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A General and Efficient Robust Control Method for Uncertain Nonlinear Mechanical Systems

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***Abstract*—This paper presents a novel and general method for control design of linear or nonlinear mechanical systems subject to disturbances and uncertainties. This method easily allows determining simple and robust control laws which force a mechanical system to track a target trajectory, with bounded acceleration, with prefixed maximum position and velocity errors and fixed maximum convergence velocity. Method efficiency is illustrated by designing a decoupled control law, like PD control, for a three-link planar robot which has the task of periodically doing some work on given points of a rotating object.**

***Index Terms*— Robot tracking, General robust control method, Robust stability, Nonlinear mechanical uncertain systems.**

I. INTRODUCTION

Control of linear and nonlinear mechanical systems, subject to disturbances and uncertainties is surely the most important and studied problem in the past and recent literature. Whenever the reference signal and/or disturbance is not a polynomial, very few theoretical results exist and often no of practically use, since either they refer to a class of systems with little relevant interest to engineers, or such design methods are quite complex, not very robust and they do not allow satisfying more than a single specification [1]-[14].

In this paper, starting from new theoretical results about practical stability for a quite general class of systems, including mechanical systems, articulated too, are formulated and demonstrated some theorems in order to allow determining easily simple and robust control laws, like PD with a possible cascade nonlinear amplifier and compensation action.

The proposed controllers structure tries to give to the error system the properties of a Butterworth system with a damping ratio of $\sqrt{2}/2$. For this reason, it depends on few design parameters which can be easily determined in order to separately satisfy robustness specifications with respect to uncertainties, specifications with respect to position and velocity precision and to convergence velocity of the tracking error of a general prefixed trajectory with bounded acceleration without chattering of the control signal.

II. SOME THEORETICAL PRELIMINARIES

Consider the following

Problem 1. Given the nonlinear dynamic plant

$$\ddot{y} = F(t, p, y, \dot{y})u + f(t, p, y, \dot{y}), \tag{1}$$

where $t \in R$ is the time, $y \in R^{\nu}$ is the output, $u \in R^{m}$ is the input control, $p \in \wp \subset R^{\mu}$ is the vector of uncertain plant parameters, with $\wp$ compact set, $F \in R^{\nu \times m}$, $f \in R^{\nu}$ are respectively continuous nonlinear matrix and vector with

$$\text{rank } (F) = \nu, \tag{2}$$

and two compact subsets $T^{-} \subset T^{+} \subseteq R^{2\nu}$,

design a continuous control law of type:

$$u = H(K_p y + K_d \dot{y}) + u_c, \tag{3}$$

where $K_p, K_d \in R^{\nu \times \nu}$ are constant matrices and $H \in R^{m \times \nu}$, $u_c \in R^{m}$, depending in general on $t, p, y, \dot{y}$, are such that (see Fig. 1)

$$\forall x_0 \in T^{+}, \quad x(t) \in T^{+} \text{ and } \lim_{t \to \infty} x(t) \in T^{-}, \tag{4}$$

where

$$x = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}. \tag{5}$$

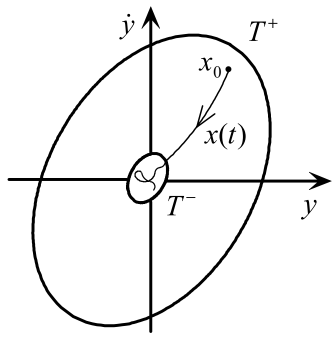

Fig. 1. Graphical representation of practical stability.

Problem 1 can be solved using the following theorem.

Theorem 1. Problem 1 can be solved by choosing the matrix H such that

$$\lambda_{\min}(FH + H^T F^T) < -1 \,, \tag{6}$$

K_p and K_d as follows:

$$K_p = a^2 I \,, \quad K_d = \sqrt{2} a I, \quad a > 0 \,, \tag{7}$$

where I denotes a $(\nu \times \nu)$ identity matrix, u_c and a are so that

$$\frac{w^T w}{\rho^2} = \left(\frac{\|w\|}{\rho}\right)^2 < \frac{a^3}{\sqrt{8}} \,, \begin{cases} \forall \rho \in (\rho^-, \rho^+] \\ \forall x \in C_{\rho+} - C_{\rho-} \\ \forall p \in \wp \end{cases} , \tag{8}$$

with

$$w = F u_c + f \,, \tag{9}$$

$$\begin{aligned} C_{\rho^-} &= \{x : d(x) \le \rho^-\} \subseteq T^- \\ C_{\rho^+} &= \{x : d(x) \le \rho^+\} \supseteq T^+ \end{aligned} \,, \tag{10}$$

