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Hybrid differential evolution method for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem under pole assignment

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Abstract— **This paper is concerned with the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem combined with robust pole placement in linear matrix inequality (LMI) regions. Based on Differential Evolution algorithms (DEs), Salomon's Evolutionary Gradient Search method (EGS) and LMIs, a hybrid algorithm is presented for numerical computation of a robust fixed-order, static or dynamic, output feedback controller. This approach does not require that all specifications are enforced by a single closed-loop Lyapunov function. This allows to reduce the conservatism of the usual existing methods. This approach can be used for synthesis of reduced or full order controllers. Examples borrowed from the literature are discussed to validate this approach.**

Keywords: **robust control, LMI, differential evolution, evolutionary gradient search.**

I. INTRODUCTION

THE mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for systems subject to parameter uncertainties by static or dynamic output feedback has been studied widely during the last two decades. The theoretic motivation for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem has been extensively discussed in [1], [2], [3], [4]. Some important results about output feedback control can be found in [5], [6], [7], [8] and the references therein. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance criterion has also been considered in conjunction with regional pole placement [5], [6]. It is well known that satisfactory transient behaviour can be achieved by placing the closed-loop poles in a suitable region of the complex plane. However, the general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem does not have yet a closed-form solution except for special cases presented in the literature.

In this work, a hybrid approach based on differential evolution (DE), Salomon's evolutionary gradient search (EGS) method [9] and LMIs is presented in order to find an internally stabilizing output feedback controller which solves the constrained mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem. By the constrained mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem it is understood the design problem with a mix of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance and pole placement in LMI regions.

The first motivation for this work arises from the fact that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust controller, in general, is not easy to be designed. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem was reduced to a convex optimization problem by considering a formulation with a common Lyapunov function in [10], [6], [11], [12], [13], nevertheless this assumption results in some degree of conservatism.

The second motivation comes from the change of variable presented in [5], [12], [13], developed to turn output feedback specification into LMIs. In this change of variable the system matrices A and B_2 are involved, thus the dynamic output feedback control design problem for system subject to polytopic uncertainties cannot be reduced to a convex optimization problem.

The last motivation, but not the least, is the interesting global search properties of the differential evolution algorithm presented in [14], [15], [16], [17], [18] and the powerful local search characteristics of the gradient search methods [19], [9]. Many evolutionary algorithms can be applied to a number of control methodologies for the improvement of the overall system performance. In [18], [20] the DE algorithm is applied to the optimization of PI and PID controllers. In [21], [22], [23], genetic algorithms, another class of the evolutionary algorithms, are used to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for SISO systems using transfer functions. Some related approaches which use Genetic Algorithms and LMIs can be found in [24], [25], [26], [27], [28].

This paper is organized as follows. Section II presents the pole placement LMIs. In section III the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is detailed. In section IV the proposed hybrid algorithm is described. In section V examples from the literature are borrowed to validate this papers approach.

II. POLE PLACEMENT IN LMI REGIONS

A LMI region is any subset $\mathcal{D}$ of the complex left-half plane that can be defined as

$$\mathcal{D} = \left\{ z \in \mathbb{C} : T + \Gamma z + \Gamma^T \bar{z} < 0 \right\} \tag{1}$$

where T and Γ are real matrices and $T = T^T$. A dynamical system $\dot{x} = Ax$ is called $\mathcal{D}$-stable if all its poles are in the LMI region $\mathcal{D}$. Many symmetric regions, with respect to the real axis, can be described as LMI regions.

