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Flow-Invariant Sets with Respect to the Markings of Timed Continuous Petri Nets

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Abstract—**The paper investigates the existence of flow-invariant sets with respect to the marking of a timed continuous Petri net (TCPN) with infinite server semantics. Such a set has the property that for any initial marking belonging to the set, the marking at any moment in the evolution of the net also belongs to the set. Thus, the traditional concept of marking invariance used in PN theory, which refers to a set of places, is complemented in the sharper sense of the individual monitoring of each place. We take into consideration several types of bounded flow-invariant sets. The join-free TCPNs are treated separately from TCPNs with joins as allowing the development of supplementary investigation tools. Subsidiary to our results we give a consistent and rigorous mathematical proof for the nonnegativeness of the marking in TCPNs.**

I. Introduction

PETRI nets (PNs) were successfully used for the modeling, analysis, synthesis and implementation of discrete event systems. The state of a PN is represented by the marking vector that changes due to transitions firings. In the original *discrete* PNs, the marking of each place in the net is restricted to be natural number, e.g. [1]. The idea of relaxing this condition in order to obtain a *continuous* model capable to describe a large class of continuous systems was introduced in [2], where non-negative real numbers were allowed for markings. Later on, works such as [3]-[5] presented several ways for timing continuous PNs, and introduced different server semantics to allow describing the token-flow through transitions.

In the following we consider *timed continuous Petri nets* (TCPNs) as defined in [4], [6], [7], given by a *topology* $\mathcal{N}$ and a vector of *firing rates* associated with the transitions in the net, $\boldsymbol{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\lambda} = [\lambda_1 \dots \lambda_m]^T > \boldsymbol{0}$.

The topology $\mathcal{N} = \langle P, T, \boldsymbol{C}^+, \boldsymbol{C}^- \rangle$ of a TCPN is defined by the disjoint sets of places, $P = \{p_1, \dots, p_n\}$, and transitions, $T = \{t_1, \dots, t_m\}$, and by the input and output incidence matrices, $\boldsymbol{C}^+ = \left[C_{ij}^+\right]$ and $\boldsymbol{C}^- = \left[C_{ij}^-\right]$, $\boldsymbol{C}^+, \boldsymbol{C}^- \in \mathbb{N}^{n\times m}$, respectively. Each place in a *continuous PN* can accommodate any non-negative real number of tokens. Each transition has at least one input place. A transition t_j is *enabled* at the marking $\boldsymbol{m} \in \mathbb{R}_+^n$ if all its input places are marked, that is $m_i > 0$ for all $p_i \in {}^\bullet t_j$ (m_i denotes the marking of p_i). The *enabling degree* of transition t_j at a certain marking $\boldsymbol{m}$ is defined as

$$e_j(\boldsymbol{m}) = \min_{p_i \in {}^\bullet t_j} \left\{ m_i \big/ C_{ij}^- \right\}. \tag{1}$$

Under the *infinite server semantics* considered in our paper, the *token-flow* through transition t_j at an arbitrary moment $\tau \in \mathbb{R}_+$, when the marking of the net is $\boldsymbol{m}(\tau)$, equals the product of its firing rate and enabling degree,

$$f_j(\tau) = \lambda_j\, e_j(\boldsymbol{m}(\tau)), \quad j = \overline{1,m}. \tag{2}$$

The *state-equation* corresponding to a TCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ describes the time-dependence of its marking:

$$\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C}\,\boldsymbol{f}(\tau), \quad \tau \ge \tau_0 \ge 0, \quad \boldsymbol{m}(\tau_0) = \boldsymbol{m}_0 \ge \boldsymbol{0}, \tag{3}$$

where $\boldsymbol{f}(\tau) = [f_1(\tau)\ f_2(\tau) \dots f_m(\tau)]^T$ (T denoting the transposition). In (3), $\boldsymbol{C} = \left[C_{ij}\right] = \boldsymbol{C}^+ - \boldsymbol{C}^- \in \mathbb{Z}^{n\times m}$ is the *incidence matrix* that characterizes the net topology $\mathcal{N}$.

A TCPN with joins (i.e. at least one transition has two or more input places) is described by a set of switching systems of linear differential equations with constant coefficients.

The evolution of the marking of a *join-free* (JF) TCPN (i.e. each transition has exactly one input place) is described by a unique linear differential equation derived from (3):

$$\dot{\boldsymbol{m}}(\tau) = \boldsymbol{A}\boldsymbol{m}(\tau), \quad \tau \ge \tau_0 \ge 0, \quad \boldsymbol{m}(\tau_0) = \boldsymbol{m}_0 \ge \boldsymbol{0}, \tag{4}$$

where

$$\boldsymbol{A} = \boldsymbol{C}\Lambda, \tag{5}$$

$$\Lambda = \left[\Lambda_{ji}\right],\ \Lambda_{ji} = \begin{cases} \lambda_j \big/ C_{ij}^-, & \text{if } {}^\bullet t_j = \{p_i\}, \\ 0, & \text{otherwise}, \end{cases} \quad i = \overline{1,n},\ j = \overline{1,m}. \tag{6}$$

Our paper focuses on the analysis of the sets in $\mathbb{R}_+^n$ that are flow-invariant *with respect to* (w.r.t.) the marking of a TCPN. To define such flow-invariant sets, let us consider two continuously differentiable, vector functions

$$\boldsymbol{g}, \boldsymbol{h} : \mathbb{R}_+ \to \mathbb{R}^n,\ 0 \le g_i(\tau) < h_i(\tau),\ i = \overline{1,n}. \tag{7}$$

