frac{1}{2}\int_{\widehat{T}_s}^{\widehat{T}} w(t)^2 H_{101}(\widehat{\lambda}(t))\,\mathrm{d}t \\ &+ \int_{\widehat{T}_s}^{\widehat{T}} w(t) L_{\zeta(t)} L_{\dot{g}_t}\widehat{\beta}(\widehat{x}_s)\,\mathrm{d}t \end{aligned}$$

on the space of those $\delta e = (x, w)$ such that

$$\begin{aligned} &\dot{\zeta}(t) = w(t)\dot{g}_t(\widehat{x}_s) \\ &\zeta(\widehat{T}_s) = x \text{ free}, \quad \zeta(\widehat{T}) = 0\,. \end{aligned}$$

Since J''_χ restricted to $\operatorname{span}\left\{\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}\right\}$ is coercive, by a result in [7], it is possible to prove that, if $\chi > 0$ is sufficiently large, then the quadratic form J''_χ is coercive on $\mathbb{R}^n$.

Using the same arguments of the proof of Theorem 2.2 in [2], we can prove that if the graph of a trajectory ξ of system (1) belongs to a suitable neighborhood of the graph of $\widehat{\xi}$ and $\xi(T) = \widehat{\xi}(\widehat{T})$, then

$$\alpha(\xi(\widehat{T}_s)) = \alpha(\xi(\widehat{T}_s)) - \alpha(\widehat{x}_s) \geq \widehat{T} - T + o(T - \widehat{T}). \quad (29)$$

For any admissible trajectory ξ define

$$q(t) = \exp(\widehat{T}_s - t)f^+(\xi(t)) \qquad t \in [0, \widehat{T}_s]$$

and let $q_1(t) = x_1(q(t))$, so that

$$q(\widehat{T}_s) = \xi(\widehat{T}_s) \text{ and } q(0) = \widehat{x}_s.$$

To prove the main theorem it suffices to show that there exists a neighborhood $\mathcal{U}$ of $\widehat{x}_s$ such that $\alpha(\xi(\widehat{T}_s)) \leq 0$ for any admissible trajectory ξ such that $q(t) \in \mathcal{U}$.

For this purpose we introduce the new function

$$\widetilde{\alpha}(x) \equiv \alpha(x) + \frac{x_1^2}{2}\,.$$

We may choose a neighborhood $\mathcal{U}$ of $\widehat{x}_s$ in M such that

$$\begin{aligned} &L_{f_{01}}\alpha(x) = x_1\eta(x) \text{ where } \eta(x) > 0\,, \quad x \in \mathcal{U}\,; \\ &L_{f_{01}}\widetilde{\alpha}(x) = x_1\eta(x) \text{ where } \widetilde{\eta}(x) > 0\,, \quad x \in \mathcal{U}\,. \end{aligned}$$

Since the bang arc is regular on $[0, \widehat{T}_s)$ and $H_{101}(\widehat{\lambda}(\widehat{T}_s)) = \lambda_3 > 0$, by continuity and possibly restricting $\mathcal{U}$, we can choose $\varepsilon > 0$ such that

$$1 - \varepsilon\eta(x) > 0\,, \quad 1 - \varepsilon\widetilde{\eta}(x) > 0\,, \quad x \in \mathcal{U}\,,$$

$$L_{g_t}\alpha(x) > 0\,, \quad L_{g_t}\widetilde{\alpha}(x) > 0\,, \quad (t, x) \in [0, \widehat{T}_s - \varepsilon] \times \mathcal{U}$$

and for any $(t, x) \in [\widehat{T}_s - \varepsilon, \widehat{T}_s] \times \mathcal{U}$

$$L_{\ddot{g}_t}\alpha(x) > 0\,, \quad L_{\ddot{g}_t}\widetilde{\alpha}(x) > 0\,, \quad L_{g_t}x_1(x) > 0\,.$$