The Lyapunov theorem can be directly applied to LMI regions [29]. If $[t_{rs}]$ and $[\tau_{rs}]$, $1 \leq r, s \leq q$, denote the entries of the matrices T and Γ respectively, a dynamic matrix A is called $\mathcal{D}$-stable if and only if there exists a positive definite matrix X_D such that [30]:

$$\left[t_{rs} X_D + \tau_{rs} A X_D + \tau_{sr} X_D A^T \right] < 0,\ 1 \leq r, s \leq q. \tag{2}$$

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III. Output-Feedback $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem with Pole Placement

Consider an uncertain time-invariant plant S described as

$$\begin{cases} \dot{x}(t)= Ax(t)+B_{1_2}w_2(t)+B_{1_\infty}w_\infty(t)+B_2u(t) \\ z_\infty(t)= C_1x(t)+D_{11_2}w_2(t)+D_{11_\infty}w_\infty(t)+D_{12}u(t) \\ z_2(t)= C_2x(t)+D_{21_\infty}w_\infty(t)+D_{22}u(t) \\ y(t) = C_yx(t)+D_{y1_\infty}w_\infty(t) \end{cases}, \quad (3)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the sensor output, $z_\infty(t) \in \mathbb{R}^{p_1}$ and $z_2(t) \in \mathbb{R}^{p_2}$ are the controlled outputs and $w_\infty(t) \in \mathbb{R}^{l_1}$ and $w_2(t) \in \mathbb{R}^{l_2}$ are the exogenous inputs. All matrices are real with appropriate dimensions. Assume that A and B_2 belong to convex-bounded domains defined as:

$$\mathcal{A} = \left\{ A\,;\, A = \sum_{i=1}^{N} \alpha_i A_i\,,\, \sum_{i=1}^{N} \alpha_i = 1\,,\, \alpha_i \geq 0 \right\}, \quad (4)$$

$$\mathcal{B} = \left\{ B\,;\, B = \sum_{j=1}^{N} \beta_j B_{2j}\,,\, \sum_{j=1}^{N} \beta_j = 1\,,\, \beta_j \geq 0 \right\}. \quad (5)$$

Assume also that all pairs (A, B_2) and (C_y, A) are stabilizable and detectable, respectively. Let the linear time invariant (LTI) output feedback controller L be described by the following space-state equations:

$$L : \begin{cases} \dot{\eta}(t) = A_K\eta(t) + B_Ky(t) \\ u(t) = C_K\eta(t) + D_Ky(t) \end{cases}, \quad (6)$$

where $A_K \in \mathbb{R}^{n_c \times n_c}$. In this approach, the controller can be chosen of reduced ($n_c < n$) or full order ($n_c \geq n$) but n_c must be fixed. Both S and L are real-rational and proper. No assumptions are necessary about singular plants with $j\omega$-axis zeros or rank deficiencies in matrices D.

The system (3) can be rewritten as follows:

$$\begin{cases} \dot{x}_f(t)=\tilde{A}x_f(t)+\tilde{B}_{1_2}w_2(t)+\tilde{B}_{1_\infty}w_\infty(t) + \tilde{B}_2u_s(t) \\ z_\infty(t)=\tilde{C}_1x_f(t)+D_{11_2}w_2(t)+D_{11_\infty}w_\infty(t) + \tilde{D}_{12}u_s(t) \\ z_2(t)=\tilde{C}_2x_f(t)+D_{21_\infty}w_\infty(t) + \tilde{D}_{22}u_s(t) \\ y_s(t) =\tilde{C}_yx_f(t)+\tilde{D}_{y1_\infty}w_\infty(t) \\ y(t) = C_{yf}x_f(t)+D_{y1f_\infty}w_\infty(t) \end{cases} \quad (7)$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0_{n_c\times n_c} \end{bmatrix}, \quad \tilde{B}_{1_i} = \begin{bmatrix} B_{1_i} \\ 0 \end{bmatrix}, \quad i = 2, \infty,$$
$$\tilde{B}_2 = \begin{bmatrix} B_2 & 0 \\ 0 & \mathbf{I}_{n_c\times n_c} \end{bmatrix}, \quad (8)$$
$$\tilde{C}_i = \begin{bmatrix} C_i & 0 \end{bmatrix}, \quad \tilde{D}_{i2} = \begin{bmatrix} D_{i2} & 0 \end{bmatrix}, \quad i = 1, 2,$$
$$\tilde{C}_y = \begin{bmatrix} C_y & 0 \\ 0 & \mathbf{I}_{n_c\times n_c} \end{bmatrix}, \quad \tilde{D}_{y1_\infty} = \begin{bmatrix} D_{y1_\infty} \\ 0 \end{bmatrix},$$