Definition 1. The n-dimensional time-dependent set defined by the Cartesian product

$$\mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]} = [g_1(\tau), h_1(\tau)] \times \dots \times [g_n(\tau), h_n(\tau)],\ \tau \in \mathbb{R}_+, \tag{8}$$

is flow-invariant w.r.t. the marking of the TCPN $(\mathcal{N}, \boldsymbol{\lambda})$ described by (3) if

$$\begin{aligned} &\forall \tau_0, \tau \in \mathbb{R},\ \tau \ge \tau_0 \ge 0, \\ &\boldsymbol{m}(\tau_0) \in \mathcal{S}_{[\boldsymbol{g}(\tau_0),\boldsymbol{h}(\tau_0)]} \Rightarrow \boldsymbol{m}(\tau) \in \mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]}, \end{aligned} \tag{9a}$$

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or, equivalently,

$$\forall \tau_0, \tau \in \mathbb{R},\ \tau \ge \tau_0 \ge 0,$$
$$g_i(\tau_0) \le m_i(\tau_0) \le h_i(\tau_0),\ i = \overline{1,n} \Rightarrow \qquad \blacksquare \ (9b)$$
$$\Rightarrow g_i(\tau) \le m_i(\tau) \le h_i(\tau),\ i = \overline{1,n}.$$

Our analysis is based on the following general result adapted from [8], pp. 74.

Lemma 1. Let $\boldsymbol{F}: \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{F}(\boldsymbol{x}) = [F_1(\boldsymbol{x})\, F_2(\boldsymbol{x}) \ldots F_n(\boldsymbol{x})]^T$, be a continuous function that guarantees the uniqueness of the solution to the Cauchy problem

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}(t)),\ \boldsymbol{x}(\tau_0) = \boldsymbol{x}_0,\ \tau_0 \ge 0. \qquad (10)$$

Let $\boldsymbol{g}, \boldsymbol{h}: \mathbb{R}_+ \to \mathbb{R}^n$ be two continuously differentiable vector functions satisfying (7). The time-dependent set $\mathcal{S}_{[\boldsymbol{g}(\tau), \boldsymbol{h}(\tau)]}$ (8) is flow-invariant w.r.t. the state-space trajectories of (10) if and only if

$$\dot{g}_i(t) \le F_i(x_1, \ldots, x_{i-1}, g_i(t), x_{i+1}, \ldots, x_n),\ i = \overline{1,n}, \qquad (11)$$

$$\dot{h}_i(t) \ge F_i(x_1, \ldots, x_{i-1}, h_i(t), x_{i+1}, \ldots, x_n),\ i = \overline{1,n}, \qquad (12)$$

for all $x_j \in [g_j(\tau), h_j(\tau)]$, $j = \overline{1,n}$, $j \ne i$. and $t \in \mathbb{R}_+$. ■

The structure of this paper is as follows. The existence of flow-invariant sets w.r.t. the marking in the general framework of TCPNs with joins is approached in Section 2. Section 3 presents results referring to flow-invariant sets of general form for JFTCPNs. For such nets, a deeper insight into some particular cases of bounded flow-invariant sets is provided in Sections 4 and 5. Some concluding remarks on the importance of our work are formulated in Section 6. All over the text, the vector (matrix) inequalities have componentwise meaning.

II. Case of TCPNs with Joins

We first give a general characterization of the sets that are flow-invariant w.r.t. the marking of a TCPN.

Theorem 1. The set $\mathcal{S}_{[\boldsymbol{g}(\tau), \boldsymbol{h}(\tau)]}$ (8) is flow-invariant w.r.t. the marking of the TCPN (with or without joins) described by (3) *iff* $\boldsymbol{0} \le \boldsymbol{g}(\tau) < \boldsymbol{h}(\tau)$, $\tau \in \mathbb{R}_+$, are solutions to:

$$\dot{g}_i(\tau) \le \sum_{j=1}^{n} C_{ij} \lambda_j e_j(m_1, \ldots, m_{i-1}, g_i(\tau), m_{i+1}, \ldots, m_n) \qquad (13)$$

$$\dot{h}_i(\tau) \ge \sum_{j=1}^{n} C_{ij} \lambda_j e_j(m_1, \ldots, m_{i-1}, h_i(\tau), m_{i+1}, \ldots, m_n) \qquad (14)$$

for all $\tau \ge 0$, $m_j \in [g_j(\tau), h_j(\tau)]$, $j = \overline{1,n}$, $j \ne i$, $i = \overline{1,n}$.

Proof: This results from the application of Lemma 1 for the model (3), written explicitly as

$$\dot{\boldsymbol{m}}(\tau) = \boldsymbol{F}(\boldsymbol{m}(\tau)),\ \boldsymbol{m}(\tau_0) = \boldsymbol{m}_0 \ge \boldsymbol{0},\ \tau \ge \tau_0 \ge 0,$$
$$\boldsymbol{F}: \mathbb{R}^n \to \mathbb{R}^n,\ \boldsymbol{F}(\boldsymbol{m}) = [F_1(\boldsymbol{m}) \ldots F_n(\boldsymbol{m})]^T, \qquad \blacksquare \ (15)$$
$$F_i(\boldsymbol{m}) = \sum_{j=1}^{n} C_{ij} f_j(\boldsymbol{m}) = \sum_{j=1}^{n} C_{ij} \lambda_j e_j(\boldsymbol{m}),\ i = \overline{1,n}.$$

Theorem 1 allows us to prove that the description by differential equation (3) (or, equivalently, (15)) of the dynamics of a TCPN with infinite server semantics is consistent with the nonnegativeness of the marking (accepted as a conjecture from the dynamics of DPNs – see, for instance, [4]).

