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Distributed Controller Design Aspects for Guaranteed Performance*

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Abstract— **In this paper we consider some design aspects of distributed controllers that guarantee a ($\mathcal{H}_\infty$ or $\mathcal{H}_2$) performance level. In particular, we consider two design problems. First is the case where, without loss of generality, there are two distributed subcontollers connected to a (generalized) plant and the interest is placed in minimizing the number of noise-free (and dynamics free) communication channels between the subcontrollers needed to provide a given performance. The second is the case where, given specific signals to be communicated among the subcontrollers, noise is present and we seek to guarantee a performance level while keeping the communication signal to noise power limited. We take an LMI approach to provide solution procedures to these problems and present examples that demonstrate their efficiency.**

Keywords: LMI, distributed, communication, convex-relaxation

I. INTRODUCTION

In large, complex and distributed systems it is often desirable to implement the controller in a distributive manner. Such a task is becoming increasingly relevant because of the use of massively parallel systems that employ large numbers of sensors and actuators. In such applications, the controller can not be realized at a single station due the associated large computational load. This issue raises the important question on how to divide the control task into the various stations and characterize the effect communication constraints and noise in the overall system performance.

Several results on distributed and structured control synthesis using recently developed techniques are by now available (e.g., [1], [3], [10].) In these works, the communication between the various control sites is assumed to be without noise or other forms of uncertainty. In the context of networked control, there are also several works that deal with the effects of channel communication limitations from a single control site to the (physical) plant and vice versa (e.g., communication delay [8], packet loss [6], power constraint [2], communication noise [2] and limited bandwidth of communication channel [5], [9], [12], [16], [17].) Research that considers communication effects between subcontroller to subcontroller in a distributed architecture has started to emerge in [13], [14], [15] where the Youla-Kucera parametrization is used to lead to convex formulations of the underlying problems. It is along this third research direction that the current paper contributes to the synthesis of distributed controllers by taking an LMI approach.

Specifically, in this paper we consider some design aspects of distributed controllers that guarantee a ($\mathcal{H}_\infty$ or $\mathcal{H}_2$) performance level. In particular, we consider two design problems. First is the case where, without loss of generality, there are two distributed subcontrollers connected to a (generalized) plant and the interest is placed in minimizing the number of noise-free (and dynamics free) communication channels between the subcontrollers needed to provide a given performance. The second is the case where, given specific signals to be communicated among the subcontrollers, noise is present and we seek to guarantee a performance level while keeping the communication signal to noise power limited. We take an LMI approach to provide solution procedures to these problems and present examples that demonstrate their efficiency.

The rest of the paper is organized as follows. In Section II, we introduce the basic setup of our design problem. In Section III, we formulate the limited link problem and propose an LMI-based iteration method to find the distributed controller with the least communication links that achieves a pre-specified performance. In Section IV, we consider $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design that accounts for communication noise, and we propose iteration-based mehtods. In Section V, we conclude.

II. BASIC SETUP

This paper deals with distributed controller design for linear time-invariant (LTI) systems taking into account communication effects such as limited links and communication noise. The LTI continuous time system G is Figure 1 is described as

$$\begin{aligned} \dot{x} &= Ax + B_w w + Bu \\ z &= C_z x + D_{zw} w + D_{zu} u \\ y &= Cx + D_{yw} w \end{aligned} \quad (1)$$

There are l measured (possibly vector) outputs $y = [y_1^{'}, y_2^{'}, \ldots, y_l^{'}]^{'}$ and l (possibly vector) control inputs $u = [u_1^{'}, u_2^{'}, \ldots, u_l^{'}]^{'}$; z and w represent respectively the outputs to be regulated and the exogenous disturbances. The standard conditions are that (A, B) is stabilizable and (A, C) is detectable.

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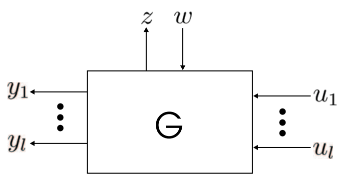

Fig. 1. The n control input-output plant

We want to design a controller $u = Ky$ in a distributed fashion such that the closed-loop map of interest $\Phi = \Phi(G, K) := w \to z$ satisfies certain design objectives. In particular, the controller K needs to be distributed to l subcontrollers K_j associated with the pairs (u_j, y_j) in a fully connected architecture. The structure of subcontroller K_i is shown in Figure 2, where $\{v_{ij}\}_{j \neq i}$ are the signals from subcontroller K_j to subcontroller K_i. For simplicity,

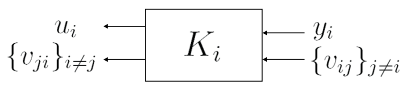

Fig. 2. The local controller K_i

we only consider the case of $l = 2$; the more general case can be treated similarly. The controller K is distributed to 2 subcontrollers K_1 and K_2 associated with the corresponding communication signals v_1, v_2 between two subcontrollers, as shown in Figure 3. The system parameters

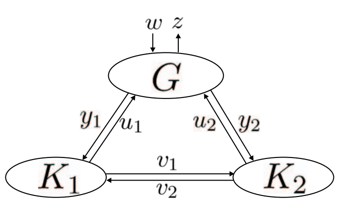

Fig. 3. The Communication Structure

are partitioned correspondingly as: $B = [B_1, B_2]$, $D_{zu} = [D_{zu1}, D_{zu2}]$, $C = [C_1', C_2']'$, $D_{yw} = [D_{yw1}', D_{yw2}']'$. The dimensions of the signals x, w, u_1, u_2, z, y_1, y_2 are n, m_2, q_1, q_2, m_1, p_1, p_2 respectively.