and $d(x)$ is the distance of x from origin, defined by the positive definite quadratic form

$$\begin{aligned} d^2(x) &= d^2(y, \dot{y}) = \sqrt{2} a y^T y + \sqrt{2} / a \dot{y}^T \dot{y} + 2 y^T \dot{y} \\ &= \sqrt{2} y_n^T \; y_n + \sqrt{2} \dot{y}_n^T \dot{y}_n + 2 y_n^T \dot{y}_n \end{aligned} \tag{11}$$

where

$$y_n = \sqrt{a} y, \;\; \dot{y}_n = \dot{y} / \sqrt{a} \,. \tag{12}$$

Moreover, if

$$\lambda_{\min}(FH + H^T F^T) \le -2 \tag{13}$$

convergence velocity to a generic hyper-circle (with respect to the metric d) C_ρ of radius ρ centered at (0,0)

$$C_\rho = \{(y, \dot{y}) : d(y, \dot{y}) \le \rho\}, \qquad \rho^- < \rho \le \rho^+ \tag{14}$$

is no more than an exponential one characterized by a time constant (see Fig. 2)

$$\tau_\rho = \frac{\tau_\infty}{1 - \dfrac{\rho^-}{\rho}} \,, \qquad \tau_\infty = \frac{\sqrt{2}}{a} \,. \tag{15}$$

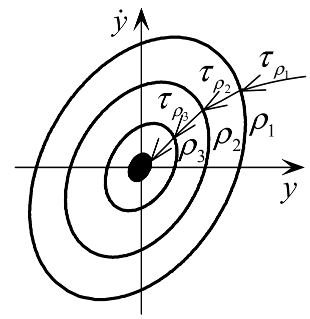

Fig. 2. Graphical representation of convergence velocity.

Proof. The proof of the theorem follows considering Lyapunov function $V = e^T P e$, with $P = \begin{pmatrix} \sqrt{2} a I & I \\ I & \dfrac{\sqrt{2}}{a} I \end{pmatrix}$.

The details about this demonstration, for brevity, are omitted and can be found in [16] and [17].

Remark 1. Considering that if $e^T P e \le \rho^2$ then $|e_i| < \sqrt{p_{ii}^{inv}}\, \rho$, where p_{ii}^{inv} is the (i,i) element of matrix P^{-1}, (see [15]), it follows that hyper-circle C_{ρ^-} can be approximated to the bigger and more practical hyper-rectangular one (see Fig. 3):

$$R_{\rho^-} = \left\{(y, \dot{y}) : |y_i| < \bar{y}_i, \; |\dot{y}_i| < \bar{\dot{y}}_i, \; i = 1,2,\ldots,\upsilon\right\}, \tag{16}$$

where

$$\begin{aligned} \bar{y}_i &= \frac{\sqrt[4]{2}}{\sqrt{a}} \rho^- \\ \bar{\dot{y}}_i &= \sqrt[4]{2} \sqrt{a} \rho^- = a \bar{y}_i \end{aligned} \,, \tag{17}$$

with

$$\begin{aligned} \rho^- &= \frac{\sqrt[4]{8}}{a\sqrt{a}} W \\ W &= \max \|w\| \Big|_{\substack{\forall (y, \dot{y}) \in \Gamma_{\rho^-} \\ \forall p \in \wp}} \end{aligned} \,, \tag{18}$$

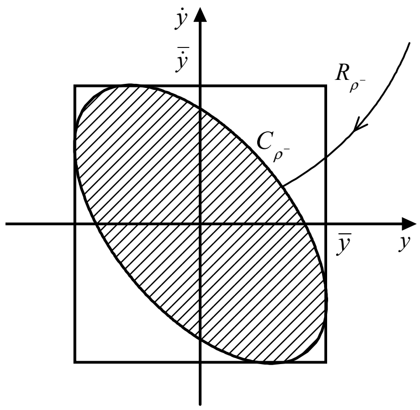

Fig. 3. C_{ρ^-} approximation with R_{ρ^-}.

where Γ_{ρ^-} denotes C_{ρ^-} boundary, that is the hyper-circumference of radius ρ^-

$$\Gamma_{\rho^-} = \{(y,\dot{y}): \quad d(y,\dot{y}) = \rho^-\}. \tag{19}$$

III. Main Results

It is well known that the dynamic model of a general mechanical system, articulated too, (a building, a robot,…) can be written in the form

$$B\ddot{q} = c + d + Tu \,, \tag{20}$$

where:

