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Optimal Tracking for MIMO Systems via Data Based System Representation

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Abstract— **The system representation in data space, which was proposed recently, leads to a new control strategy for a linear time-invariant plant. The control input is computed directly from the input-output data of the plant without using any traditional mathematical model, such as transfer function or state space equation. In this paper, a dead-beat tracking for arbitrary reference signals is considered for multi-input multi-output systems, and the control input which minimizes a quadratic performance index is computed in that framework.**

I. Introduction

Behaviors of a plant contain rich knowledge of its dynamics, and the plant can be represented by the behaviors themselves. That is, if we start from input-output data of the plant, we do not have to introduce any traditional mathematical models of the plant such as a transfer function, a state equation [1], or a kernel representation [2]. Based on a sufficient number of the observed data, we can derive a control input directly [3].

From this point of view, a system representation and a control strategy in data space are proposed recently in [3], [4], [5], [6]. This approach employs a *data based system representation* of the plant, where the plant dynamics is represented as a set of basis vectors whose elements are input-output data of the plant. That is, the plant dynamics is represented by its behaviors themselves. Then, with this system representation, dead-beat optimal regulation is investigated in [4], while dead-beat optimal tracking is considered in [5], [6].

We here focus our attention on dead-beat optimal tracking in [5], [6], which is a control that makes tracking error for arbitrary reference signal zero within a finite number of time steps. This problem has been considered, under the assumption that *the relative degree of the plant is given*, for general single-input single-output (SISO) plants [5], and for *a class* of multi-input multi-output (MIMO) plants [6] with *uniform rank* [7]. However, one of the advantage of data based method is to proceed a control strategy without a special knowledge of the plant. Thus, it is desirable to extend the control strategy for *general* MIMO plants with *unknown relative degree*.

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This paper follows this line of research. The objective of this paper is to show that, in the framework of data based system representation, dead-beat optimal tracking is actually possible for MIMO plants without the assumptions. To this end, in the first part of this paper, we investigate a structure of the data space which is useful for dead-beat optimal tracking. Then, in the second part of this paper, we actually give a procedure for computing the control input which minimizes a quadratic performance index subject to dead-beat tracking. It is also shown that the vector representing the plant dynamics should be longer than that of the previous literature where the relative degree of the plant is assumed to be known. The proofs of the theorems are given in the appendix.

In closing this section, we remark that the control strategy proposed in this paper requires neither the mathematical model of the plant nor that of the controller. This is a significant feature relative to the other results [8], [9], [10], [11] concerning optimal control based on input-output data, which are interested in deriving a difference equation of the controller.

II. System Representation in Data Space

A. Data Space

In this paper, we consider a causal, finite dimensional, linear, discrete time, shift invariant plant with p inputs and m outputs. Throughout the paper, n denotes the MacMillan degree of the plant. We assume that the plant is right invertible, which is a necessary condition for dead-beat tracking, independently of system representation of the plant.

Let us introduce a *data vector* which consists of input-output data with ℓ steps from time k

$$z = \left[\begin{array}{ccc;{2pt/2pt}ccc} y_k^{\mathrm{T}} & y_{k+1}^{\mathrm{T}} \cdots & y_{k+\ell-1}^{\mathrm{T}} & u_k^{\mathrm{T}} & u_{k+1}^{\mathrm{T}} \cdots & u_{k+\ell-1}^{\mathrm{T}} \end{array}\right]^{\mathrm{T}} \quad (1)$$

where $y_k \in \mathcal{R}^m$ is the output at time k, $u_k \in \mathcal{R}^p$ is the input at time k. We call the set of all z generated by the plant the *data space*, which is denoted by $\mathcal{Z}$.

We remark that input data to time $k+\ell-1$ are considered as [4], while input data to time $k+\ell-r-1$ have been considered in [5], [6], where the relative degree r of the plant has been assumed to be known.

All admissible data are constrained by the plant dynamics, thus $\mathcal{Z}$ belongs to a subspace of the $\ell(m+p)$ dimensional vector space $\mathcal{R}^{\ell(m+p)}$. Here, we state the following fact [4].

Proposition 1: If $\ell \geq \mu$, then

$$\dim(\mathcal{Z}) = \ell p + n$$

where $\mu \in \mathbb{N}$, one of which exists in $[n/m, n]$.

We therefore see that any data vector can be represented as a linear combination of a basis of $\mathcal{Z}$ if $\ell \geq \mu$. This means that we can regard a set of $\ell p + n$ basis vector of $\mathcal{Z}$ as a system representation of the plant.

Based on this system representation, we further develop a comprehensive framework as dynamical system theory and consider a dead-beat optimal tracking control strategy.

Throughout this paper, we assume that data vectors consist of a time series with ℓ steps. Furthermore, we use μ as a constant defined by Proposition 1 and assume $\ell \geq \mu$.