Corollary 1. For any choice of the firing rates $\boldsymbol{\lambda} > \boldsymbol{0}$ associated with the transitions of a TCPN (with or without joins), the continuous marking of the net remains nonnegative, i.e. it satisfies:

$$\boldsymbol{m}(\tau) \ge \boldsymbol{0},\ \forall \tau \ge \tau_0, \qquad (16)$$

for any initial marking $\boldsymbol{m}(\tau_0) = \boldsymbol{m}_0 \ge \boldsymbol{0}$.

Proof: (13) is used for $g_i(\tau) \equiv 0$, $\forall \tau \ge 0$, $i = \overline{1,n}$. If $p_i \notin {}^\bullet t_j$, then $C_{ij} \ge 0$ and $e_j(m_1, \ldots, m_{i-1}, g_i(\tau), m_{i+1}, \ldots, m_n)$ does not depend on $g_i(\tau)$; therefore it has a nonnegative value when $m_j \in [0, h_j(\tau)]$, $j = \overline{1,n}$, $j \ne i$. If $p_i \in {}^\bullet t_j$, then $e_j(m_1, \ldots, m_{i-1}, g_i(\tau), m_{i+1}, \ldots, m_n)$ depends on $g_i(\tau)$; hence it vanishes for $g_i(\tau) \equiv 0$. ■

The result proved in Theorem 1 is far from triviality as illustrated by the following example.

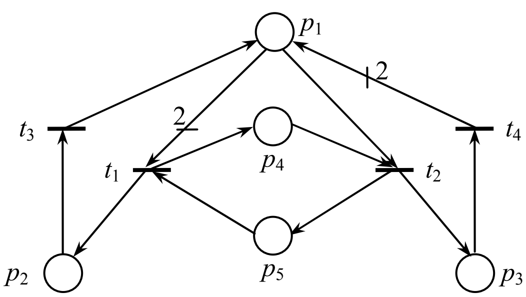

Fig. 1. Topology of the TCPN with joins used in Example 1.

Example 1. Let us consider the TCPN with joins [6] whose topology is depicted in fig. 1. For a given vector of firing rates $\boldsymbol{\lambda} = [\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5]^T > \boldsymbol{0}$ associated with the transitions in the net, we investigate the existence of flow-invariant sets of the form $\mathcal{S}_{[\boldsymbol{0}, \boldsymbol{d}]}$, $\boldsymbol{d} > \boldsymbol{0}$. Taking Corollary 1 into account, the usage of (14) in Theorem 1 leads to the algebraic inequalities:

$$-2\lambda_1 \min\left\{\frac{d_1}{2}, m_5\right\} - \lambda_2 \min\{d_1, m_4\} + \lambda_3 m_2 + 2\lambda_4 m_3 \le 0;$$

$$\lambda_1 \min\left\{\frac{m_1}{2}, m_5\right\} - \lambda_3 d_2 \le 0; \qquad \lambda_2 \min\{m_1, m_4\} - \lambda_4 d_3 \le 0;$$

$$\lambda_1 \min\left\{\frac{m_1}{2}, m_5\right\} - \lambda_2 \min\{m_1, d_4\} \le 0;$$

$$-\lambda_1 \min\left\{\frac{m_1}{2}, d_5\right\} + \lambda_2 \min\{m_1, m_4\} \le 0,$$

whose fulfillment is required for any $m_j \in [0, d_j]$, $j = \overline{1,5}$. This is impossible; for instance, maximizing the left hand side of the first inequality leads to $\lambda_3 d_2 + 2\lambda_4 d_3 \le 0$, condition which cannot be satisfied for any $\boldsymbol{d} > \boldsymbol{0}$ and $\boldsymbol{\lambda} > \boldsymbol{0}$. Consequently, there exist no sets of the form $\mathcal{S}_{[\boldsymbol{0}, \boldsymbol{d}]}$, $\boldsymbol{d} > \boldsymbol{0}$, flow-invariant w.r.t. the marking of the TCPN, although, for different firing rates and initial markings, simulation shows that constant steady states are reached. ■

III. Case of Join-Free TCPNs

Let us notice that for a JFTCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ described by (4)-(6), matrix $\boldsymbol{C}^{-}\Lambda$ is diagonal, since $\boldsymbol{C}^{-}$ has a single non-zero element in each column and the non-zero entries of Λ are placed in the same position as the non-zero entries of $(\boldsymbol{C}^{-})^{T}$. Consequently, matrix $\boldsymbol{A}$ (5) is a Metzler matrix, using the nomenclature in [9], i.e. all its off-diagonal elements of are non-negative. Some important properties of Metzler matrices are presented in the Appendix.

Theorem 2. The set $\mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]}$ (8) is flow-invariant w.r.t. the marking of the JFTCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ (4) *iff* $\boldsymbol{0} \le \boldsymbol{g}(\tau) < \boldsymbol{h}(\tau)$, $\tau \in \mathbb{R}_{+}$, satisfy the differential inequalities:

$$\dot{\boldsymbol{g}}(\tau) \le \boldsymbol{A}\boldsymbol{g}(\tau), \tag{17}$$

$$\dot{\boldsymbol{h}}(\tau) \ge \boldsymbol{A}\boldsymbol{h}(\tau). \tag{18}$$

Proof: By using Theorem 1 we get the equivalence between the flow-invariance of $\mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]}$ w.r.t. the marking of JFTCPN (4) and the inequalities:

$$\dot{g}_i(\tau) \le a_{ii} g_i(\tau) + \sum_{j=1,\, j\ne i}^{n} a_{ij} m_j,\ i = \overline{1,n}, \tag{19}$$