$q \in S_L \subseteq R^\nu$ is the vector of Lagrangian coordinates, being S_L the joint work space delimited by physical constraints depending on system and/or external environment,

$B(p,q)$ is the inertia matrix, in witch $p \in \wp \subset R^\mu$ is the vector of uncertain mechanical system parameters ,

$c = C(p,q,\dot{q})\dot{q}$ is the vector of the centrifugal, Coriolis and friction generalised forces,

$d = d(t,p,q)$ is the vector of gravitational generalised forces and various disturbances acting on the system,

u is the vector of control generalised forces provided by the actuators,

T is the transmission matrix of the control generalised forces.
In this work, for brevity, it is assumed that $T = I$.

Assume that $\hat{q}(t)$ is the desired trajectory for the mechanical system.
By posing that

$$\begin{array}{ll} e = \hat{q} - q, & q = \hat{q} - e \\ \dot{e} = \dot{\hat{q}} - \dot{q}, & \dot{q} = \dot{\hat{q}} - \dot{e} \end{array}, \tag{21}$$

where e is the position error and $\dot{e}$ is the velocity error, system (20) can be written in the form

$$\ddot{e} = Fu + f \,, \tag{22}$$

where:

$$\begin{array}{l} F(t,p,e) = -B^{-1} = -B^{-1}(p,q) = -B^{-1}(p,\hat{q}-e) \\ f(t,p,e,\dot{e}) = \ddot{\hat{q}} - B^{-1}(c+d) \end{array}. \tag{23}$$

It is easy to note that tracking error model (22) and (23) is a particular case of system (1); then, Problem 1 can be formulated for it and Theorem 1 can be conveniently applied in order to easily determine some good solutions.

In this way, it is important to note that, using the control law

$$u = H(K_p e + K_d \dot{e}) + u_c \,, \tag{24}$$

with

$$K_p = a^2 I\,, \quad K_d = \sqrt{2}aI, \quad a > 0 \,, \tag{25}$$

condition (13) of robust stability, which allows obtaining a τ_∞ given by

$$\tau_\infty = \frac{\sqrt{2}}{a} \,, \tag{26}$$

considering that inertia matrix B is symmetric, becomes

$$\lambda_{\min}[B^{-1}(H + H^T)] \ge 2 \,, \tag{27}$$

while, on the other hand, the vector w which allows computing radii ρ^+ *e* ρ^-, using (8), and/or only

$$\rho^- = \frac{\sqrt[4]{8}}{a\sqrt{a}} W, \quad W = \max\|w\| \Big|_{\substack{\forall (y,\dot{y}) \in \Gamma_{\rho^-} \\ \forall p \in \wp}} \,, \tag{28}$$

which making possible estimate maximum steady-state position error $\bar{e}_i$ and velocity error $\bar{\dot{e}}_i$ values by the relations

$$\bar{e}_i = \frac{2}{a^2} W \qquad , \quad i = 1,2,...,\upsilon \ , \qquad (29)$$
$$\bar{\dot{e}}_i = \frac{2}{a} W = a\bar{e}_i$$

turns out to be

$$\begin{aligned} w = w(t,p,e,\dot{e}) &= \ddot{\hat{q}} - B^{-1}(u_c + c + d) = \\ &= B^{-1}(p,\hat{q}-e)(u_c + C(p,\hat{q}-e,\dot{\hat{q}}-\dot{e})(\dot{\hat{q}}-\dot{e}) + \\ &+ d(t,p,\hat{q}-e)). \end{aligned} \qquad (30)$$

That said, it is possible to state some important results.

Theorem 2. Some possible choices of matrix H that guarantee stability robustness condition (13) and a τ_∞ given by (26) are as follows.

A. If it is desired to use a PD control law (24) (or state feedback if it is available besides the measure of e even that of $\dot{e}$) decoupled, linear and time-invariant then:

A.1 $H = hI$, (31)

with

$h \geq \lambda_{\max}(B)$; (32)

A.2 $H = diag(h_1, h_2, ..., h_\nu)$, (33)

where $h_1, h_2, ..., h_\nu$ are such that

$$\lambda_{\max}\left[diag\left(\frac{1}{h_1}, \frac{1}{h_2}, ..., \frac{1}{h_\nu}\right) B \right] \leq 1 \ . \qquad (34)$$

B. On the other hand, if it is desired a coupled PD controller then:

B.1 $H = B$, (35)

if B is perfectly known and its on-line calculation is not onerous, whereas

$$\begin{aligned} \text{B.2} \quad H = \tilde{B} diag(h_{10}, h_{20}, ..., h_{\nu 0}) + \\ + diag(h_1, h_2, ..., h_\nu) \ , \end{aligned} \qquad (36)$$

with $\tilde{B}$ (suitable approximation of B), h_{i0} and h_i, $i = 1,2,...,\upsilon$, satisfying (27),

if B is not perfectly known and/or its on-line calculation is onerous.