III. The Minimal Link Problem

Our design problem is stated as follows. Given a desired system $\mathcal{H}_\infty$ performance level γ, i.e., $\|w \to z\| < \gamma$, find conditions for the existence of a K with a distributed architecture $\{K_1, K_2\}$ and the least number of communication links, i.e. minimize the dimension of $(v_1', v_2')'$.

Suppose the two subcontrollers have orders n_{K_1}, n_{K_2} respectively and state-space forms as follows.

$$K_i: \quad \begin{aligned} \dot{x}_{K_i} &= A_{K_i} x_{K_i} + B_{K_{ii}} y_i + B_{K_{ij}} v_j \\ u_{K_i} &= C_{K_{i1}} x_{K_i} + D_{K_{i1i}} y_i + D_{K_{i1j}} v_j \\ v_i &= C_{K_{i2}} x_{K_i} + D_{K_{i2i}} y_j \end{aligned} \tag{2}$$

where $i, j \in \{1, 2\}$ and $i \neq j$.

Define

$$\Theta_{11} := \begin{bmatrix} A_{K_1} & B_{K_{11}} \\ C_{K_{11}} & D_{K_{111}} \end{bmatrix}, \Theta_{12} := \begin{bmatrix} B_{K_{12}} \\ D_{K_{112}} \end{bmatrix} [C_{K_{22}} \ D_{K_{222}}]$$

$$\Theta_{22} := \begin{bmatrix} A_{K_2} & B_{K_{22}} \\ C_{K_{21}} & D_{K_{212}} \end{bmatrix}, \Theta_{21} := \begin{bmatrix} B_{K_{21}} \\ D_{K_{211}} \end{bmatrix} [C_{K_{12}} \ D_{K_{121}}]$$

and

$$\Theta := \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

Then the overall controller can be formulated as:

$$\begin{aligned} \dot{x}_K &= L_1 \Theta L_2 x_k + L_1 \Theta L_3 y \\ u &= L_4 \Theta L_2 x_k + L_4 \Theta L_3 y \end{aligned} \tag{3}$$

where

$$L_1 = \begin{bmatrix} I_{n_{K_1}} & 0 & 0 & 0 \\ 0 & 0 & I_{n_{K_2}} & 0 \end{bmatrix} \quad L_3 = \begin{bmatrix} 0 & I_{p_1} & 0 & 0 \\ 0 & 0 & 0 & I_{p_2} \end{bmatrix}'$$

$$L_2 = \begin{bmatrix} I_{n_{K_1}} & 0 & 0 & 0 \\ 0 & 0 & I_{n_{K_2}} & 0 \end{bmatrix}' \quad L_4 = \begin{bmatrix} 0 & I_{q_1} & 0 & 0 \\ 0 & 0 & 0 & I_{q_2} \end{bmatrix}$$

Hence, the closed-loop system is:

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x}_K \end{bmatrix} &= \begin{bmatrix} A + BL_4 \Theta L_3 C & BL_4 \Theta L_2 \\ L_1 \Theta L_3 C & L_1 \Theta L_2 \end{bmatrix} \begin{bmatrix} x \\ x_K \end{bmatrix} \\ &+ \begin{bmatrix} B_w + BL_4 \Theta L_3 D_{yw} \\ L_1 \Theta L_3 D_{yw} \end{bmatrix} w \\ z &= [C_z + D_{zu} L_4 \Theta L_3 C, \quad D_{zu} L_4 \Theta L_2] \begin{bmatrix} x \\ x_K \end{bmatrix} \\ &+ [D_{zw} + D_{zu} L_4 \Theta L_3 D_{yw}] w \end{aligned} \tag{4}$$

Define

$$A_0 = \begin{bmatrix} A & \\ & 0 \end{bmatrix}, \mathscr{B} = \begin{bmatrix} BL_4 \\ L_1 \end{bmatrix}, B_0 = \begin{bmatrix} B_w \\ 0 \end{bmatrix}$$

$$\mathscr{C} = [L_3 C, \ L_2], \mathscr{D}_{12} = D_{zu} L_4, \mathscr{D}_{21} = L_3 D_{yw}$$

$$C_0 = [C_w, \ 0]$$

In [11] it is shown that the $\mathcal{H}_\infty$ performance is less than γ if and only if there exists a matrix X_{cl} and Θ such that

$$\begin{bmatrix} (A_0 + \mathscr{B}\Theta\mathscr{C})' X_{cl} + * & * & * \\ (B_0 + \mathscr{B}\Theta\mathscr{D}_{21})' X_{cl} & -\gamma I & * \\ (C_0 + \mathscr{D}_{12}\Theta\mathscr{C})' & (D_{11} + \mathscr{D}_{12}\Theta\mathscr{D}_{21}) & -\gamma I \end{bmatrix} < 0 \tag{5}$$