Let ξ be an admissible trajectory such that $q(t) \in \mathcal{U}$ for any $t \in [0, \widehat{T}_s]$. As a consequence of

$$\dot{q}_1(t) = L_{\dot{q}(t)}x_1 = (u(t) - 1)L_{g_t}x_1(q(t))\,,$$

$q_1(t)$ is a decreasing function in $[\widehat{T}_s - \varepsilon, \widehat{T}_s]$. Moreover

$$\begin{aligned} 0 &= \widetilde{\alpha}(\widehat{x}_s) = \widetilde{\alpha}(q(0)) = \widetilde{\alpha}(q(\widehat{T}_s)) \\ &+ \int_{\widehat{T}_s}^{0} (u(t)-1) L_{g_t}\widetilde{\alpha}(q(t))\,\mathrm{d}t \\ &\geq \widetilde{\alpha}(q(\widehat{T}_s)) + \int_{\widehat{T}_s-\varepsilon}^{\widehat{T}_s} (1-u(t)) L_{g_t}\widetilde{\alpha}(q(t))\,\mathrm{d}t \end{aligned}$$

If $L_{g_t}\widetilde{\alpha}(q(t)) \geq 0$ for all $t \in [\widehat{T}_s - \varepsilon, \widehat{T}_s]$, then we are done.

If there exists $\bar{t} \in \left[\widehat{T}_s - \varepsilon, \widehat{T}_s\right]$ such that $L_{g_{\bar{t}}}\widetilde{\alpha}(q(\bar{t})) = 0$, then, using the Taylor expansion of $L_{g_t}\widetilde{\alpha}(x)$ at $t = \widehat{T}_s$, it is not difficult to see that $q_1(\bar{t}) < 0$. Therefore $q_1(t) < 0$ for any $t \in [\bar{t}, \widehat{T}_s]$ and we can compute

$$\begin{aligned} 0 &= \widetilde{\alpha}(\widehat{x}_s) \geq \widetilde{\alpha}(q(\bar{t})) \geq \alpha(q(\bar{t})) = \\ &= \alpha(q(\widehat{T}_s)) + \int_{\bar{t}}^{\widehat{T}_s} (1-u(t)) L_{g_t}\alpha(q(t))\,\mathrm{d}t \\ &\geq \alpha(q(\widehat{T}_s)) + \int_{\bar{t}}^{\widehat{T}_s} (1-u(t))(t-\widehat{T}_s) q_1(t)\eta(q(t))\,\mathrm{d}t \\ &\geq \alpha(q(\widehat{T}_s)). \end{aligned}$$

This completes the proof of Theorem 1.1.

VI. Examples

Example 1

Consider the simple example

$$\dot{x}_1 = u\,, \quad \dot{x}_2 = x_1\,, \quad \dot{x}_3 = x_1^2, \quad u \in [-1,1]\,. \tag{30}$$

In this case

$$\begin{aligned} & f_0 = x_1 \frac{\partial}{\partial x_2} + x_1^2 \frac{\partial}{\partial x_3}\,,\ f_1 = \frac{\partial}{\partial x_1}\,, \\ & f_{01} = -\frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3}\,,\ f_{001} \equiv 0\,,\ f_{101} = -2\frac{\partial}{\partial x_3}\,. \end{aligned}$$

f_1 and f_{01} are linearly independent on the whole $\mathbb{R}^3$ and $H_{001} \equiv 0$. Moreover it is not difficult to see that every normal bang–singular extremal satisfies SGLC (12), therefore Corollary 4.3 applies.

Since for every bang–singular extremal

$$g_t(\widehat{x}_s) = f_0(\widehat{x}_s) + (t - \widehat{T}_s) f_{01}(\widehat{x}_s),$$

then we obtain:

every bang–singular normal extremal for the minimum–time problem with fixed end–points for system (30) *is locally optimal in the strong topology.*

Example 2

Consider the control system driven by the equations

$$\begin{aligned} & \dot{x}_1 = x_2 \\ & \dot{x}_2 = (1-x_1^2)x_2 - x_1 + u \\ & \dot{x}_3 = x_1^2 + x_2^2 \\ & u \in [-1,1]\,. \end{aligned} \tag{31}$$

Applying Corollary 4.3 we can show:

every bang–singular normal extremal for the minimum–time problem with fixed end–points for system (31) *is locally optimal in the strong topology.*