and the investigated control law becomes $u_s(t) = L_K y_s(t)$ with

$$L_K = \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{(m+n_c)\times(p+n)}, \quad (9)$$

with $A \in \mathcal{A}$ and $B_2 \in \mathcal{B}$. Thus, the dynamic output feedback problem can be treat as a static output feedback one. L_K is called a $\mathcal{D}-$stable controller if the closed-loop system (7) is $\mathcal{D}-$stable.

Defining $T_{z_\infty w_\infty}(s)$ as the closed-loop transfer matrix from w_∞ to z_∞ and $T_{z_2w_2}(s)$ the one from w_2 to z_2, that is

$$\begin{aligned} T_{z_\infty w_\infty}(s) &= (\tilde{C}_1 + \tilde{D}_{12}L_K\tilde{C}_y)[s\mathbf{I} - (\tilde{A} + \tilde{B}_2L_K\tilde{C}_y)]^{-1} \\ &\quad (\tilde{B}_{1_\infty} + \tilde{B}_2L_K\tilde{D}_{y1_\infty}) + (D_{11_\infty} + \tilde{D}_{12}L_K\tilde{D}_{y1_\infty}), \\ T_{z_2w_2}(s) &= (\tilde{C}_2 + \tilde{D}_{22}L_K\tilde{C}_y)[s\mathbf{I} - (\tilde{A} + \tilde{B}_2L_K\tilde{C}_y)]^{-1}\tilde{B}_{1_2}, \end{aligned} \quad (10)$$

the considered constrained mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem can be formulated as follows.

Find a proper, real rational, admissible, fixed structure, $\mathcal{D}$-stable controller L_K that minimizes the $\mathcal{H}_2$ norm $\|T_{z_2w_2}\|_2$ subject to the $\mathcal{H}_\infty$ norm constraint $\|T_{z_\infty w_\infty}\|_\infty < \gamma$, for a given achievable $\mathcal{H}_\infty$-norm bound γ, $\forall A \in \mathcal{A}$ and $\forall B_2 \in \mathcal{B}$.

Using the bounded real lemma [31], the concept of quadratic stability [32], and the $\mathcal{H}_2$ performance measure [1], [33], the suboptimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem can be formulated in matrix inequalities form as follows:

$$\min_{\substack{X_2 > 0,\, X_\infty > 0,\\ X_D > 0,\, L_K}} \quad \mathbf{Trace}(\tilde{C}_{2f}X_2\tilde{C}_{2f}^T) \quad (11)$$

subject to:

$$\tilde{A}_fX_2 + X_2\tilde{A}_f^T + \tilde{B}_{1_2}\tilde{B}_{1_2}^T < 0 \quad (12)$$

$$\begin{bmatrix} \tilde{A}_fX_\infty + X_\infty\tilde{A}_f^T & \tilde{B}_{1f_\infty} & X_\infty\tilde{C}_{1f}^T \\ \tilde{B}_{1f_\infty}^T & -\mathbf{I} & \tilde{D}_{11f_\infty}^T \\ \tilde{C}_{1f}X_\infty & \tilde{D}_{11f_\infty} & -\gamma^2\mathbf{I} \end{bmatrix} < 0 \quad (13)$$

$$\left[t_{rs}X_D + \tau_{rs}\tilde{A}_fX_D + \tau_{rs}X_D\tilde{A}_f^T\right] < 0,\ 1 \leq r, s \leq q \quad (14)$$