B. Reachable Data Space

We define the *initial series* of a data vector as its inputs and outputs in the first μ steps. Let us consider a data vector whose initial series is 0, i.e.,

$$\begin{aligned} z_F = [\ 0 \ \ \cdots \ \ 0 \ \ y_{k+\mu}^{\mathrm{T}} \ \ \cdots \ \ y_{k+\ell-1}^{\mathrm{T}} \ ; \\ 0 \ \ \cdots \ \ 0 \ \ u_{k+\mu}^{\mathrm{T}} \ \ \cdots \ \ u_{k+\ell-1}^{\mathrm{T}} \ ]^{\mathrm{T}}. \end{aligned} \quad (2)$$

We call the set of all z_F generated by the plant the *reachable data space*, which is denoted by $\mathcal{Z}_F$.

Obviously, $\mathcal{Z}_F$ is a subspace of $\mathcal{Z}$. We can obtain the following fact [4].

Proposition 2:

$$\dim(\mathcal{Z}_F) = (\ell - \mu)p.$$

That is, the dimension of $\mathcal{Z}_F$ is identical to the degrees of freedom of u_k in z_F.

If we rewrite the data space $\mathcal{Z}$ as a direct sum

$$\mathcal{Z} = \mathcal{Z}_I \oplus \mathcal{Z}_F \quad (3)$$

then, from Proposition 1 and 2,

$$\dim(\mathcal{Z}_I) = \mu p + n.$$

The relation (3) means that any data vector z has a unique decomposition

$$z = z_I + z_F$$

where $z_I \in \mathcal{Z}_I$ whose initial series is identical to that of z, and $z_F \in \mathcal{Z}_F$ whose initial series is 0.

C. Output Controllable Data Space

We define the *output terminal series* of an data vector as its outputs in the last s steps where $s \in \mathbb{N}$. This output terminal series denotes arbitrary reference signal in dead-beat tracking control, and its length denotes time interval whose outputs is forced to the reference signal. Let us consider a data vector whose output terminal series is 0, i.e.,

$$\begin{aligned} z_{Py} = [\ y_k^{\mathrm{T}} \ \ \cdots \ \ y_{k+\ell-s-1}^{\mathrm{T}} \ \ 0 \ \ \cdots \ \ 0 \ ; \\ u_k^{\mathrm{T}} \ \ u_{k+1}^{\mathrm{T}} \ \ \cdots \ \ u_{k+\ell-1}^{\mathrm{T}} \ ]^{\mathrm{T}} \end{aligned} \quad (4)$$

where we assume $\ell \geq s$. We call the set of all z_{Py} generated by the plant the *output controllable data space*, which is denoted by $\mathcal{Z}_{Py}$.

Obviously, $\mathcal{Z}_{Py}$ is a subspace of $\mathcal{Z}$. We can obtain the following theorem.

Theorem 1: If $\ell \geq \nu + s$, then

$$\dim(\mathcal{Z}_{Py}) = \ell p + n - sm$$

where $\nu \in \mathbb{N}$, one of which exists in $[n/p, n]$.

If we rewrite the data space $\mathcal{Z}$ as a direct sum

$$\mathcal{Z} = \mathcal{Z}_{Ty} \oplus \mathcal{Z}_{Py} \quad (5)$$

then, from Proposition 1 and Theorem 1,

$$\dim(\mathcal{Z}_{Ty}) = sm.$$

The relation (5) means that any data vector z has a unique decomposition

$$z = z_{Ty} + z_{Py}$$

where $z_{Ty} \in \mathcal{Z}_{Ty}$ whose output terminal series is identical to that of z, and $z_{Py} \in \mathcal{Z}_{Py}$ whose output terminal series is 0.

D. Intersection of Reachable and Output Controllable Data Spaces

Let us consider a behavior which concatenates a given initial series and a given output terminal series. Notice first that we can prove the following theorem.

Theorem 2: If $\ell \geq \mu + \nu + s$, then

$$\dim(\mathcal{Z}_F \cap \mathcal{Z}_{Py}) = (\ell - \mu)p - sm.$$

We therefore see that

$$\begin{aligned} &\dim(\mathcal{Z}_F + \mathcal{Z}_{Py}) \\ &= \dim(\mathcal{Z}_F) + \dim(\mathcal{Z}_{Py}) - \dim(\mathcal{Z}_F \cap \mathcal{Z}_{Py}) \\ &= \dim(\mathcal{Z}) \end{aligned}$$

from Propositions 1-2 and Theorem 1 when the condition in Theorem 2 is satisfied. This implies

$$\mathcal{Z} = \mathcal{Z}_F + \mathcal{Z}_{Py}.$$

Then, we rewrite the data space $\mathcal{Z}$ as a direct sum

$$\mathcal{Z} = \mathcal{Z}_{IPy} \oplus \mathcal{Z}_{TyF} \oplus \mathcal{Z}_{Cy} \quad (6)$$

under the condition of Theorem 2 where

$$\mathcal{Z}_{Cy} = \mathcal{Z}_{Py} \cap \mathcal{Z}_F$$

and the subspaces $\mathcal{Z}_{IPy}$ and $\mathcal{Z}_{TyF}$ satisfy

$$\begin{aligned} \mathcal{Z}_{Py} &= \mathcal{Z}_{IPy} \cap \mathcal{Z}_{Cy} \\ \mathcal{Z}_F &= \mathcal{Z}_{TyF} \cap \mathcal{Z}_{Cy}. \end{aligned}$$