$$\dot{h}_i(\tau) \ge a_{ii} h_i(\tau) + \sum_{j=1,\, j\ne i}^{n} a_{ij} m_j,\ i = \overline{1,n}, \tag{20}$$

for all $m_j \in [g_j(\tau), h_j(\tau)]$, $j = \overline{1,n}$, $j \ne i$, $\tau \in \mathbb{R}_{+}$. Since $\boldsymbol{A}$ (5) is a Metzler matrix, $a_{ij} g_j(\tau) \le a_{ij} m_j \le a_{ij} h_j(\tau)$ for $m_j \in [g_j(\tau), h_j(\tau)]$, $i, j = \overline{1,n}$, $i \ne j$. Inequalities (19) and (20) are equivalent to (17) and (18), respectively. ■

Remark 1. Due to the linearity of differential inequalities (17) and (17), if the set $\mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]}$ is flow-invariant w.r.t. the marking of a JFTCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$, then any set $\mathcal{S}_{[k_g \boldsymbol{g}(\tau), k_h \boldsymbol{h}(\tau)]}$ with arbitrary $0 < k_g \le k_h$ is also flow-invariant w.r.t. the marking of $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$. ■

An equivalent form of Theorem 2 can be obtained via a direct exploitation of the solutions to the differential inequalities (17) and (18).

Theorem 3. The set $\mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]}$ (8) is flow-invariant w.r.t. the marking of the JFTCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ (4) *iff* for any $\tau \in \mathbb{R}_{+}$ the following inequalities hold:

$$\boldsymbol{0} \le \boldsymbol{g}(\tau) \le \mathrm{e}^{\boldsymbol{A}\tau}\, \boldsymbol{g}(0),\ \tau \ge 0, \tag{21}$$

$$\boldsymbol{h}(\tau) \ge \mathrm{e}^{\boldsymbol{A}\tau}\, \boldsymbol{h}(0),\ \tau \ge 0, \tag{22}$$

when $\boldsymbol{0} \le \boldsymbol{g}(0) < \boldsymbol{h}(0)$.

Proof: Sufficiency. If $\boldsymbol{0} \le \boldsymbol{g}(0) < \boldsymbol{h}(0)$ are two arbitrary positive vectors, then any continuously differentiable vector functions $\boldsymbol{g}(\tau)$, $\boldsymbol{h}(\tau)$ fulfilling (21) and (22), respectively, define a set that is flow-invariant w.r.t. the marking of the TCPN. This is because $\boldsymbol{g}(\tau)$, $\boldsymbol{h}(\tau)$ given by (21), (22) are solutions to the differential inequalities (17), (18), and, on the other hand, for $\boldsymbol{0} \le \boldsymbol{g}(0) < \boldsymbol{h}(0)$, we have

$$\forall \tau \in \mathbb{R}_{+},\ \boldsymbol{0} \le \mathrm{e}^{\boldsymbol{A}\tau}\, \boldsymbol{g}(0) < \mathrm{e}^{\boldsymbol{A}\tau}\, \boldsymbol{h}(0). \tag{23}$$

This is becuse $\boldsymbol{O} \le \mathrm{e}^{\boldsymbol{A}\tau}$, $\forall \tau \in \mathbb{R}_{+}$, when $\boldsymbol{A}$ is a Metzler matrix [9], and each row of matrix $\mathrm{e}^{\boldsymbol{A}\tau}$ contains at least one positive element. *Necessity.* From the comparison theory [10], taking into consideration the quasi-monotonicity of $\boldsymbol{F}: \mathbb{R}^{n} \to \mathbb{R}^{n}$, $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$, when $\boldsymbol{A}$ is a Metzler matrix, it results that any solutions to the differential inequalities (10) and (11), fulfilling $\boldsymbol{0} \le \boldsymbol{g}(0) < \boldsymbol{h}(0)$, also satisfy conditions (21) and (22). ■

Corollary 1 and the proof of Theorem 2 also point out the following property of the marking vectors of a JFTCPN.

Proposition 1. Let $\boldsymbol{m}^{1}(\tau)$, $\boldsymbol{m}^{2}(\tau)$ be the markings of the JFTCPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ (4), corresponding to two initial markings $\boldsymbol{m}^{1}(\tau_0) = \boldsymbol{m}_0^{1}$ and $\boldsymbol{m}^{2}(\tau_0) = \boldsymbol{m}_0^{2}$, respectively. If $\boldsymbol{0} \le \boldsymbol{m}_0^{1} < \boldsymbol{m}_0^{2}$, then $\boldsymbol{0} \le \boldsymbol{m}^{1}(\tau) < \boldsymbol{m}^{2}(\tau)$ for all $\tau \ge \tau_0$. ■

Remark 2. Theorem 2 provides a technique for constructing time-dependent sets $\mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]}$ of form (8) that are flow-invariant for the marking of the JFTCPN (4) starting from arbitrarily fixed $\boldsymbol{0} \le \boldsymbol{g}(0) < \boldsymbol{h}(0)$. In particular, according to Proposition 1, the functions $\boldsymbol{g}(\tau)$ and $\boldsymbol{h}(\tau)$ can be chosen as solutions to the linear differential equation (4) that characterizes the behavior of the JFTCPN, meaning that $\boldsymbol{g}(\tau)$ and $\boldsymbol{h}(\tau)$ represent true markings. ■

A characterization for the flow-invariant sets considered of the form $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{h}(\tau)]}$ can also be obtained from Theorems 1 and 2 with $\boldsymbol{g}(\tau) \equiv \boldsymbol{0}$, $\forall \tau \in \mathbb{R}_{+}$.