where

$$\begin{aligned} \tilde{A}_f &= \tilde{A}_i + \tilde{B}_{2j}L_K\tilde{C}_y \\ \tilde{B}_{1f_\infty} &= \tilde{B}_{1_\infty} + \tilde{B}_{2j}L_K\tilde{D}_{y1_\infty} \\ \tilde{C}_{if} &= \tilde{C}_i + \tilde{D}_{i2}L_K\tilde{C}_y,\ i = 1, 2 \\ \tilde{D}_{11f_\infty} &= \tilde{D}_{12}L_K\tilde{D}_{y1_\infty}, \end{aligned} \quad (15)$$

with $i = 1, 2, ..., N$, $j = 1, 2, ..., M$ and t_{rs} and τ_{rs} defined as in (2).

The problem is not jointly convex in the variables $(X_2, X_\infty, X_D, L_K)$, but it is still convex for a fixed controller L_K. This performance criterion gives an upper bound of the optimal $\mathcal{H}_2$ performance subject to the $\mathcal{H}_\infty$ norm constraint and pole placement constraints.

It is important to point out that this approach does not require the hypothesis of common Lyapunov matrices $X_2 = X_\infty = X_D$. This reduces the conservatism and can provide better results. Furthermore, the usual change of variable necessary to recast the mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem as a LMI problem [12], [13] can also be avoided. It allows to readily solve the dynamic or static output feedback control case for plants subject to uncertainties. In the change of variable introduced in [13] using matrices R and S, the system matrices A and B_2 are involved, hence certain limitations are imposed on extending this result to deal with control synthesis problem for polytopic uncertain systems. Relaxing these assumptions, the design of both reduced and full-order controllers can also be considered in an unified state-space framework by this approach.

IV. The Hybrid Differential Evolution Algorithm (HDE)

In this section a hybrid design procedure of robust output feedback controller for solving the constrained mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem presented in the preceeding section is introduced.

The differential evolution (DE) algorithm, a branch of the evolutionary algorithms, was first developed by Storn & Price [14] for real number optimization problems and it is well known for its robustness and efficacy [16], [15]. The gradient search methods were first proposed by Newton and these methods have the general structure $x_{j+1} = x_j - \eta\nabla f(x_j)$ where η denotes a small constant and j denotes the iteration number. This so called steepest-descent method starts at an initial point x_0 and then repeatedly subtracts a small fraction η of the locally calculated gradient $\nabla f(x_j)$ from the current point x_j [17]. The greatest advantages of these methods are the accuracy, efficency, speed and proof of convergence. On the other hand, the drawbacks of these algorithms are that they can be applied only to continuously differentiable objective functions, which is not the case, and that they generally stop at the next local optimum.

Salomon proposed a method of estimating the gradient without the computation of the derivatives of the function [9]. This method is easily implemented and under some constraints can produce a good estimation value.

The HDE algorithm was developed with the goal of combining the global search characteristics and reliability of the differential evolution method [16], [15], the local search properties of the classic gradient search methods [19], [17] and the accuracy and flexibility provided by the LMI formulation [32], [34]. This combination allows the algorithm to evolve the solutions locally and globally increasing the probability of finding the global optimum. There is, although, no proof that this global optimum is reached.

In order to link both methods, two different operators were created: the Differential Evolution Operator (DEO; see IV-C) based on the differential evolution algorithm from [16], [17], and the Estimated Gradient Search Operator (EGSO see IV-D) based on the evolutionary gradient search procedure proposed in [9].

Based on DEs, EGS and LMIs the HDE algorithm searches for an optimal $\mathcal{D}$-stable robust controller L_K (9) and consequently determines X_2, X_∞ and X_D that solve the optimization problem (11) satisfying (12)-(14). Hence the algorithm works with a population of candidate controllers (individuals)$[L_K]$. This choice has been made in order to maintain simplicity and effectiveness.