Proof. The proof of the theorem easily follows noting that the inertia matrix B is always positive definite and that $eig(A) = 1./eig(A^{-1})$, $\forall\, A \in R^{n \times n}$ nonsingular.

Remark 2. If $H = B$ it is simple to prove that, if w is independent of $(e, \dot{e})$, system (22) is linear, time-invariant and composed by ν unrelated subsystems that, considering e_i as output, have d.c. gain $1/a^2$, angular eigenfrequency a, time constant $\tau = \tau_\infty = \sqrt{2}/a$, damping ratio $\varsigma = \sqrt{2}/2$, overshoot $s = 0.43\%$ and, considering $\dot{e}_i$ as output, have resonance peak $1/\sqrt{2}a$.

Theorem 3. If friction, centrifugal, Coriolis, gravitational forces and disturbances can be neglected, or are perfectly compensated by an action

$u_c = -c - d$, (37)

then

$$\rho^+ = \infty \qquad (38)$$
$$\rho^- = \frac{\sqrt[4]{8}}{a\sqrt{a}} \max \|\ddot{\hat{q}}\| \ ,$$

and hence

$$|e_i| \leq \frac{2}{a^2} \max \|\ddot{\hat{q}}\| = \bar{e}_i \qquad , \quad i = 1,2,...,\upsilon \ . \qquad (39)$$
$$|\dot{e}_i| \leq \frac{2}{a} \max \|\ddot{\hat{q}}\| = a\bar{e}_i$$

Proof. The proof easily follows from (8), (28), (29), observing that B is always nonsingular and that, if $c + d = 0$ and $u_c = 0$ or u_c satisfies the relationship $c + d + u_c = 0$, using (30), it is

$w = \ddot{\hat{q}}, \quad \forall t, p, e, \dot{e}$. (40)

Remark 3. Hypotheses of Theorem 3 are satisfied without any compensation in the case of many systems of relevant interest to engineers, e.g. several mechanical systems composed by rigid bodies connected by elastic, with high stiffness and negligible mass elements (buildings, …), some robots (Cartesian with suitable counterweight used for balancing the gravitational forces, …).

Remark 4. The relations (24) and (39) allow making several important remarks; e.g., in order to reduce steady-state errors it is necessary to increase PD gains or to lower maximum acceleration of the tracked trajectory.

Theorem 4. If $\left\|\ddot{\hat{q}}\right\|$ and the disturbances are limited, $u_c = 0$, friction is bounded with respect to a neighborhood $\Im_\delta$ of $(\hat{q}, \dot{\hat{q}})$ of radius δ then

$$\lim_{a\to\infty} \rho^- = 0 \ , \qquad (41)$$

i.e. steady-state errors can be reduced as one likes.

Moreover, if friction forces are continuous in the neighborhood $\Im_\delta$ and a is high enough then

$$\rho^- \cong \frac{\sqrt[4]{8}}{a\sqrt{a}} \max\left\|w_n\right\| \ , \qquad (42)$$

where

$$w_n = w(t, p, 0, 0) = \ddot{\hat{q}} - B^{-1}(\hat{q})(C(p, \hat{q}, \dot{\hat{q}})\dot{\hat{q}} + \\ + d(t, \hat{q})) \ . \qquad (43)$$

Proof. It is important to note that, if $\left\|\ddot{\hat{q}}\right\|$ and disturbances are bounded and $u_c = 0$, since inertia matrix and gravitational forces are continuous with respect to e, centrifugal and Coriolis forces are continuous with respect to $(e, \dot{e})$ and friction forces are bounded with respect to a neighborhood $\Im_\delta$ of $(\hat{q}, \dot{\hat{q}})$, there certainty exists a W such that

$$\left\|w\right\| < W, \quad \forall (e, \dot{e}) \in \Im_\delta \ . \qquad (44)$$