Corrolary 2. The set $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{h}(\tau)]}$ is flow-invariant w.r.t. the marking of the JFTCPN (4) *iff* $\boldsymbol{h}(\tau) > \boldsymbol{0}$, $\tau \in \mathbb{R}_{+}$, is a solution to inequality (18) or (22). ■

Our main interest focuses on *bounded sets* that are flow-invariant w.r.t. the JFTCPN marking, subject which is approached in the section.

IV. Bounded Flow-Invariant Sets of the Form $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{h}(\tau)]}$

The following result focuses on constant sets that are flow-invariant w.r.t. the marking of a JFTCPN.

Corrolary 3. The set $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{d}]}$ defined with $\boldsymbol{d} \in \mathbb{R}^{n}$, $\boldsymbol{d} > \boldsymbol{0}$, is flow-invariant w.r.t. the marking of the JFTCPN (4) *iff* $\boldsymbol{d}$ satisfies the algebraic inequality:

$$\boldsymbol{A}\boldsymbol{d} \le \boldsymbol{0}. \tag{24}$$

Proof: For the vector function $\boldsymbol{h}(\tau) \equiv \boldsymbol{d}$, $\forall \tau \in \mathbb{R}_{+}$, the differential inequality (18) is equivalent to (24). ■

The interpretation of Corollary 3 is as follows. Vector $\boldsymbol{d}$ may be regarded as a vector of capacities assigned to the places of the JFTCPN. Condition (24) is necessary and sufficient for ensuring that the capacities of the places are not exceeded at any moment during the evolution of the marking. This condition also ensures that the token-flow through each transition does not go above a certain value.

Remark 3. A direct consequence of the existence of a positive vector $\boldsymbol{d} > \boldsymbol{0}$ satisfying (24) is that the *spectral abscissa* of matrix $\boldsymbol{A}$ (see the Appendix) is nonpositive, i.e.

$$\sigma_{\max}(\boldsymbol{A}) \leq 0 . \qquad (25)$$

The converse statement is not generally true. Therefore, condition (25) is necessary for the existence of a constant set $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{d}]}$, $\boldsymbol{d} > \boldsymbol{0}$, flow-invariant w.r.t. the marking of the JFTCPN (4). ■

The next theorem refers to a context when condition (25) is also sufficient for the existence of a *bounded* flow-invariant set.

Theorem 4. Let $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ be a JFTCPN described by (4) for which $\mathcal{N}$ is a strongly connected graph w.r.t. the place-type nodes. There exists a bounded positive vector function $\boldsymbol{h}(\tau) > \boldsymbol{0}$, $\tau \in \mathbb{R}_+$, so that $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{h}(\tau)]}$ is flow-invariant w.r.t. the marking of the JFTCPN *iff* one of the following equivalent conditions is fulfilled:

(a) $\exists \boldsymbol{d} \in \mathbb{R}^n$, $\boldsymbol{d} > \boldsymbol{0}$: $\boldsymbol{A}\boldsymbol{d} \leq \boldsymbol{0}$; (26)

(b) $\sigma_{\max}(\boldsymbol{A}) \leq 0$. (27)

Proof. As stated in Remark 3, since $\boldsymbol{A}$ is a Metzler matrix, condition (26) implies (27). If $\mathcal{N}$ is a strongly connected graph w.r.t. the place-type nodes, then matrix $\boldsymbol{A}$ is irreducible (see [9]). In this case, by choosing vector $\boldsymbol{d} > \boldsymbol{0}$ as the Perron vector of the nonnegative matrix $\boldsymbol{A} + s\boldsymbol{I}$, with $s + a_{ii} \geq 0$, $i = \overline{1,n}$, condition (27) involves (26). *Sufficiency.* If (26) is valid, then the constant function $\boldsymbol{h}(\tau) \equiv \boldsymbol{d}$, $\forall \tau \in \mathbb{R}_+$, is bounded and the set $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{d}]}$ is flow-invariant w.r.t. the marking of the JFTCPN. *Necessity.* Let $\boldsymbol{v} > \boldsymbol{0}$, $\| \boldsymbol{v} \|_\infty = 1$, be the eigenvector of the irreducible matrix $\boldsymbol{A}$ corresponding to the simple eigenvalue $\sigma_{\max}(\boldsymbol{A})$. By choosing the initial marking so that $0 \leq \boldsymbol{m}(0) = \boldsymbol{m}_0 = \alpha \boldsymbol{v} \leq \boldsymbol{h}(0)$ for some $\alpha > 0$, the corresponding marking evolution is given by $\boldsymbol{m}(\tau) = \mathrm{e}^{\boldsymbol{A}\tau} \alpha \boldsymbol{v} = \mathrm{e}^{\sigma_{\max}(\boldsymbol{A})\tau} \alpha \boldsymbol{v}$, $\tau \geq 0$. In case that $\sigma_{\max}(\boldsymbol{A}) > 0$, the marking $\boldsymbol{m}(\tau)$ cannot be bounded by $\boldsymbol{h}(\tau)$, contradicting the hypothesis that $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{h}(\tau)]}$ is flow-invariant w.r.t. the marking of the JFTCPN. ■

The following theorem discusses the case when the boundedness of the vector function $\boldsymbol{h}(\tau)$ generating the flow-invariant set $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{h}(\tau)]}$ is replaced by the stronger property of approaching $\boldsymbol{0}$ for infinite time-horizon.