At each iteration the optimization problem (11) is solved using the Matlab package LMI-Lab [29] for all candidate individuals of the population P^M of size M. The algorithm stops when a certain number of generations J is reached. Remember that for fixed $L_K = [l_{ij}]_{(m+n_c)\times(p+n_c)}$ the constrained mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem (11)-(14) is convex. The $\mathcal{H}_\infty$ norm constraint γ is given but nevertheless the more general constrained mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust control problem defined with the performance cost the trade-off criterion:

$$\min_{L_K} \quad \alpha \left\| T_{z_\infty w_\infty} \right\|_\infty^2 + \beta \left\| T_{z_2 w_2} \right\|_2^2, \tag{16}$$

with $\alpha > 0$ and $\beta > 0$ as defined in [29] and constraints (11)-(14) can be obtained with this algorithm. The HDE procedure $HDE(M, J, p_{DE}, p_{EGS})$ returns an evolved population of controllers L_K and it is described as follows.

1) initiate the population P^M with M individuals;
2) while $j < J$, where j denotes the current iteration and J the termination iteration, do
   a) while $m < M$, $m = 1, \ldots, M$ do
      i) randomly select $L \in P_j^M$;
      ii) $L_{DE} = DEO(L, p_{DE})$;
      iii) $L_{EGS} = EGSO(L_{DE}, p_{EGS})$;
      iv) substitute L by L_{EGS} in the population P_j^M;

where p_{DE} and p_{GS} represent the probabilities of occurrence of the DEO and $EGSO$ operators respectively. L_{DE} and L_{EGS} are the controllers returned by its respective operator, in an analogy with the recombination and mutation operators from the evolutionary algorithms.

An elitist strategy was implemented such that the worst element L_{j+1} of the population P_{j+1} is substituted by the best element L_j of the population P_j. The elitism improved the convergence of the method. The Matlab interior points LMI solver is used to test the feasibility of each generated solution finding the respective matrices X_2, X_D and X_∞.

A. Objective and Fitness Function

The objective (cost) function $J(L_K)$ provides the mechanism for evaluating each individual L_K. The objective function corresponds to the $\mathcal{H}_2$ norm $J(L_K) = \mathbf{Trace}(\tilde{C}_{2f} X_2 \tilde{C}_{2f}^T)$ (11) subject to (12)-(14).

To maintain uniformity over different problems, the fitness function was rescaled and it is defined as:

$$fitness(L_K) = \frac{1}{1 + J(L_K)}. \tag{17}$$

Minimizing the objective function $J(L_K)$ is equivalent to getting a maximum fitness value in the genetic search. No penalty functions are used in the algorithm.

B. Initial Population

Since the search space is not convex and its bounds are unknown, the classic random initial population generation of the DEs [14], [16], [15] requires a strong computational effort. An element is called feasible if it satisfies the matrix inequalities (12)-(14), otherwise it is called infeasible. In order to reduce the computational effort generating feasible elements, an approach described in [35] was implemented. This approach gives a sufficient condition to find feasible controllers using the Lyapunov inequality associated to the $\mathcal{H}_2$-norm for the state feedback control problem. The procedure description is given as follows.

Consider the following system from (7)

$$\begin{cases} \dot{x}_f = \tilde{A}x_f + \tilde{B}_{1_2}w_2 + \tilde{B}_2 u_s \\ z_2 = \tilde{C}_2 x_f + \tilde{D}_{22} u_s \\ y_s = \tilde{C}_y x_f \end{cases}, \tag{18}$$

and the static output feedback control law

$$u_s = L_K y_s. \tag{19}$$

If $rank(C_y)$ is not full, generate a random matrix N such that

$$M = \begin{bmatrix} N \\ C_y \end{bmatrix} \tag{20}$$

has full rank. The matrix N carries the diversity of the initial population, since that for each generated M a distinct initial solution is found;