Let

$$\Psi_{X_{cl}} = \begin{bmatrix} A_0' X_{cl} + X_{cl} A_0 & X_{cl} B_0 & C_0' \\ B_0' X_{cl} & -\gamma I & D_{11}' \\ C_0 & D_{11} & -\gamma I \end{bmatrix}$$

$$\mathscr{P} = [\mathscr{B}', 0, \mathscr{D}_{12}'], \mathscr{P}_{X_{cl}} = [\mathscr{B}' X_{cl}, 0, \mathscr{D}_{12}'], \mathscr{Q} = [\mathscr{C}, \mathscr{D}_{21}, 0]$$

then Equation (5) becomes

$$\Psi_{X_{cl}} + \mathscr{Q}' \Theta' \mathscr{P}_{X_{cl}} + \mathscr{P}_{X_{cl}}' \Theta \mathscr{Q} < 0 \tag{6}$$

The relevant problem we would like to solve is to find the matrices X_{cl} and Θ to satisfy the inequality (6) for given γ and minimize

$$\text{rank}(\Theta_{12}) + \text{rank}(\Theta_{21}).$$

which is equivalent to minimize the dimension of the communication signals v_1 and v_2.

In general, this is a non-convex problem. We first introduce the following lemma to convert it to a more convenient form for design.

Lemma 3.1: [7] Let $X \in \Re^{m\times n}$ be a given matrix. Then $\text{rank}(X) \leq r$ if and only if there exist matrices $Y = Y^{'} \in \Re^{m\times m}$ and $Z = Z^{'} \in \Re^{n\times n}$ such that

$$\text{rank}(Y) + \text{rank}(Z) \leq 2r, \quad \begin{bmatrix} Y & X \\ X^{'} & Z \end{bmatrix} \geq 0$$

Through Lemma 3.1, the non-convex objective $\min \text{rank}(\Theta_{12}) + \text{rank}(\Theta_{21})$ is equivalent to

$$\begin{aligned} &\min \text{rank}(Y_1) + \text{rank}(Z_1) + \text{rank}(Y_2) + \text{rank}(Z_2) \\ &\begin{bmatrix} Y_1 & \Theta_{12} \\ \Theta_{12}^{'} & Z_1 \end{bmatrix} \geq 0, \begin{bmatrix} Y_2 & \Theta_{21} \\ \Theta_{21}^{'} & Z_2 \end{bmatrix} \geq 0 \end{aligned} \tag{7}$$

Due to the hardness of rank minimization, we could try to solve an alternate problem, i.e.

$$\begin{aligned} &\min \text{Tr}(Y_1) + \text{Tr}(Z_1) + \text{Tr}(Y_2) + \text{Tr}(Z_2) \\ &\begin{bmatrix} Y_1 & \Theta_{12} \\ \Theta_{12}^{'} & Z_1 \end{bmatrix} \geq 0, \begin{bmatrix} Y_2 & \Theta_{21} \\ \Theta_{21}^{'} & Z_2 \end{bmatrix} \geq 0 \\ &\Psi_{X_{cl}} + \mathscr{Q}^{'}\Theta^{'}\mathscr{P}_{X_{cl}} + \mathscr{P}_{X_{cl}}^{'}\Theta\mathscr{Q} < 0 \end{aligned} \tag{8}$$

Note that problem (8) is still non-convex since the second inequality constraint is a bilinear matrix inequality (BMI) in X_{cl} and Θ. We now propose the following algorithm to solve this constrained optimization problem.

i. Solve the unconstrained problem, i.e. no rank constraints. From the Projection Lemma [4], there exists a matrix Θ satisfying Equation (6) if and only if the inequalities

$$\begin{aligned} \mathscr{Q}^{\perp'}\Psi_{X_{cl}}\mathscr{Q}^{\perp} < 0 \\ \mathscr{P}_{X_{cl}}^{\perp'}\Psi_{X_{cl}}\mathscr{P}_{X_{cl}}^{\perp} < 0 \end{aligned} \tag{9}$$

both hold. Hence, we can get arbitrarily close to the optimal performance γ^* and the corresponding X_{cl}.