Therefore, using (8), a possible value of ρ^- is

$$\rho^- = \frac{\sqrt[4]{8}}{a\sqrt{a}} W \qquad (45)$$

if

$$\Gamma_{\rho^-} = \left\{(e, \dot{e}) : d(e, \dot{e}) = \rho^-\right\} \subseteq \Im_\delta \ . \qquad (46)$$

Since, considering Remark 1 and relations (29), $\Gamma_{\rho^-} \subseteq R_{\rho-}$, where

$$R_{\rho^-} = \left\{ \begin{array}{l} (e, \dot{e}) : \left|e_i\right| \le \frac{2}{a^2} W, \left|\dot{e}_i\right| \le \frac{2}{a} W, \\ i = 1,2 \ldots, \nu \end{array} \right\}, \qquad (47)$$

it follows that (46) can be surely satisfied if a is chosen big enough. From this comment and (45), equation (41) follows. Moreover, if friction forces are continuous with respect to $(e, \dot{e})$ in the neighborhood $\Im_\delta$ of $(\hat{q}, \dot{\hat{q}})$, it follows that if $a \to \infty$, then $w \to w_n$ and after (42).

Remark 5. It is possible to track trajectories with high rate and low errors even if a is not very high using a suitable compensation u_c in order to reduce w.

Remark 6. With regard to the calculation of ρ^+, for brevity, it is important to note that it increases with a, i.e. with PD gains, and/or with a higher compensation.
Moreover, it is likewise important to note that, in practice, in order to reduce the control forces amplitude, not always tolerable by the system and requiring expensive and bulky actuators, it is advised to set the control system in motion by possibly limiting the tracking errors.

IV. CASE STUDY

The results presented in the previous sections are used for designing a controller for a three link planar robot which has the task of periodically executing an outward trajectory, given by an arc of circumference, and an inward one, given by an arc of ellipse with trapezoidal velocity profiles (see Figs. 4 and 5).
This type of task is very common in industrial applications, e.g. it is used to realize cages for concrete piles (see Fig. 6), to do some work on given points of objects moving on a belt having a circular part, etc.

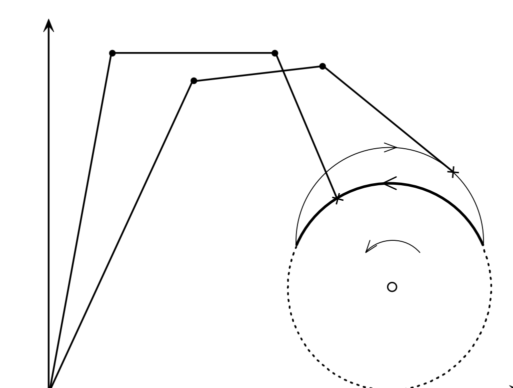

Fig. 4. Outward and inward trajectory.

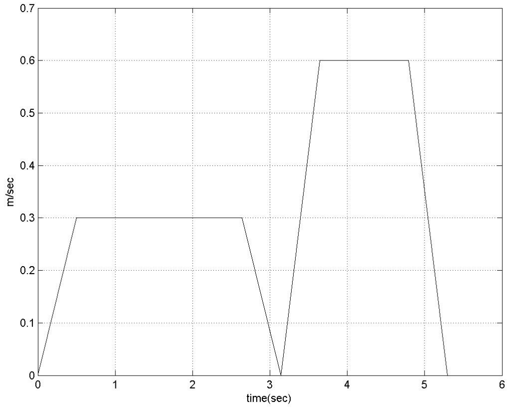

Fig. 5. Velocity profile.

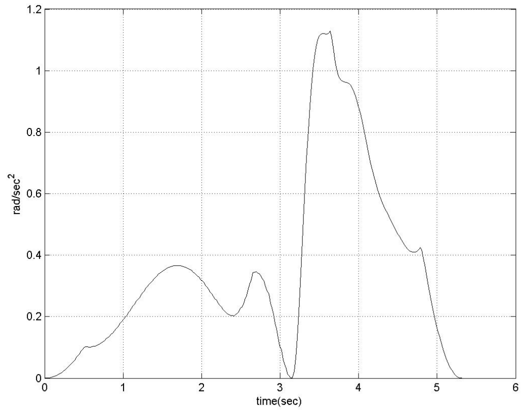

Fig. 7. Time history of $\|w_n\|$.