Perform a similarity transformation $\bar{x} = Mx_f$ then the system (18) is given by:

$$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}_{1_2}w_2 + \bar{B}_2 u_s \\ z_2 = \bar{C}_2\bar{x} + D_{22} u_s \\ y_s = \bar{C}_y \bar{x} \end{cases}, \tag{21}$$

where $\bar{A} = M\tilde{A}M^{-1}$, $\bar{B}_{1_2} = M\tilde{B}_{1_2}$, $\bar{B}_2 = M\tilde{B}_2$, $\bar{C}_2 = \tilde{C}_2 M^{-1}$ and $\bar{C}_y = \tilde{C}_y M^{-1}$. The resulting control law is given by

$$u_s = L\bar{C}_y M^{-1}\bar{x} = \bar{L}\bar{x}, \tag{22}$$

where the new controller $\bar{L}$ has the following structure

$$\bar{L} = \begin{bmatrix} 0 & L_K \end{bmatrix}; \tag{23}$$

In order to find feasible controllers $\bar{L}$, the inequalities (12) and (14) are converted to LMI form using the change of variables $\bar{L}\bar{W}_1 = \bar{W}_2$ with fixed structure matrices

$$\bar{W}_1 = \begin{bmatrix} \bullet & 0 \\ 0 & \bullet \end{bmatrix}; \; \bar{W}_2 = \begin{bmatrix} 0 & \bullet \end{bmatrix}, \tag{24}$$

assuming $X_2 = X_D$. This reduced convex problem is solved with help of the Matlab LMI-Lab [29];

Extract L_K from $\bar{L}$ and compute X_2, X_∞ and X_D solving the convex problem (11)-(14);

Save the feasible solution L_K, if any, and return to step 1 until the initial population is fulfilled.

C. Differential Evolution Operator (DEO)

This operator was extracted from the classic DE theory. Some modifications were necessary due to the complexity of the problem. The procedure $L_{DE} = DEO(L, p_{DE})$ is as follows:

1) select $L_1 \in P_j^M$ via rank selection method;
2) randomly select L_2 and L_3;
3) if $random\,[0,1) < p_{DE}$ then $L' = L_1 + F \times (L_2 - L_3)$ else $L' = L_1$;
4) $L_{DE} = L'$ if $fitness(L') > fitness(L)$, else $L_{DE} = L$; L is an argument passed to the DEO operator;
5) return L_{DE} if feasible, else update F and repeat 1 to 4 up to 5 times.

In this procedure no repair rule is applied to the resulting offspring L_{DE} since the geometry of the problem is unknown. Moreover, different from the crossover operator of the classic differential evolution algorithms, the DEO operates over the full variable vector L which corresponds to the controller being optimized (for details [14], [17], [18]). Simulations showed that applying the operator partially over L' resulted in many infeasible elements reducing the efficency of the algorithm.

The classic DE algorithms [14], [16] select L_1 randomly. Many simulations showed that if the controller L_1 is selected via a stochastic selection method, rather then randomly, a better convergence is achieved. Therefore a linear ranking selection algorithm [36] for L_1 was implemented.

The variable $F \in [0,1)$ is randomly selected through the expression $F = \frac{10 \times random[0,1)}{10^\alpha}$ where α is a counter with initial value 1. If the offspring L_{DE} is infeasible, α is increased, F is updated and the procedure is repeated up to 5 times. This technique helps the algorithm to produce a feasible offspring L_{DE} since it reduces the magnitude of F at each try. In general, the value 5 allows a satisfactory trade-off between the algorithm's performance and the number of feasible offsprings.