ii. For the given performance $\gamma(\geq \gamma^*)$, use the matrix X_{cl} in step i or iii and γ to solve the following LMI to get Θ:

$$\begin{aligned} \min_{\Theta} \quad &\text{Tr}(Y_1) + \text{Tr}(Z_1) + \text{Tr}(Y_2) + \text{Tr}(Z_2) \\ &\Psi_{X_{cl}} + \mathscr{Q}^{'}\Theta^{'}\mathscr{P}_{X_{cl}} + \mathscr{P}_{X_{cl}}^{'}\Theta\mathscr{Q} < 0 \\ &\begin{bmatrix} Y_1 & \Theta_{12} \\ \Theta_{12}^{'} & Z_1 \end{bmatrix} \geq 0, \begin{bmatrix} Y_2 & \Theta_{21} \\ \Theta_{21}^{'} & Z_2 \end{bmatrix} \geq 0, \end{aligned} \tag{10}$$

iii. Fix γ and Θ in step ii, solve the following LMI to get X_{cl}:

$$\Psi_{X_{cl}} + \mathscr{Q}^{'}\Theta^{'}\mathscr{P}_{X_{cl}} + \mathscr{P}_{X_{cl}}^{'}\Theta\mathscr{Q} < 0 \tag{11}$$

iv. Repeat step ii and step iii to get a convergent result.

Let r_1=rank(Θ_{21}) and r_2=rank(Θ_{12}). Conduct the SVDs of Θ_{12} and Θ_{21}, i.e. $\Theta_{21} = U_1\Lambda_1V_1^{'}, \Theta_{12} = U_2\Lambda_2V_2^{'}$.

Define

$$\begin{aligned} U_{ii} = U_i\begin{bmatrix} I_{r_i} \\ 0 \end{bmatrix}, V_{ii} = V_i\begin{bmatrix} I_{r_i} \\ 0 \end{bmatrix} \\ \Lambda_{r_i} = \begin{bmatrix} I_{r_i} \\ 0 \end{bmatrix}^{'}\Lambda_i\begin{bmatrix} I_{r_i} \\ 0 \end{bmatrix}. \end{aligned} \tag{12}$$

Then one possible choice of signals and parameters is:

$$\begin{aligned} v_1 = \Lambda_{r_1}^{\frac{1}{2}}V_{11}^{'}\begin{bmatrix} x_{K_1} \\ y_1 \end{bmatrix}, \ v_2 = \Lambda_{r_2}^{\frac{1}{2}}V_{22}^{'}\begin{bmatrix} x_{K_2} \\ y_2 \end{bmatrix} \\ B_{K_{12}} = [I,0]U_{22}\Lambda_{r_2}^{\frac{1}{2}}, \ D_{K_{112}} = [0,I]U_{22}\Lambda_{r_2}^{\frac{1}{2}} \\ B_{K_{21}} = [I,0]U_{11}\Lambda_{r_1}^{\frac{1}{2}}, \ D_{K_{211}} = [0,I]U_{11}\Lambda_{r_1}^{\frac{1}{2}} \end{aligned} \tag{13}$$

Other parameter matrices can be determined from the matrix Θ. Other choices for v_1, v_2 are possible, but their dimensions are kept constant, i.e., r_1, r_2 respectively.

A. Remarks

Any centralized controller of state dimension $n_{K_1} + n_{K_2}$ can be split to K_1 and K_2 as in Equation (2). If $n_{K_1} + n_{K_2} \geq n$, then there are always feasible solutions for any $\gamma \geq \gamma^*$, where γ^* is the optimal centralized performance of the map $\|w \to z\|$.

We have the freedom to choose n_{K_1}, n_{K_2}, and the choice may affect the final solution. For the decentrally stabilizable systems, the off-diagonal sub-matrices of Θ will converge to 0 for sufficiently large γ and a suitable initial value of F.

We only considered the case of $l = 2$. If $l > 2$, then the controller parameter matrix becomes $\Theta = \begin{bmatrix} \Theta_{11} & \cdots & \Theta_{1l} \\ \vdots & \cdots & \vdots \\ \Theta_{l1} & \cdots & \Theta_{ll} \end{bmatrix}$ and we need minimize the rank of each off-diagonal matrix Θ_{ij}, where $i \neq j$. The same relaxation procedure will work.

B. Example

Consider a randomly generated LTI system described by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1.9598 & -1.7492 & -0.0791 & -0.1176 \\ 2.2433 & -0.7607 & -0.1640 & 0.5221 \\ 1.1384 & 0.7075 & 1.2717 & 1.5545 \\ -0.8888 & -0.0649 & 1.0258 & -0.2202 \end{bmatrix} x \\ &+ \begin{bmatrix} -2.8639 & -0.0833 \\ 1.0665 & -0.3118 \\ -3.0461 & -0.6737 \\ 0.1857 & 0.6033 \end{bmatrix} w + \left[\begin{array}{c|c} 1.2769 & 0.9920 \\ 0.5591 & 0.5773 \\ 0.4344 & -2.1403 \\ -0.1493 & 0.3521 \end{array}\right] u \\ z &= \begin{bmatrix} 0.4858 & -0.6656 & 0.3187 & 0.5090 \\ -0.4877 & -0.1422 & -1.1188 & -0.8549 \\ -0.0609 & -0.1512 & 0.4833 & 0.0357 \end{bmatrix} x \\ &+ \begin{bmatrix} -0.3563 & 0.7599 \\ 0.8490 & -1.4784 \\ 0.1354 & 0.2198 \end{bmatrix} w + \left[\begin{array}{c|c} -1.2054 & 0.6365 \\ 0.2081 & -0.8083 \\ -0.6790 & 0.0812 \end{array}\right] u \\ y &= \left[\begin{array}{cccc} 0.1469 & 1.6788 & -0.6627 & -2.2385 \\ \hline -0.2290 & 1.2381 & -1.9932 & -0.8608 \end{array}\right] x \\ &+ \left[\begin{array}{cc} -2.1272 & 0.6508 \\ \hline -0.2457 & 1.4827 \end{array}\right] w \end{aligned} \tag{14}$$