Theorem 5. Let $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ be a JFTCPN described by (4). There exists a positive vector function $\boldsymbol{h}(\tau) > \boldsymbol{0}$, $\tau \in \mathbb{R}_+$, satisfying the condition

$$\lim_{\tau \to \infty} \boldsymbol{h}(\tau) = \boldsymbol{0} \qquad (28)$$

such that the set $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{h}(\tau)]}$ is flow-invariant w.r.t. the marking of the JFTCPN (4) *iff* one of the following equivalent conditions is fulfilled:

(a) $\exists \boldsymbol{d} \in \mathbb{R}^n$, $\boldsymbol{d} > \boldsymbol{0}$: $\boldsymbol{A}\boldsymbol{d} < \boldsymbol{0}$; (29)

(b) $\sigma_{\max}(\boldsymbol{A}) < 0$ (i.e. $\boldsymbol{A}$ is Hurwitz stable). (30)

Proof: For the Metzler matrix $\boldsymbol{A}$ (not necessarily irreducible), Lemma A3 ensures that (29) implies (30). To prove the converse statement, apply Lemma A2 for $\boldsymbol{\Theta} = \boldsymbol{A}$ and an arbitrary $r \in \mathbb{R}$ satisfying $\sigma_{\max}(\boldsymbol{A}) < r < 0$. *Sufficiency.* A particular solution to the differential inequality (18) with $\boldsymbol{h}(0) > \boldsymbol{0}$ is given by $\boldsymbol{h}(\tau) = \mathrm{e}^{\boldsymbol{A}\tau} \boldsymbol{h}(0) > \boldsymbol{0}$, $\tau \geq 0$. When matrix $\boldsymbol{A}$ is Hurwitz stable, this solution satisfies (28). *Necessity.* If a positive solution to (18) can be found satisfying (28), then there exists $\tau^* > 0$ so that $\boldsymbol{h}(\tau^*) < \boldsymbol{h}(0)$. Based on (22) we get $\mathrm{e}^{\boldsymbol{A}\tau^*} \boldsymbol{h}(0) < \boldsymbol{h}(0)$; this shows that the eigenvalues of $\mathrm{e}^{\boldsymbol{A}\tau^*}$, denoted by $\sigma_i(\mathrm{e}^{\boldsymbol{A}\tau^*})$, fulfill $|\sigma_i(\mathrm{e}^{\boldsymbol{A}\tau^*})| < 1$ equivalent to $\mathrm{Re}[\sigma_i(\boldsymbol{A})] < 0$, $i = \overline{1,n}$. ■

A special case of (28) that deserves our attention is when $\boldsymbol{h}(\tau)$ is of exponential form:

$$\boldsymbol{h}(\tau) = \boldsymbol{d}\, \mathrm{e}^{r\tau}, \ \tau \in \mathbb{R}_+, \ r < 0, \ \boldsymbol{d} = [d_1, \ldots, d_n]^T > \boldsymbol{0}. \qquad (31)$$

Corollary 4. The set $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{d}\,\mathrm{e}^{r\tau}]}$ defined with $r < 0$, $\boldsymbol{d} > \boldsymbol{0}$, is flow-invariant w.r.t. the marking of the JFTCPN (4) *iff* $r < 0$ and $\boldsymbol{d} > \boldsymbol{0}$ are solutions to the algebraic inequality:

$$\boldsymbol{A}\boldsymbol{d} \leq r\boldsymbol{d} . \qquad (32)$$

Proof: If (31) is used in the differential inequality (18) we get the algebraic inequality (32). ■

The previous results motivate us to develop a sharper investigation of the quantitative link between the Hurwitz stability of matrix $\boldsymbol{A}$, expressed by condition (30) in Theorem 5, and the exponentially decaying rate $r < 0$ involved in relation (32) in Corollary 4. The equivalence between the two conditions is stated as follows.

Corollary 5. There exist exponentially decaying flow-invariant sets $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{d}\,\mathrm{e}^{r\tau}]}$, $r < 0$, $\boldsymbol{d} > \boldsymbol{0}$, w.r.t. the marking of the JFTCPN (4) *iff* matrix $\boldsymbol{A}$ defined by (5) satisfies (30). ■

Remark 4. Corollary 5 shows that whenever the marking of the JFTCPN approaches 0 for $\tau \to \infty$, it also remains inside some flow-invariant sets whose decaying rate can be taken as close to $\sigma_{\max}(\boldsymbol{A})$ as needed. Note that the flow-invariant sets $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{d}\,\mathrm{e}^{r\tau}]}$ cannot decay faster than $\sigma_{\max}(\boldsymbol{A})$. ■

V. Bounded Flow-Invariant Sets of the Form $\mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]}$

Corollary 6. Let $\langle \mathcal{N},\boldsymbol{\lambda}\rangle$ be a JFTCPN described by (4). There exists $\boldsymbol{\zeta}\in\mathbb{R}^n$, $\boldsymbol{\zeta}>\boldsymbol{0}$, so that for any initial marking $\boldsymbol{m}(\tau_0)=\boldsymbol{m}_0\geq\boldsymbol{\zeta}$, the marking of net remains lower bounded by $\boldsymbol{\zeta}$, i.e. $\boldsymbol{m}(\tau)\geq\boldsymbol{\zeta}$, $\forall\tau\geq\tau_0$, *iff* $\boldsymbol{\zeta}$ is a solution to the algebraic inequality

$$\boldsymbol{A}\boldsymbol{\zeta}\geq\boldsymbol{0}. \quad (33)$$

Proof: For the vector function $\boldsymbol{g}(\tau)\equiv\boldsymbol{\zeta}$, $\forall\tau\in\mathbb{R}_+$, the differential inequality (17) is equivalent to (33). ■

The interpretation of Corollary 6 is as follows. Vector $\boldsymbol{\zeta}$ may be regarded as a vector of lower bounds for the markings of the places in the JFTCPN. Condition (33) is necessary and sufficient for ensuring that the marking of each place remains above the corresponding component of $\boldsymbol{\zeta}$ at any moment during the evolution of the marking. This condition also ensures that the token-flow through each transition does not go below a certain value.