Here, from Proposition 2 and Theorems 1-2, the dimensions of the subspaces are

$$\begin{aligned} \dim(\mathcal{Z}_{IPy}) &= \mu p + n \\ \dim(\mathcal{Z}_{TyF}) &= sm. \end{aligned}$$

The relation (6) means that any data vector z has a unique decomposition

$$z = z_{IPy} + z_{TyF} + z_{Cy} \tag{7}$$

where $z_{IPy} \in \mathcal{Z}_{IPy}$ whose initial series is identical to that of z and whose output terminal series is 0, $z_{TyF} \in \mathcal{Z}_{TyF}$ whose initial series is 0 and whose output terminal series is identical to that of z, and $z_{Cy} \in \mathcal{Z}_{Cy}$ whose initial and output terminal series are both 0. Utilizing this fact, we can solve a dead-beat optimal tracking control problem based on the system representation in data space.

Remark 1: The minimums of μ and ν such that Propositions 1-2 and Theorems 1-2 hold are observability index μ^* and controllability index ν^* for a minimal realization of the plant, which can be seen in the proofs of these theorems. For an SISO plant, $m = p = 1$, thus $\mu^* = \nu^* = n$. On the other hand, $\mu^* \leq n$ and $\nu^* \leq n$ for an MIMO plant, and μ^* and ν^* are less than n in general.

Remark 2: Note that $\mu \leq n$ and $\nu \leq n$. Thus, regardless of whether μ and ν are unknown or not, we see that Propositions 1-2 hold for $\ell \geq n$, Theorem 1 holds for $\ell \geq n + s$, and Theorem 2 holds for $\ell \geq 2n + s$.

Remark 3: For the case the relative degree is unknown, $\mathcal{Z}_C \neq 0$ even if ℓ is selected as the minimum $\mu^* + \nu^* + s$ in Theorem 2. That is, a behavior concatenating a given initial series and a given output terminal series is not uniquely determined. On the other hand, for the case of the relative degree of the plant is known, it can be shown that the behavior is determined uniquely if data vector is shortened by the relative degree. The details can be found in [5], [6].

Remark 4: The condition in Theorem 1 is related to the number of the constraints between output terminal series and input data from time k to $k + \ell - 1$. It also appears in the context of "a delayed inverse [12]" of the plant.

III. Optimal Tracking via Data Based System Representation

A. Optimal Tracking in Data Space

In this section, based on the structures of the data space, we consider dead-beat optimal tracking control as an optimal control with finite horizon. We use a performance index

$$J = (z_R - z)^{\mathrm{T}} Q (z_R - z) \tag{8}$$

where $z \in \mathcal{Z}$ is a data vector of the plant, $z_R \in \mathcal{R}^{\ell(m+p)}$ is a given *reference data vector*, and $Q \in \mathcal{R}^{\ell(m+p)\times\ell(m+p)}$ is a given positive definite matrix.

Hereinafter, we assume $\ell \geq \mu + \nu + s$. Suppose that the plant have behaved until the time $k + \mu$, which means that the initial series of the plant is specified. Then, we consider forcing the output from $k + \ell - s$ to $k + \ell - 1$ to a reference signal, that is, the given output terminal series. In this case, $\hat{z}_{IPy} \in \mathcal{Z}_{IPy}$ and $\hat{z}_{TyF} \in \mathcal{Z}_{TyF}$ in (7) is uniquely determined. Then, dead-beat optimal tracking control problem is formulated as follows.

Problem 1: For given $z_R \in \mathcal{R}^{\ell(m+p)}$, $\hat{z}_{IPy} \in \mathcal{Z}_{IPy}$ and $\hat{z}_{TyF} \in \mathcal{Z}_{TyF}$, find the optimal data vector $z_{opt} \in \mathcal{Z}$ which minimizes the performance index J of (8) subject to

$$z = \hat{z}_{IPy} + \hat{z}_{TyF} + z_{Cy} \tag{9}$$

where $z_{Cy} \in \mathcal{Z}_{Cy}$ is the decision variable.

Since the quadratic form $x^{\mathrm{T}} Q x$ can be regarded as a metric in the inner product space $\mathcal{R}^{\ell(m+p)}$, the optimal data vector $z_{Cyopt} \in \mathcal{Z}_{Cy}$ which minimizes the performance index J of (8) can be represented as

$$z_{Cyopt} = P_{Cy}(z_R - \hat{z}_{IPy} - \hat{z}_{TyF})$$

where P_{Cy} is the orthogonal projection onto $\mathcal{Z}_{Cy}$ in $\mathcal{R}^{\ell(m+p)}$. It is given by

$$P_{Cy} = H_{Cy}(H_{Cy}^{\mathrm{T}} Q H_{Cy})^{-1} H_{Cy}^{\mathrm{T}} Q \tag{10}$$

where H_{Cy} is a matrix whose columns consists of a basis of $\mathcal{Z}_{Cy}$. Then, we can obtain the following theorem.