The centralized $\mathcal{H}_\infty$ performance is $\gamma^* \simeq 8.0196$. Assume both the two sub-controllers have order 2. Running the proposed algorithm, we get the results shown in Figure 4. In this figure we plot the number of links v.s. the system performance γ.

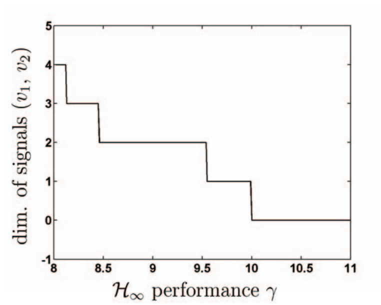

Fig. 4. System performance vs signal dimension

Remarks: If $\gamma = \gamma^*$, the number of communication links is 4 instead of 6 in the centralized case, which means that the optimal performance γ^* is still achieved with less communication links.

As γ increases, the number of communication links decreases, which means that, as expected, the less links we use, the worse performance the system achieves.

As $\gamma \geq 10$, the number of communication links comes to 0, which means the system is stabilized in a decentralized way.

IV. Communication Noise

In this section we turn our attention to the situation where there is communication noise in the transmission of information between the nodes in the distributed setting. For simplicity, we only consider the state-feedback problem, shown in Figure 5. In particular, assume the communication

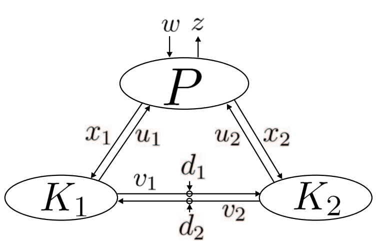

Fig. 5. State feedback stabilization

signals are: $v_1 = F_{21}x_1$, $v_2 = F_{12}x_2$, which is a special case of the setup in Section III. Then the actual control signals are $u_1 = F_{11}x_1 + F_{12}x_2 + d_2$, $u_2 = F_{21}x_1 + F_{22}x_2 + d_1$, and the closed loop system is

$$\dot{x} = (A + BF)x + (B_w,\, B)\begin{bmatrix} w \\ d \end{bmatrix}$$

$$z = (C_z + D_{zu}F)x + (D_{zw},\, D_{zu})\begin{bmatrix} w \\ d \end{bmatrix}$$

$$v = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} & F_{12} \\ F_{21} & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $d = \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}$, $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$. Therefore,

$$A_c = A + BF,\; B_c = (B_w,\, B)$$
$$C_c = \begin{bmatrix} C_z + D_{zu}F \\ \begin{bmatrix} & F_{12} \\ F_{21} & \end{bmatrix} \end{bmatrix}, D_c = \begin{bmatrix} D_{zw} & D_{zu} \\ 0 & 0 \end{bmatrix}. \quad (15)$$

We are interested in the interplay between closed loop performance and communication power needed to achieve it. If there is no communication noise, any centralized performance level γ can be achieved with arbitrarily small communication power. One such scheme is shown in Figure 6: Given any K_1 and K_2 that achieve a performance level γ we can scale down by ϵ the (original) corresponding transmission signals v_1 and v_2 before transmitted; at the receiving end we scale back the received signal by $\frac{1}{\epsilon}$ to recover the original signal. Clearly that does not change the w to z system. As ϵ can be arbitrarily small, the power of the (actually) transmitted signals $\tilde{v}_1$, $\tilde{v}_2$ becomes arbitrarily small.

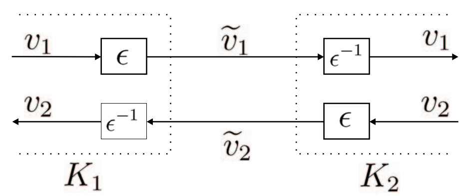

Fig. 6. Communication scheme with small communication power

If there is no constraint on the power of v_1 and v_2, arbitrarily close to any centralized performance level can be achieved (see Figure 7): Contrary to the above case, we

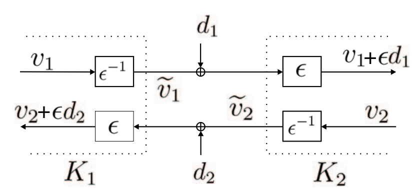

Fig. 7. Communication with high power

scale up by a parameter ϵ^{-1} the (original) corresponding transmission signals v_1 and v_2 before transmitted; at the receiving end we scale down the received signal by ϵ. Equivalently the effect of the noise d_1 and d_2 on z is scaled down by ϵ. As ϵ can be arbitrarily small, the (actual) closed loop performance $\|[w', d']' \to v\|$ is arbitrarily close to γ.