Remark 5. A direct consequence of the existence of a positive vector $\boldsymbol{\zeta}>\boldsymbol{0}$ satisfying (33) is that the spectral abscissa of matrix $\boldsymbol{A}$ is nonnegative, i.e.

$$\sigma_{\max}(\boldsymbol{A})\geq 0. \quad ■ \ (34)$$

The next result considers flow-invariant sets defined with constant positive vectors and is a direct consequence of Corollaries 3 and 5.

Corollary 7. Let $\langle \mathcal{N},\boldsymbol{\lambda}\rangle$ be a JFTCPN described by (4). The constant set $\mathcal{S}_{[\boldsymbol{\zeta},\boldsymbol{d}]}$, defined with $\boldsymbol{0}<\boldsymbol{\zeta}<\boldsymbol{d}$, is flow-invariant w.r.t. the marking of the net *iff* $\boldsymbol{0}<\boldsymbol{\zeta}<\boldsymbol{d}$ are solutions to the algebraic inequalities (24) and (33). ■

Remark 6. The simultaneous satisfaction of inequalities (24) and (33) by some positive vectors $\boldsymbol{0}<\boldsymbol{\zeta}<\boldsymbol{d}$ implies $\sigma_{\max}(\boldsymbol{A})=0$ as a necessary condition. If $\mathcal{N}$ is a strongly connected graph w.r.t. the place-type nodes (i.e. matrix $\boldsymbol{A}$ is irreducible) and $\sigma_{\max}(\boldsymbol{A})=0$, then the Perron-Frobenius theorem ensures that the corresponding eigenvector $\boldsymbol{v}$, $\|\boldsymbol{v}\|_\infty=1$, can be chosen with positive elements, $\boldsymbol{v}>\boldsymbol{0}$. Hence, the sets $\mathcal{S}_{[\boldsymbol{\zeta},\boldsymbol{d}]}$, defined by $\boldsymbol{\zeta}=k_{\boldsymbol{\zeta}}\boldsymbol{v}$, $\boldsymbol{d}=k_{\boldsymbol{d}}\boldsymbol{v}$, with $0<k_{\boldsymbol{\zeta}}<k_{\boldsymbol{d}}$, are flow-invariant w.r.t. the marking of the JFTCPN $\langle \mathcal{N},\boldsymbol{\lambda}\rangle$. Moreover, for any initial marking $\boldsymbol{m}_0$, so that $\boldsymbol{\zeta}\leq\boldsymbol{m}_0\leq\boldsymbol{d}$, there exists the steady-state marking $\boldsymbol{m}_{ss}$, given by $\boldsymbol{A}\boldsymbol{m}_{ss}=\boldsymbol{0}$, and it also fulfills $\boldsymbol{\zeta}\leq\boldsymbol{m}_{ss}\leq\boldsymbol{d}$. ■

Following the same presentation plan as in the previous section, next we explore the case when the two vector functions $\boldsymbol{g}(\tau)$ and $\boldsymbol{h}(\tau)$ generating the flow-invariant set $\mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]}$ approach $\boldsymbol{0}$ for infinite time-horizon.

Theorem 6. Let $\langle \mathcal{N},\boldsymbol{\lambda}\rangle$ be a JFTCPN described by (4). There exists positive vector functions $\boldsymbol{h}(\tau)>\boldsymbol{g}(\tau)>\boldsymbol{0}$, $\tau\in\mathbb{R}_+$, satisfying the condition

$$\lim_{\tau\to\infty}\boldsymbol{g}(\tau)=\boldsymbol{0},\ \lim_{\tau\to\infty}\boldsymbol{h}(\tau)=\boldsymbol{0}, \quad (35)$$

so that the set $\mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]}$ is flow-invariant w.r.t. the marking of the net *iff* matrix $\boldsymbol{A}$ given by (5) satisfies (30).

Proof: *Necessity*. If $\mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]}$ is a flow-invariant set, then $\mathcal{S}_{[\boldsymbol{0},\boldsymbol{h}(\tau)]}$ is a flow-invariant set and Theorem 5 can be applied. *Sufficiency*. If $\boldsymbol{A}$ is Hurwitz stable, then we can use (18) with $\boldsymbol{g}(0)<\boldsymbol{h}(0)$ and construct a flow-invariant set $\mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]}$ with $\boldsymbol{g}(\tau)$, $\boldsymbol{h}(\tau)$ satisfying (35). ■

A special case of (35) that deserves our attention is when $\boldsymbol{g}(\tau)$, $\boldsymbol{h}(\tau)$ are exponentially decaying vector functions

$$\boldsymbol{g}(\tau)=\boldsymbol{\zeta}\,\mathrm{e}^{\rho\tau},\boldsymbol{h}(\tau)=\boldsymbol{d}\,\mathrm{e}^{r\tau},\ \tau\in\mathbb{R}_+,\rho\leq r<0,\ \boldsymbol{0}<\boldsymbol{\zeta}<\boldsymbol{d}. \quad (36)$$