Theorem 3: For given $z_R \in \mathcal{R}^{\ell(m+p)}$, $\hat{z}_{IPy} \in \mathcal{Z}_{IPy}$ and $\hat{z}_{TyF} \in \mathcal{Z}_{TyF}$, there exists a unique $z_{opt} \in \mathcal{Z}$ which minimizes the performance index J of (8) subject to (9), and it is given by

$$z_{opt} = (I - P_{Cy})(\hat{z}_{IPy} + \hat{z}_{TyF}) + P_{Cy} z_R.$$

The elements of z_{opt} corresponding to $u_{k+\mu}, u_{k+\mu+1}, \ldots$, are the optimal inputs at the times $k + \mu$, $k + \mu + 1$, ...

B. Bases of Data Spaces

We give procedures to derive the bases of the data space which are used in the proposed control strategy.

Let us introduce a block Hankel matrix of y_i and u_i

$$H = \left[\begin{array}{ccccc} y_0 & y_1 & \cdots & y_i & \cdots \\ y_1 & y_2 & \cdots & y_{i+1} & \cdots \\ \vdots & \vdots & & \vdots & \cdots \\ y_{\ell-1} & y_\ell & \cdots & y_{i+\ell-1} & \cdots \\ \hdashline u_0 & u_1 & \cdots & u_i & \cdots \\ u_1 & u_2 & \cdots & u_{i+1} & \cdots \\ \vdots & \vdots & & \vdots & \cdots \\ u_{\ell-1} & u_\ell & \cdots & u_{i+\ell-1} & \cdots \end{array}\right]. \tag{11}$$

When a sufficient number of data are available, we can select $\ell p + n$ independent columns of H from Proposition 1. Then, we set $H_Z \in \mathcal{R}^{\ell(m+p)\times(\ell p+n)}$ whose column consist of the vectors. Here we can derive the following theorem, from Proposition 2 and Theorems 1-2.

Theorem 4: If ℓ satisfies the condition in Theorem 2, then a column-equivalent matrix H_Z given by elementary column operations is represented as

$$\left[\begin{array}{c|c|c} H_{IPy} & H_{TyF} & H_{Cy} \end{array}\right] = \left[\begin{array}{c|c|c} U_{Iy} & 0 & 0 \\ * & * & * \\ 0 & I_{sm} & 0 \\ \hdashline U_{Iu} & 0 & 0 \\ * & * & * \end{array}\right]$$

where H_{IPy}, H_{TyF} and H_{Cy} are bases of $\mathcal{Z}_{IPy}$, $\mathcal{Z}_{TyF}$ and $\mathcal{Z}_{Cy}$ respectively. The matrices $U_{Iy} \in \mathcal{R}^{\mu m \times (\mu p+n)}$, $U_{Iu} \in \mathcal{R}^{\mu p \times (\mu p+n)}$ and $*$ are appropriate ones determined by the operations, and

$$\mathrm{rank}(U_I) = \mu p + n, \quad U_I = \begin{bmatrix} U_{Iy} \\ U_{Iu} \end{bmatrix}.$$

From this theorem, we obtain P_{Cy}. Suppose that time is $k+\mu$. Then, based on the data obtained by this time, we define *initial series vector* as

$$x_I = \begin{bmatrix} y_k^{\mathrm{T}} & y_{k+1}^{\mathrm{T}} & \cdots & y_{k+\mu-1}^{\mathrm{T}} & \vdots \\ u_k^{\mathrm{T}} & u_{k+1}^{\mathrm{T}} & \cdots & u_{k+\mu-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

Then, we have

$$\hat{z}_{IPy} = H_{IPy}(U_I^{\mathrm{T}} U_I)^{-1} U_I^{\mathrm{T}} x_I.$$

Similarly, if a desirable *output terminal series vector*

$$x_{Ty} = \begin{bmatrix} y_{k+\ell-s}^{\mathrm{T}} & y_{k+\ell-s+1}^{\mathrm{T}} & \cdots & y_{k+\ell-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

is given, we have

$$\hat{z}_{TyF} = H_{TyF} x_{Ty}.$$

Using these formulae, for given z_R and these we can find the solution z_{opt} of Problem 1 based on Theorem 4.

IV. Numerical Example

In this section, we summarize the procedure proposed in this paper thorough a numerical example. We here consider a plant of the MacMillan degree 3 with 2 inputs and 2 outputs.