D. Estimated Gradient Search Operator (EGSO)

This operator is based on the Evolutionary-Gradient-Search Procedure proposed by Salomon [9]. The aim of this procedure is to approximate the gradient of the function being optimized through a weighted sum without making use of the derivatives. The procedure can result in great estimation errors and the conditions for a good approximation are found in [9]. The procedure $L_{EGS} = EGSO(L_{DE}, p_{EGS})$ is as follows:

1) if $random\,[0,1) > p_{EGS}$ return $L_{EGS} = L_{DE}$ else:
2) create N test candidates (offspring) L_i such that $L_i = L_{DE} + Z_i$, $1, \ldots, N$ where Z_i is a matrix with Gaussian distributed elements of mean 0 and standard deviation $\frac{\sigma_m}{\sqrt{n}}$ and n is the number of entries of the controllers L;
3) evaluate $G = \sum_{i=1}^{n} (fitness\,(L_i) - fitness\,(L_{DE})) \times (L_i - L_{DE})$;

4) compute the vector $E = \frac{G}{\|G\|_2}$;
5) compute $L_{EGS} = L_{DE} - \sigma_m E$;
6) return L_{EGS} if feasible, else repeat $(1)-(5)$ up to 5 times;
7) self adapt σ such that if $fitness(L_{EGS} - \sigma_m \zeta E) \leq fitness(L_{EGS} - (\sigma_m/\zeta)E)$ then $\sigma_{m+1} = \sigma_m \zeta$ else $\sigma_{m+1} = \sigma_m/\zeta$.

The initial values $\sigma_0 = 1$, $\zeta = 2$ and $N = 60$ were arbitrarily chosen after many simulations. For the interested reader, the proof of the convergence of the algorithm to the real gradient when $N \to \infty$ is also demonstrated in [9].

V. Numerical Examples

In order to validate our approach, some numerical examples were borrowed from the literature.

Example 1: This example is described in details in [37]. It corresponds to an inverted pendulum system. The matrices are:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 & a_6 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, C_y = \mathbf{I}_4,$$

$$B_{1_2}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, D_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} I_4 \\ 0_{1\times 4} \end{bmatrix}, \quad B_{1_\infty} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$D_{11_2} = 0, \quad C_2 = 0.1 * \mathbf{I}_4, \quad D_{11_\infty} = 0,$$
$$D_{y1_\infty} = 0, \quad D_{21_\infty} = 0, \quad D_{22} = 0,$$

where $a_1 = -0.0369$, $a_2 \in [-0.1183, -0.0968]$, $a_3 \in 10^{-3}[0.5336, 0.6522]$, $a_4 = 30.7744$, $a_5 \in [0.3025, 0.3697]$ and $a_6 \in [-0.5444, -0.4454]$. The parameters a_2 and a_5 as well as the parameters a_3 and a_6 derive from only two distinct uncertain parameters [37]. Therefore the matrix A generates only four vertices. The desired pole placement region is a disk centered in $(-9.8, 0)$ with radius $r = 9$. The reduced order $(n_c = 2)$ output feedback controller found by the algorithm with $p_{DE} = 0.8$, $p_{EGS} = 0.3$, $J = 50$, and $M = 20$ is:

$$A_K = \begin{bmatrix} -11.4167 & -18.0374 \\ -1.6352 & -3.9691 \end{bmatrix}, D_K^T = \begin{bmatrix} -0.0172 \\ 2.2823 \end{bmatrix},$$

$$B_K = \begin{bmatrix} -1.2874 & 50.8211 \\ -0.2362 & 9.8441 \end{bmatrix}, C_K^T = \begin{bmatrix} -0.3425 \\ -0.0698 \end{bmatrix},$$

and the associated guaranteed cost$\|H\|_2^2 = 0.0345$ and $\gamma = 2.5199$. In [25] the guaranteed cost found is $\|H\|_2^2 = 0.0530$ and $\gamma = 3.1506$. The full order $(n_c = 4)$ output feedback controller obtained by the algorithm with $p_{DE} = 1$, $p_{EGS} = 0.6$, $J = 50$, and $M = 20$ is:

$$A_K = \begin{bmatrix} -20.1948 & -13.2019 & -9.2737 & -1.0663 \\ 1.7711 & -1.5262 & 1.1661 & 0.2192 \\ 7.9938 & 6.2597 & 1.4597 & 0.7734 \\ 21.1485 & 16.4875 & 11.4439 & -1.2163 \end{bmatrix},$$