In a more realistic situation, a design objective should account for the trade offs between closed loop performance and communication power. A natural problem to investigate therefore can be posed as

$$\inf_F \; \| \begin{bmatrix} w \\ d \end{bmatrix} \to z \|$$
$$\text{s.t. } \| \begin{bmatrix} w \\ d \end{bmatrix} \to v \| \leq \gamma_p \quad (16)$$

where γ_p is a power level constraint. Rather than using this formulation we adopt for convenience a simpler design objective. Let $T = \begin{bmatrix} T_{zw} & T_{zd} \\ T_{vw} & T_{vd} \end{bmatrix} := \begin{bmatrix} w \\ d \end{bmatrix} \rightarrow \begin{bmatrix} z \\ v \end{bmatrix}$. Our basic design problem is defined as

$$\inf_F \|T\| \tag{17}$$

We note that by looking at a weighted version

$$\inf_F \; \| \begin{bmatrix} I & \\ & \alpha I \end{bmatrix} T \| \tag{18}$$

and adjusting the parameter α upper bounds to problem (16) can be obtained and the essential trade-offs can be revealed.

A. $\mathcal{H}_\infty$ control

For the $\mathcal{H}_\infty$ control case, A_c is stable and $\|T\|_\infty$ is smaller than γ iff there exists a symmetric P such that [11]

$$\begin{bmatrix} A_c^{'}P + PA_c & PB_c & C_c^{'} \\ B_c^{'}P & -\gamma I & D_c^{'} \\ C_c & D_c & -\gamma \end{bmatrix} < 0, \quad P > 0. \tag{19}$$

In general, this is a non-convex problem and there is not any standard method to solve it. If we choose P to be a block diagonal matrix, then it can be converted to a convex problem. Instead, we propose an iteration-based method to get a sub-optimal result.

i. Given A and B, find F s.t. $A + BF$ is stable. One possible choice is to take an (arbitrarily close to) optimal centralized solution F^*, which solves the problem $\inf_F \|w \rightarrow z\|$.

ii. Using F obtained in step i or iii, we can solve the following optimization problem:

$$\min_{P>0} \; \gamma$$
$$\begin{bmatrix} A_c^{'}P + PA_c & PB_c & C_c^{'} \\ B_c^{'}P & -\gamma I & D_c^{'} \\ C_c & D_c & -\gamma \end{bmatrix} < 0, \quad P > 0. \tag{20}$$

iii. Fix P, try to find the minimal γ and the corresponding matrix F such that Equation (19) holds.

iv. Repeat step ii and step iii to get a convergent result.

1) Example: Consider a randomly generated system

$$\dot{x} = \left[\begin{array}{cc|cc} -1.2705 & -0.5412 & -0.0113 & -0.2640 \\ -1.6636 & -1.3335 & -0.0008 & -1.6640 \\ \hline -0.7036 & 1.0727 & -0.2494 & -1.0290 \\ 0.2809 & -0.7121 & 0.3966 & 0.2431 \end{array}\right] x$$
$$+ \left[\begin{array}{cc} -1.2566 & -1.0211 \\ -0.3472 & -0.4017 \\ \hline -0.9414 & 0.1737 \\ -1.1746 & -0.1161 \end{array}\right] w + \left[\begin{array}{c|c} 1.0641 & 0.0714 \\ -0.2454 & 0.3165 \\ \hline -1.5175 & 0.4998 \\ 0.0097 & 1.2781 \end{array}\right] u$$
$$z = \left[\begin{array}{cc|cc} -0.5478 & -0.5803 & -1.4095 & -1.1190 \\ 0.2608 & 2.1363 & 1.7701 & 0.6204 \\ -0.0132 & -0.2576 & 0.3255 & 1.2698 \end{array}\right] x$$
$$+ \begin{bmatrix} 0.5362 & 0.3144 \\ -0.7164 & 0.1068 \\ -0.6556 & 1.8482 \end{bmatrix} w + \left[\begin{array}{c|c} -0.8960 & -1.1634 \\ 0.1352 & 1.1837 \\ -0.1390 & -0.0154 \end{array}\right] u$$

The centralized optimal performance γ^* of the map $w \rightarrow z$ is approximately 2.0154. Applying the above algorithm to this example, the optimization procedure terminates after 60 iterations, and the $\mathcal{H}_\infty$ performance converges to $\bar{\gamma}_1 \simeq 3.1284$, shown in Figure 8.