Let us consider an input series to the plant

$$\begin{aligned}
&u_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, & &u_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, & &u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \\
&u_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & &u_4 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, & &u_5 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\
&u_6 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, & &u_7 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, & &u_8 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \\
&u_9 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, & &u_{10} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, & &u_{11} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \\
&u_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & &u_{13} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, & &u_{14} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \\
&u_{15} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, & &u_{16} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, & &u_{17} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\
&u_{18} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, & &u_{19} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, & &u_{20} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \\
&u_{21} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, & &u_{22} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, & &u_{23} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \\
&u_{24} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, & &u_{25} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & &u_{26} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\end{aligned} \tag{12}$$

and corresponding output series

$$\begin{aligned}
&y_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & &y_1 = \begin{bmatrix} -2.00 \\ 1.00 \end{bmatrix}, & &y_2 = \begin{bmatrix} -0.70 \\ -1.50 \end{bmatrix}, \\
&y_3 = \begin{bmatrix} -0.59 \\ -0.25 \end{bmatrix}, & &y_4 = \begin{bmatrix} -0.64 \\ -0.88 \end{bmatrix}, & &y_5 = \begin{bmatrix} -0.69 \\ -0.56 \end{bmatrix}, \\
&y_6 = \begin{bmatrix} -0.72 \\ 1.28 \end{bmatrix}, & &y_7 = \begin{bmatrix} 1.87 \\ -1.64 \end{bmatrix}, & &y_8 = \begin{bmatrix} -1.32 \\ 1.82 \end{bmatrix}, \\
&y_9 = \begin{bmatrix} -0.41 \\ -1.91 \end{bmatrix}, & &y_{10} = \begin{bmatrix} -0.46 \\ 1.96 \end{bmatrix}, & &y_{11} = \begin{bmatrix} -1.98 \\ 0.02 \end{bmatrix}, \\
&y_{12} = \begin{bmatrix} -0.64 \\ 0.99 \end{bmatrix}, & &y_{13} = \begin{bmatrix} 2.05 \\ -1.49 \end{bmatrix}, & &y_{14} = \begin{bmatrix} -1.20 \\ 1.75 \end{bmatrix}, \\
&y_{15} = \begin{bmatrix} -0.35 \\ 0.13 \end{bmatrix}, & &y_{16} = \begin{bmatrix} 0.18 \\ 0.94 \end{bmatrix}, & &y_{17} = \begin{bmatrix} 0.46 \\ 0.53 \end{bmatrix}, \\
&y_{18} = \begin{bmatrix} 0.60 \\ 0.73 \end{bmatrix}, & &y_{19} = \begin{bmatrix} -1.32 \\ 0.63 \end{bmatrix}, & &y_{20} = \begin{bmatrix} 1.71 \\ -1.32 \end{bmatrix}, \\
&y_{21} = \begin{bmatrix} 0.63 \\ -0.34 \end{bmatrix}, & &y_{22} = \begin{bmatrix} -0.03 \\ 1.17 \end{bmatrix}, & &y_{23} = \begin{bmatrix} -1.78 \\ 0.41 \end{bmatrix}, \\
&y_{24} = \begin{bmatrix} -0.55 \\ -1.21 \end{bmatrix}, & &y_{25} = \begin{bmatrix} -2.50 \\ 1.60 \end{bmatrix}, & &y_{26} = \begin{bmatrix} 1.00 \\ -1.80 \end{bmatrix}.
\end{aligned} \tag{13}$$

In the following, we compute an input series for dead-beat optimal control directly from the data (12) and (13).

We set $\ell = 8$, and substitute these data into y_k and u_k of H in (11). Then, the matrix H with 19 columns is of full column rank. Since $\ell p + n = 19$, we can set this H as H_Z. Choosing $\mu = \nu = n$ and $s = 2$, we can derive a column-equivalent matrix of H_Z with elementary column operations as is shown in Theorem 4. When we choose Q in the performance index as

$$Q = \text{block diag}\{I_{\ell m}, 2I_{\ell p}\}$$

we can compute the orthogonal projection P_{Cy} onto $\mathcal{Z}_{Cy}$ in $\mathcal{R}^{\ell(m+p)}$ using (10). We choose the reference data vector z_R as

$$z_R = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 & 1 & 1 & 0.5 & 0.5 & 0 & 0 \\ -0.5 & -0.5 & -1 & -1 & -0.5 & -0.5 & \vdots & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}.$$

Since the data by the time 26 is available, input-output data from the time 24 to the time 26 is the initial series and we start the optimal control from the time 27, i.e., the initial series vector x_I is

$$x_I = \begin{bmatrix} 1 & -1 & 1 & 1 & -1 & 1 & \vdots \\ -0.55 & -1.21 & -2.50 & 1.60 & 1.00 & -1.80 \end{bmatrix}^{\mathrm{T}}.$$