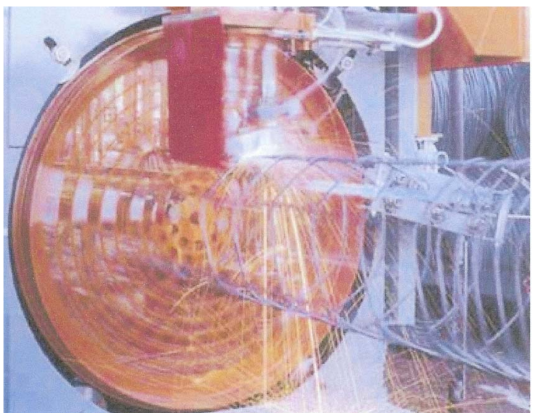

Fig. 6. Welding phase of a coil with a bar.

Consider a robot with

$$\begin{array}{l} L_1 = 1m,\, m_1 = 1Kg/m,\, M_1 = 1Kg \\ L_2 = L_3 = 0.5m,\, m_2 = m_3 = 0.5Kg/m \\ M_2 = M_3 = 0.5Kg \end{array} , \qquad (48)$$

and a working object moving on a circumference of radius

$$R = 0.3m , \qquad (49)$$

in the hypothesis of only gravity compensation or a horizontal work plane of the robot, the time history of $\|w_n\|$ is reported in Fig. 7. Using a controller constituted by three decoupled PD actions with $h = 5.7$ and $a = 10$ it is

$$\tau_\infty = 0.1414 \sec, \quad \bar{e}_i = 0.02W, \quad \bar{\dot{e}}_i = 0.2W , \qquad (50)$$

where W is the maximum value of $\|w_n\|$.

In Fig. 8 the position and velocity errors for the first link are reported, assuming zero initial conditions. In Fig. 9 the same quantities are reported, assuming that the initial position error is $0.1rad$.

These simulation results and others, obtained with disturbances such that maximum of $\|w_n\|$ during the outward and inward trajectories is the same, confirm the estimated results about maximum errors and convergence velocity.

In Fig. 10 control torque applied to the first link with null initial conditions is reported. Note that, because of lack of chattering and because of ς value for the ideal model, the maximum control torque is not too high.

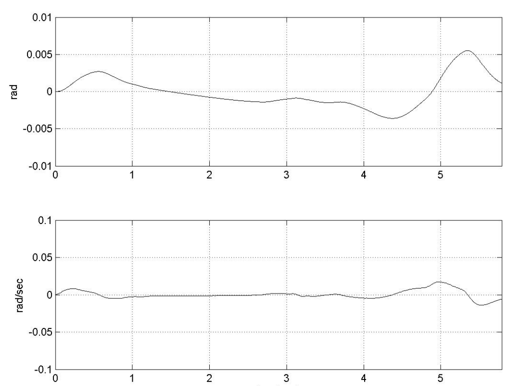

Fig. 8. Position and velocity errors for the first link.

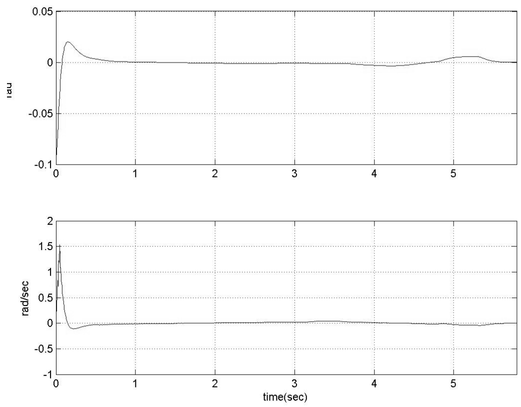

Fig. 9. Position and velocity errors for the first link with a not null initial condition.

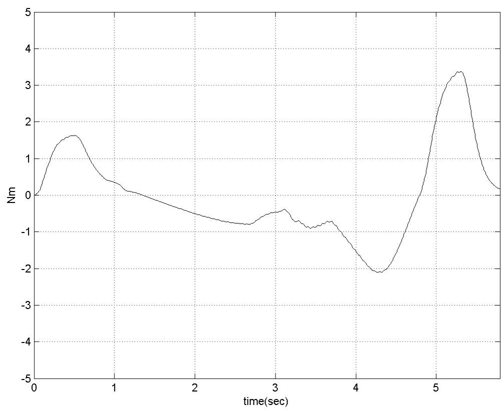

Fig. 10. Control torque applied to the first link with null initial conditions.

V. Conclusions

In this paper some theorems have been stated and proved which allow designing control laws, like PD with a possible compensation, easily satisfying robust stability specifications with regard to uncertainties (acting on H), to convergence velocity of the tracking error (acting on a), to position and velocity precision (acting on a and on compensation eventually).

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