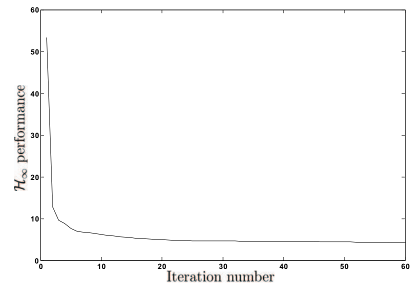

Fig. 8. $\mathcal{H}_\infty$ performance v.s. iteration

2) Asymptotic Recovery of Centralized Performance: In this subsection we are interested in the asymptotic performance of the previously presented design algorithm when the communication noise becomes arbitrarily small. In particular, if $d_i = \epsilon \bar{d}_i$ with $\|\bar{d}_i\| \leq 1$ we would like to know if the proposed design can recover the best possible centralized $w \rightarrow z$ performance γ^*, while the communication power is bounded. To this end we consider a weighted version of our problem shown in Figure 9.

$$\gamma_\epsilon := \min_F \| \begin{bmatrix} w \\ \bar{d} \end{bmatrix} \rightarrow \begin{bmatrix} z \\ \tilde{v} \end{bmatrix} \|_\infty \tag{21}$$

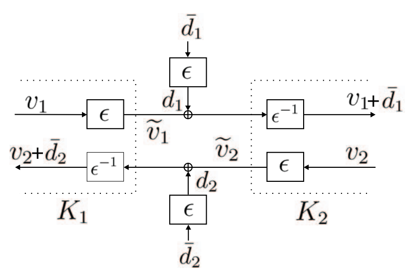

Fig. 9. Communication structure of asymptotic recovery

For any fixed ϵ our algorithm produces a design with performance $\bar{\gamma}_\epsilon$. In general $\bar{\gamma}_\epsilon \geq \gamma_\epsilon \geq \gamma^*$ since our algorithm may terminate at a local minimum. The communication power ratio of the design for a given ϵ is captured by the norm $\|(w, \bar{d}) \rightarrow \tilde{v}\|$.

Proposition 4.1: As $\epsilon \rightarrow 0$, $\bar{\gamma}_\epsilon \rightarrow \gamma^*$. Moreover, $\|(w, \bar{d}) \rightarrow \tilde{v}\| \rightarrow 0$.

Proof: Let $\bar{\gamma}_\epsilon(F)$ denote the system performance with given F matrix. If with a given $\epsilon > 0$ we choose the optimal centralized solution F^*, which solves the problem $\inf_F \|w \rightarrow z\|$, as an initial value of iteration, then $\bar{\gamma}_\epsilon(F^*) \geq \bar{\gamma}_\epsilon$. Since $\bar{\gamma}_\epsilon(F^*) \rightarrow \gamma^*$ as $\epsilon \rightarrow 0$ and $\bar{\gamma}_\epsilon \geq \gamma_\epsilon \geq \gamma^*$, it follows that $\bar{\gamma}_\epsilon \rightarrow \gamma^*$. Since the closed loop system is stable and $\|T_{vw}\|$, $\|T_{vd}\|$ are bounded uniformly in ϵ, it follows that

$$\|(w,\bar{d}) \rightarrow \widetilde{v}\| = \left\|(\epsilon T_{vw} \ \epsilon^2 T_{vd})\right\| \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \qquad \blacksquare$$

An implication of the above proposition is that, for small noise, the algorithm will lead to near optimal centralized performance with small communication power. Note that this is consistent with the earlier discussion in the section for the case of no noise.

We consider the same system as in Section IV-A.1 and run the above proposed algorithm. As ϵ decreases from 1 to 0.01, the system performance $\bar{\gamma}_\epsilon$ converges to the centralized optimal performance $\gamma^* \simeq 2.0154$, as shown in Figure 10. This is in accordance with above discussion: the smaller the communication noise is, the better performance the system achieves. It also means that, though the iteration-based method is not globally optimal, for small ϵ, the local sub-optimal solution converges to the optimal solution.

From Figure 10 we can also see that the proposed procedure converges very fast, especially for small ϵ.

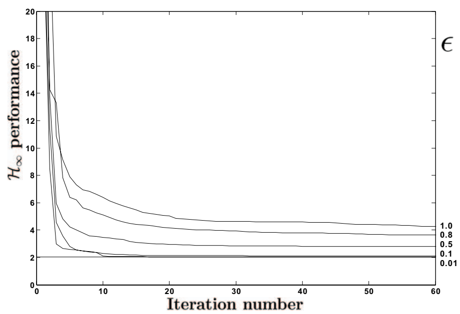

Fig. 10. $\mathcal{H}_\infty$ performance v.s. iteration for different ϵ

B. Remarks

In Section IV-A, we have considered the $\mathcal{H}_\infty$ minimization problem, similar algorithm works for the $\mathcal{H}_2$ case.