Moreover, we choose $s = 2$, then the terminal series vector x_{Ty} is

$$x_{Ty} = \begin{bmatrix} -1 & -1 & -0.5 & -0.5 \end{bmatrix}^{\mathrm{T}}$$

from the series of the reference signal contained in z_R. Using the above, we can obtain the solution z_{opt} of Problem 1 from Theorem 3 as

$$
\begin{aligned}
z_{opt} = [\ & -0.55\ \ -1.21\ \ -2.50\ \ 1.60\ \ 1.00\ \ -1.80 \\
& 0.25\ \ -0.10\ \ -0.14\ \ -0.17\ \ -1.02 \\
& 0.48\ \ -1\ \ -1\ \ -0.5\ \ -0.5 \\
& -1\ \ -1\ \ 1\ \ 1\ \ -1\ \ 1\ \ -0.13 \\
& 0.22\ \ -0.41\ \ -0.40\ \ -1.34 \\
& 0.76\ \ -0.79\ \ 1.00\ \ 0\ \ 0\ ]^{\mathrm{T}}.
\end{aligned}
$$

That is, the optimal control inputs from the time 27 to the time 31 are

$$u_{27} = \begin{bmatrix} -0.13 \\ 0.22 \end{bmatrix}, \quad u_{28} = \begin{bmatrix} -0.41 \\ -0.40 \end{bmatrix}, \quad u_{29} = \begin{bmatrix} -1.34 \\ 0.76 \end{bmatrix},$$
$$u_{30} = \begin{bmatrix} -0.79 \\ 1.00 \end{bmatrix}, \quad u_{31} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

V. Concluding Remarks

In this paper, we have shown that dead-beat optimal tracking for MIMO plants can be solved via the data based system representation.

We have only assumed that the plant is right invertible, while in the literature [5], [6] it is further assumed that the relative degree of the plant is given. Then, we have shown that the dimensions of each data spaces can be calculated if the data vector is long enough. A similar condition appears in delayed inverse construction, where some finite delay should be introduced in order to obtain a causal (approximate) inverse of the plant.

References

[1] R. E. Kalman: On the General Theory of Control Systems, Proceedings of the first World Congress of IFAC, pp. 481–492 (1960)

[2] J. C. Willems: Paradigms and Puzzles in the Theory of Dynamical Systems, IEEE Transactions on Automatic Control, Vol 36, No. 3, pp. 259–294 (1991)

[3] M. Ikeda, Y. Fujisaki and N. Hayasi: A Model-less Algorithm for Tracking Control Based on Input-Output Data, Nonlinear Analysis: Theory, Methods & Applications, Vol. 47, No. 3, pp. 1953–1960 (2001)

[4] Y. Fujisaki, Y. Duan and M. Ikeda: System Representation and Optimal Control in Input-Output Data Space, Preprints of the 10th IFAC/IFORS/IMACS/IFIP Symposium on Large Scale Systems, pp. 197–202 (2004)

[5] Y. Fujisaki, Y. Duan and M. Ikeda: System Representation and Optimal Tracking in Data Space, Preprints of the 16th IFAC World Congress (2005) (to appear)

[6] T. Kai and Y. Fujisaki: Dead-Beat Optimal Tracking Control for Multi-Input Multi-Output Systems in Input-Output Data Space, Preprints of the 33rd SICE Symposium on Control Theory, pp. 399–402 (2004) (in Japanese)

[7] U. Shaked: Nearly Singular Filtering for Uniform and Non-uniform Rank Linear Continuous Systems, Int. J. Control, Vol. 38, No. 4, pp. 811–830 (1983)

[8] R. E. Skelton and G. Shi: The Data-Based LQG Control Problem, Proceedings of the 33rd Conference on Decision and Control, pp. 1447-1452 Vol. 31, pp. 1317–1324 (1995)

[9] K. Furuta and M. Wongsaisuwan: Discrete-time LQG Dynamic Controller Design Using Plant Markov Parameters, Automatica, Vol. 31, pp. 1317–1324 (1995)

[10] J. T. H. Chan: Data-based Synthesis of a Multivariable Linear Quadratic Regulator, Automatica, Vol. 32, No. 3, pp. 403–407 (1996)

[11] Y. Kawamura: Direct Construction of LQ Regulator Based on Orthogonalization of Signals: Dynamical Output Feedback, Systems & Control Letters, Vol. 34, pp. 1–9 (1998)

[12] M. K. Sain and J. L. Massey: Invertibility of Linear Time-Invariant Dynamical Systems, IEEE Transaction on Automatic Control, Vol. AC-14, No. 2, pp. 141–149 (1969)

Appendix

The dynamical system to be studied can also be represented as a minimal realization

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k \\
y_k &= Cx_k + Du_k
\end{aligned} \tag{14}
$$

where $u_k \in \mathcal{R}^p$ is the input, $x_k \in \mathcal{R}^n$ is the state, $y_k \in \mathcal{R}^m$ is the output, and A, B, C, D are constant matrices.