In fact system (31) fits in our framework with

$$\begin{aligned} & f_0 = x_2 \frac{\partial}{\partial x_1} + \left[(1-x_1^2)x_2 - x_1\right] \frac{\partial}{\partial x_2} + \left(x_1^2 + x_2^2\right) \frac{\partial}{\partial x_3}\,, \\ & f_1 = \frac{\partial}{\partial x_2}\,. \end{aligned}$$

It is not difficult to see that

$$\begin{aligned} & \mathcal{S} \equiv \left\{(p,x) \in \left(\mathbb{R}^3\right)^* \times \mathbb{R}^3 \colon p_2 = 0\,,\ p_1 = -2p_3 q_2\right\}, \\ & H_{101}(p,x) > 0 \Longleftrightarrow p_3 < 0\,, \end{aligned}$$

and that every normal singular extremal for the minimum–time problem with fixed end–points satisfies SGLC (12). Moreover it is a straightforward calculation to show that

$$\begin{aligned} & f_s = x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + \left(x_1^2 + x_2^2\right) \frac{\partial}{\partial x_3}\,, \\ & f_{s1} \equiv [f_s, f_1] = -\frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3}\,, \\ & f_{ss1} \equiv [f_s, f_{s1}] = \frac{\partial}{\partial x_2} = f_1\,, \\ & \mathrm{ad}^n_{f_s} f_1 = \begin{cases} f_{s1} & \text{if } n \text{ is even,} \\ f_1 & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \exp(\widehat{T}_s - t) f_{s*}(\widehat{\xi}(t)) &= \sum_{n=0}^{+\infty} \frac{(t-\widehat{T}_s)^n}{n!} \mathrm{ad}^n_{f_s} f_1(\widehat{x}_s) \\ &= \cosh(t-\widehat{T}_s) f_1(\widehat{x}_s) + \sinh(t-\widehat{T}_s) f_{s1}(\widehat{x}_s)\,. \end{aligned}$$

Since $\exp(\widehat{T}_s - t) f_{s*}(\widehat{\xi}(t))$ cannot be parallel to $f_{s1}(\widehat{x}_s)$, Corollary 4.3 applies for any bang–singular normal extremal.

Example 3

An unknown referee has pointed to our attention the following minimum time problem from [8]:

minimize T

subject to

$$\begin{aligned} & \dot{x} = \cos(\Phi)\,, \quad \dot{y} = \sin(\Phi)\,, \quad \dot{\Phi} = u\,, \\ & u \in [-1,1]\,, \end{aligned} \tag{32}$$

with fixed initial point equal to $(4, 0, \pi/2)$ and final point constrained to the line $x = y = 0$.

Although this problem does not fit in our framework, because it is not a problem with fixed end–points, using Hamiltonian methods and Corollary 4.3, we are able to prove that

every bang–singular normal extremal for the minimum–time problem for system (32) *with fixed initial point and final point constrained to a line* $x(T) = x_f$, $y(T) = y_f$, $\Phi(T)$ *free, is time–optimal.*

In fact, using the same arguments as in the proof of Theorem 2.2 in [2], and due to the fact that the manifold of the final constraints N_f is an integral curve of the controlled vector field f_1, we are able to prove that, *if the strongly extended second variation for the problem with fixed end–points is coercive*, then inequality (29) holds true for every trajectory of the control system starting in a sufficiently small neighborhood of $\widehat{x}_s$ and ending on N_f. Therefore we can apply Theorem 1.1 and its Corollary 4.3. Notice that

$$f_0 = \cos(\Phi)\frac{\partial}{\partial x} + \sin(\Phi)\frac{\partial}{\partial y}\,,\ f_1 = \frac{\partial}{\partial \Phi}\,,$$
$$f_{01} = \sin(\Phi)\frac{\partial}{\partial x} - \cos(\Phi)\frac{\partial}{\partial y}\,,\ f_{001} = 0\,, \quad f_{101} = f_0\,.$$

f_1 and f_{01} are linearly independent on the whole $\mathbb{R}^3$, $H_{001} \equiv 0$ and every normal extremal satisfies SGLC (12).

Since
$$g_t = -(\widehat{T}_s - t)f_{01} + f_1\,,$$
then g_t is never parallel to f_{01}.

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