If the number of the subcontrollers l is greater than 2, the design problem has a similar structure. The only difference is that C_c has more complicated structure, i.e.

$$C_c = \begin{bmatrix} C_z \\ \begin{bmatrix} & F_{12} & & \\ & & \ddots & \\ & & & F_{1l} \\ \vdots & \cdots & & \vdots \\ F_{l1} & & & \\ & \ddots & & \\ & & F_{ll-1} & \end{bmatrix} \end{bmatrix} \qquad (22)$$

The proposed algorithms still work in both $\mathcal{H}_\infty$ and $\mathcal{H}_2$ cases.

To account more directly for the communication noise in the design, one can adopt an LMI approach (e.g., as in [11]) for the originally formulated problem in (16). A similar iteration design procedure can be obtained.

We also like to note that the case where the transmission signals are $x_i^{'}$s as opposed to $F_{ij}x_i^{'}$s is dual to what we presented and can be handled similarly.

V. Conclusions

In this paper, we took a design point of view to study the performance of closed-loop control systems with information exchange between different sub-controllers through networks. We provided LMI-based procedures that directly account to communication link constraints in order to optimize system performance. The two forms of constraints considered were the case of limited number of links and the case of communication noise. The numerical examples illustrated the efficacy of the methods.

References

[1] B. Bamieh, F. Paganini, M.A. Dahleh, "Distributed control of spatially-invariant systems," *IEEE Trans. Automat. Control,* Vol. 47, No. 7, pp. 1091-1107, 2002.

[2] J.H. Braslavsky, R.H. Middleton, and J.S. Freudenberg, "Feedback Stabilization over Signal-to-Noise Ratio Constrained Channels," *Proceedings of the 2004 American Control Conference,* Boston, Massachusetts, June 30, 2004.

[3] R. D'Andrea and G. Dullerud, "Distributed control of spatially interconnected systems,", *IEEE Trans. Automat. Control,* Vol. 48, No. 9, pp. 1478-1495, 2003.

[4] G.E. Dullerud and F. Paganini, *A Course in Robust Control Theory: A Convex Approach,* Springer-Verlag New York, 2000.

[5] N. Elia and S.K. Mitter, "Stabilization of linear systems with limited information," *IEEE Transactions on Automatic Control,* Vol. 46, No. 9, pp. 1384-1400, September 2001.

[6] N. Elia and J.N. Eisenbeis, "Limitations of linear remote control over packet drop networks," *Proceedings of the 43rd IEEE Conference on Decision and Control,* Bahamas, December 2004.

[7] M. Fazel, H. Hindi, and S.P. Boyd, "Log-det heuristic for matrix rank minimization with applications to Hankel and Euclidean distance matrices," *Proceedings of the 2003 American Control Conference,* Vol. 3, June 2003.

[8] F. Lian, J.R. Moyne and D.M. Tilbury, "Control performance study of a networked machining cell," *Proceedings of the 2000 American Control Conference,* Chicago, June 2000.

[9] L. Lu, L. Xie, and M. Fu, "Optimal Control of Networked Systems with Limited Communication: a Combined Heuristic Search and Convex Optimization Approach,"*Proceedings of the 42nd IEEE conference on Decision and Control*, Vol. 2, pp. 1194-1199. December 2003.

[10] X. Qi, M.V. Salapaka, P.G. Voulgaris, and M. Khammash, "Structured optimal and robust control with multiple criteria: a convex solution," *IEEE Transaction on Automatic Control,* Vol. 49, No. 10, October 2004.

[11] C. Scherer, P. Gahinet, and M. Chilali, "Multiobjective Output-Feedback Control via LMI Optimization," *IEEE Transaction on Automatic Control,* Vol. 42, No. 7, July 1997.

[12] S. Tatikonda, A. Sahai and S.K. Mitter, "Control of LQG systems under communication constraints," *Proceedings of the 1999 American Control Conference,* Vol. 4, pp. 2778-2782, 1999.

[13] P.G. Voulgaris and S. Jiang, "Failure-Robust Distributed Controller Architectures," *Proceedings of the 43rd IEEE Conference on Decision and Control,* Bahamas, December 2004.

[14] V. Yadav, P.G. Voulgaris, and M.V. Salapaka, "Stabilization of nested systems with uncertain subsystem communication channels,"*Proceedings of the 42nd IEEE Conference on Decision and Control,* Maui, Hawaii, December 2003..

[15] V. Yadav, P.G. Voulgaris, and M.V. Salapaka, "Distributed architectures and implementations of observer based controllers for performance optimization," *Proceedings of ACC,* June 2005.

[16] J.K. Yook, D.M. Tilbury, and N.R. Soparkar, " Trading communication for bandwidth: Reducing communication in distributed control systems using state estimators," *IEEE Transactions on Control Systems Technology,* Vol. 10, No. 4, July 2002.

[17] S. Yuksel and T. Basar, "Quantization and coding for decentralized LTI systems," *Proceedings of the 42nd IEEE Conference on Decision and Control,* Maui, Hawaii, December 2003.