Define a $i \times j$ block matrix $\Gamma_{i,j}$ as

$$\Gamma_{i,j} = \begin{bmatrix} CA^{j-i-1}B & \cdots & D & 0 & \cdots & 0 \\ CA^{j-i}B & \cdots & CB & D & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ CA^{j-2}B & \cdots & CA^{i-2}B & CA^{i-2}B & \cdots & D \end{bmatrix}$$

where $CA^{-1}B$ is regarded as D. Define a block matrix $\mathcal{O}_i$ as

$$\mathcal{O}_i = \left[\ C^{\mathrm{T}} \quad (CA)^{\mathrm{T}} \quad \cdots \quad (CA^{i-1})^{\mathrm{T}}\ \right]^{\mathrm{T}}.$$

For $U \in \mathcal{R}^{q \times \tilde{s}}$, we introduce $U^{\perp}$ satisfying

$$U^{\perp} \in \mathcal{R}^{(q-r)\times q},\ U^{\perp}U = 0,\ U^{\perp}(U^{\perp})^{\mathrm{T}} > 0$$

where $\mathrm{rank}(U) = r$. Furthermore, $M_{i,j}$ denotes an appropriate $i \times j$ block matrix.

From the state space equation (14), for all data vector z and state x_k,

$$\left[\ I_{\ell m} \quad -\Gamma_{\ell,\ell}\ \right] z = \mathcal{O}_\ell x_k. \tag{15}$$

If $\mu \geq \mu^*$, where μ^* is observability index, $\mathrm{rank}(\mathcal{O}_\ell) = n$ from $\ell \geq \mu$ and (C, A) is an observable pair. In fact, x_k is uniquely determined by z. This implies we can eliminate x_k from the equation (15). Multiplying (15) by $\mathcal{O}_\ell^{\perp} \in \mathcal{R}^{(\ell m - n)\times \ell m}$, we have

$$\Theta z = 0 \tag{16}$$

where

$$\Theta = \mathcal{O}_\ell^{\perp} \left[\ I_{\ell m} \quad -\Gamma_{\ell,\ell}\ \right]. \tag{17}$$

We need a preliminary lemma about $\Gamma_{i,j}$.

Lemma 1: If the plant is right invertible, then

$$\mathrm{rank}(\Gamma_{s,\,\nu^*+s}) = sm, \quad \forall s \in \mathbb{N}$$

where $\nu^* \in [n/p, n]$ is controllability index which is the smallest number satisfying

$$\mathrm{rank}\left[\ A^{\nu^*-1}B \quad \cdots \quad AB \quad B\ \right] = n.$$

proof: We first note that, if the plant is right invertible, then

$$\mathrm{rank}(\Gamma_{s,\,s+n}) = sm, \quad \forall s \in \mathbb{N}.$$

This fact can be obtained by tracing the proof in Theorem 3 in [12]. Define a block matrix

$$V_i = \left[\begin{array}{cccccccc} I & & & & & & & \\ T_{\nu^*-1} & \ddots & & & & & & \\ \vdots & \ddots & I & & & & \mathbf{0} & \\ T_0 & & T_{\nu^*-1} & I & & & & \\ 0 & \ddots & \vdots & 0 & \ddots & & & \\ & \ddots & T_0 & & \ddots & & & \\ \vdots & & 0 & \vdots & & & & \\ & & \vdots & & & & \ddots & \ddots \\ 0 & \cdots & 0 & 0 & & \cdots & 0 & I \end{array}\right]$$

where $T_0, \ldots, T_{\nu^*-1}$ are the matrices such that

$$\left[\begin{array}{cccc} A^{\nu^*}B & A^{\nu^*-1}B & \cdots & B \end{array}\right]\left[\begin{array}{cccc} I & T_{\nu^*-1} & \cdots & T_0 \end{array}\right]^{\mathrm{T}} = 0.$$

Then,

$$\Gamma_{s,\,s+n}V_{s+n} = \left[\begin{array}{cc} 0_{s,n-\nu^*} & \Gamma_{s,\,s+\nu^*} \end{array}\right].$$

Since V_i is non-singular, $\Gamma_{s,\,s+\nu^*}$ is of full row rank if and only if $\Gamma_{s,\,s+n}$ is row full rank. This completes the proof of Lemma 1. □

(**Proof of Proposition 1**) From the constraint of (16),

$$\mathcal{Z} = \mathrm{Ker}\,\Theta.$$

Since $\mathcal{O}_\ell^\perp$ is full row rank, $\mathrm{rank}\,\Theta = \ell m - n$. Hence, $\dim(\mathcal{Z}) = \ell p + n$.