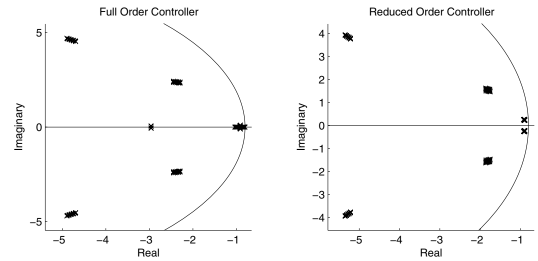

Fig. 1. Closed Loop Root Loci

$$B_K = \begin{bmatrix} -1.5921 & -23.7237 \\ 0.1028 & 2.5391 \\ 0.6680 & 10.7916 \\ 1.8529 & 28.5738 \end{bmatrix}, C_K^T = \begin{bmatrix} 1.7219 \\ 1.3651 \\ 0.9287 \\ 0.1577 \end{bmatrix},$$

$$D_K = \begin{bmatrix} 0.1317 & 2.7653 \end{bmatrix},$$

and the associated guaranteed cost $\|H\|_2^2 = 0.0489$ and $\gamma = 1.7624$. The minimal guaranteed cost found in [37] is $\|H\|_2^2 = 13.2738$, and in [25] is $\|H\|_2^2 = 0.0421$ and $\gamma = 2.1077$.

Fig. 1 shows the closed-loop poles for 26 pairs of the two uncertain parameters for the obtained full and reduced order controllers.

Example 2: This example was adapted from [24] and [38]. It represents the model of the linearized dynamic equation of the VTOL helicopter. In [38] only the optimal $\mathcal{H}_2$ control by output feedback is considered.

A reduced order $n_c = 2$ controller is designed. The algorithm with parameters $p_{DE} = 1.0$, $p_{EGS} = 0.6$, $J = 70$, and $M = 20$ obtained the following controller

$$A_K = \begin{bmatrix} -13.4542 & -13.1431 \\ -11.7492 & -13.3828 \end{bmatrix},$$

$$B_K = \begin{bmatrix} 23.8588 \\ -12.0188 \end{bmatrix}, D_K = \begin{bmatrix} 1.9157 \\ -5.7611 \end{bmatrix},$$

$$C_K = \begin{bmatrix} 5.2611 & 5.7074 \\ -2.5016 & -1.9502 \end{bmatrix},$$

and the associated guaranteed cost $\|H\|_2^2 = 12.8885$ and $\gamma = 13.5711$. The guaranteed cost found in [28] is $\|H\|_2^2 = 13.6839$, $\gamma = 13.6338$ and in [24] is $\|H\|_2^2 = 18.9888$, $\gamma = 19.9714$.

VI. Conclusion

In this work the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem with LMI region pole placement has been investigated. As solution for the problem an algorithm based on DEs, the Evolutionary Gradient Search of Salomon and LMIs has been proposed. The algorithm is capable of dealing with dynamic and static output feedback synthesis and searches for near optimum controllers that minimize the $\mathcal{H}_2$ norm under $\mathcal{H}_\infty$ norm and pole placement constraint.

Several examples borrowed from the control literature have been analyzed. Although this hybrid algorithm does not guarantee the finding of the optimal solution, the obtained simulation results have demonstrated that this strategy is efficient and produces good results. In the developed examples presented in the paper, the norms calculated are smaller than those cited by the references.

The proposed algorithm is flexible to deal with reduced or full order dynamic output controller design. In this approach the assumption $X_2 = X_\infty = X_D$ is not required reducing the conservatism and allowing better solutions.

The flexibility of the algorithm allows a wide range of LMI pole placement regions. The pole placement constraints enable the control engineer to design a better controller in terms of transient behavior specifications.

Future works will be dedicated in order to implement other classes of evolutionary algorithms for comparison purposes. Also, some effort in applying this method to discrete systems will be spent.

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