(**Proof of Proposition 2**) Define

$$J_F = \text{block diag}\{I_{\mu m}, 0_{(\ell-\mu)m}, I_{\mu p}, 0_{(\ell-\mu)p}\} \tag{18}$$

corresponding to z_F of (2). Then, since the constraint that initial series is 0 is given by

$$J_F z_F = 0,$$

we have

$$\mathcal{Z}_F = \mathrm{Ker}\,\Theta \cap \mathrm{Ker}\,J_F.$$

We rewrite the matrices of (15) as block matrices

$$\mathcal{O}_\ell = \left[\begin{array}{c} \mathcal{O}_\mu \\ \mathcal{O}_{\ell-\mu}A^\mu \end{array}\right],\ \Gamma_{\ell,\ell} = \left[\begin{array}{cc} \Gamma_{\mu,\mu} & 0 \\ M_{i,j} & \Gamma_{\ell-\mu,\ell-\mu} \end{array}\right].$$

Since $\mathcal{O}^\mu$ is of full column rank if $\mu \geq \mu^*$, we can choose

$$\mathcal{O}_\ell^\perp = \left[\begin{array}{cc} \mathcal{O}_\mu^\perp & 0 \\ -\mathcal{O}_{\ell-\mu}A^\mu\mathcal{O}_\mu^+ & I_{(\ell-\mu)m} \end{array}\right]. \tag{19}$$

Then, substituting this into (17), we have

$$\Theta = \left[\begin{array}{cc:cc} * & 0 & * & 0 \\ * & I_{(\ell-\mu)m} & * & * \end{array}\right]$$

where $*$ is appropriate ones determined by the operations. Noting the structure of J_F, we see that

$$\mathrm{rank}\left[\begin{array}{c} \Theta \\ J_F \end{array}\right] = \ell m + \mu p.$$

Hence, $\dim(\mathcal{Z}_F) = (\ell - \mu)p$.

(**Proof of Theorem 1**) Define

$$J_{Py} = \text{block diag}\{0_{(\ell-s)m}, I_{sm}, 0_{\ell p}\} \tag{20}$$

corresponding to z_{Py} of (4). Since the constraint that output terminal series is 0 is given by

$$J_{Py}z_{Py} = 0,$$

we have

$$\mathcal{Z}_{Py} = \mathrm{Ker}\,\Theta \cap \mathrm{Ker}\,J_{Py}.$$

We write $\mathcal{O}_\ell^\perp = [\,X_1 \quad X_2\,]$ where $X_1 \in \mathcal{R}^{(\ell m-n)\times(\ell-s)m}$ and $X_2 \in \mathcal{R}^{(\ell m-n)\times sm}$. We rewrite the matrices of (15) as block matrices

$$\Gamma_{\ell,\ell} = \left[\begin{array}{c} M_{\ell-s,\ell} \\ \Gamma_{s,\ell} \end{array}\right].$$

Then, we have

$$\begin{aligned} \mathrm{rank}\left[\begin{array}{c} \Theta \\ J_{Py} \end{array}\right] &= \mathrm{rank}\left[\begin{array}{cc:c} X_1 & X_2 & -X_1M_{\ell-s,\,\ell} - X_2\Gamma_{s,\,\ell} \\ 0 & I_{sm} & 0 \end{array}\right] \\ &= \mathrm{rank}\left[\begin{array}{cc} X_1 & -X_2\Gamma_{s,\,\ell} \end{array}\right] + sm \\ &= \mathrm{rank}\left(\mathcal{O}_\ell^\perp\left[\begin{array}{cc} I_{(\ell-s)m} & 0 \\ 0 & -\Gamma_{s,\,\ell} \end{array}\right]\right) + sm. \end{aligned}$$

By Lemma 1, $\Gamma_{s,\ell}$ is of full row rank if $\nu \geq \nu^*$ and $\ell \geq \nu + s$. Noting that $\mathcal{O}_\ell^\perp$ is of full row rank, we see

$$\mathrm{rank}\left[\begin{array}{c} \Theta \\ J_{Py} \end{array}\right] = \ell m + sm - n.$$

Hence, $\dim(\mathcal{Z}_{Py}) = \ell p + n - sm$.

(**Proof of Theorem 2**) From J_F of (18) and J_{Py} of (20)

$$\mathcal{Z}_{Cy} = \mathrm{Ker}\,\Theta \cap \mathrm{Ker}\,J_{Py} \cap \mathrm{Ker}\,J_F.$$

We rewrite the matrices of (15) as block matrices

$$\Gamma_{\ell,\ell} = \left[\begin{array}{cc} \Gamma_{\mu,\mu} & 0 \\ M_{\ell-\mu-s,\mu} & M_{\ell-\mu-s,\ell-\mu} \\ M_{s,\mu} & \Gamma_{s,\ell-\mu} \end{array}\right].$$

Substituting this and $\mathcal{O}_\ell^\perp$ of (19) into (17), we obtain

$$\Theta = \left[\begin{array}{ccc:cc} * & 0 & 0 & * & 0 \\ * & I_{(\ell-\mu-s)m} & 0 & * & * \\ * & 0 & * & * & \Gamma_{s,\ell-\mu} \end{array}\right].$$

By Lemma 1, the matrix $\Gamma_{s,\ell-\mu}$ is of full row rank if $\nu \geq \nu^*$ and $\ell \geq \mu + \nu + s$. Noting the structure of J_F and J_{Py}, we see

$$\mathrm{rank}\left[\begin{array}{c} \Theta \\ J_{Py} \\ J_F \end{array}\right] = \ell m + \mu p + sm.$$

Hence, $\dim(\mathcal{Z}_{Cy}) = (\ell - \mu)p - sm$.