Corollary 8. A set $\mathcal{S}_{[\boldsymbol{\zeta}\mathrm{e}^{\rho\tau},\boldsymbol{d}\mathrm{e}^{r\tau}]}$ is flow-invariant w.r.t. the marking of the JFTCPN (4) *iff* $\rho\leq r<0$ and $\boldsymbol{0}<\boldsymbol{\zeta}<\boldsymbol{d}$ are solutions to the algebraic inequalities

$$\boldsymbol{A}\boldsymbol{\zeta}\geq\rho\boldsymbol{\zeta},\ \boldsymbol{A}\boldsymbol{d}\leq r\boldsymbol{d}. \quad (37)$$

Proof: If $\boldsymbol{g}(\tau)$ and $\boldsymbol{h}(\tau)$ defined by (36) are used in (17) and (18), respectively, inequalities (37) are obtained. ■

Remark 7. The two decaying rates $\rho\leq r<0$ from (36) fulfill the inequality $\rho\leq\sigma_{\max}(\boldsymbol{A})\leq r<0$. If $\mathcal{N}$ is a strongly connected graph w.r.t. the place-type nodes, matrix $\boldsymbol{A}$ has a positive eigenvector $\boldsymbol{v}>\boldsymbol{0}$, $\|\boldsymbol{v}\|_\infty=1$, associated with $\sigma_{\max}(\boldsymbol{A})$, and we can use $\rho=r=\sigma_{\max}(\boldsymbol{A})$ and $\boldsymbol{\zeta}=k_{\boldsymbol{\zeta}}\boldsymbol{v}$, $\boldsymbol{d}=k_{\boldsymbol{d}}\boldsymbol{v}$, with $0<k_{\boldsymbol{\zeta}}<k_{\boldsymbol{d}}$, in (36). ■

Remark 8. Our results show that $\sigma_{\max}(\boldsymbol{A})<0$ (30) is not only a necessary and sufficient condition for the asymptotic (equivalently, exponential) stability of the continuous marking of a JFTCPN, as resulting from the classical qualitative analysis of linear dynamical systems. In addition, we prove that inequality (28) is also necessary and sufficient for the existence of flow-invariant sets $\mathcal{S}_{[\boldsymbol{g}(\tau),\boldsymbol{h}(\tau)]}$ defined by positive vector functions fulfilling (35). ■

The following example illustrates the investigation of flow-invariant sets w.r.t. the marking of a given JFTCPN.

Example 2. Let us consider the JFTCPN whose topology $\mathcal{N}$, depicted in fig. 2. Three vectors of firing rates are considered $\boldsymbol{\lambda}_i=[2\ \lambda_2^i\ 3\ 4]^T$, $i=1,2,3$, with $\lambda_2^1=1$, $\lambda_2^2=2$ and $\lambda_2^3=4$. The dynamics of JFTCPN $\langle \mathcal{N},\boldsymbol{\lambda}_i\rangle$ is described by (4) with matrix $\boldsymbol{A}=\boldsymbol{A}_i$ given by:

$$\boldsymbol{A}_i=\begin{bmatrix}-3-\lambda_2^i & 4 & 1\\ 1+\lambda_2^i & -4 & 0\\ 2+\lambda_2^i & 0 & -2\end{bmatrix},\ i=1,2,3.$$

For $i=1$ matrix $\boldsymbol{A}_1$ is Hurwitz stable and Remark 8 may be applied for finding flow-invariant set with exponentially decaying bounds. For $i=2$, the spectral abscissa of matrix

A_2 is 0 and Remark 6 may be applied for finding flow-invariant set with constant bounds. Since $\sigma_{\max}(A_3) > 0$, the marking of the JFTCPN is unbounded. In figure 3, dashed lines are used to represent the evolution of the bounds of the flow-invariant sets w.r.t. the markings of te JFTCPN and solid lines are used to represent the evolution of the continuous marking when initiated inside the set.

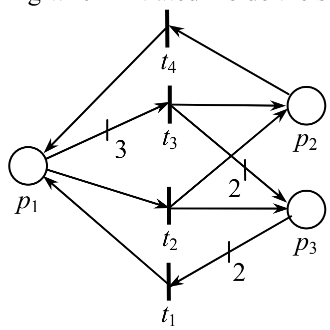

Fig. 2. Topology of the JFTCPN used in Example 2.

VI. Conclusions

This paper provides necessary and sufficient conditions for the existence of flow-invariant sets w.r.t. the marking of a TCPN with infinite server semantics. Such a set allows a qualitative characterization of the net dynamics at the level of the marking in each place of the TCPN. This novel point of view complements the classical concept of marking invariance used for discrete Petri nets, which refers to a set of places. Theorems 1 and 2 give general results that characterize the flow-invariant sets w.r.t. the marking of a general TCPN (with or without joins) and, respectively, of a join-free TCPN. Some specialized results are derived from Theorem 2 referring to particular types of bounded flow-invariant sets. An example illustrates the key elements of the flow-invariant sets analysis for a JFTCPN and offers relevant graphical representations supporting the intuitive insight.

Appendix

Lemma A1. Let us denote by $\sigma_i(\Theta)$, $i = \overline{1,n}$, the eigenvalues of a square matrix $\Theta = [\theta_{ij}] \in \mathbb{R}^{n\times n}$. If Θ is a Metzler matrix, then Θ has a real eigenvalue (simple or multiple), denoted by $\sigma_{\max}(\Theta)$ and called *spectral abscissa*, which fulfils the dominance condition $\mathrm{Re}[\sigma_i(\Theta)] \le \sigma_{\max}(\Theta)$, $i = \overline{1,n}$. ■

Lemma A2: If $\Theta \in \mathbb{R}^{n\times n}$ is a Metzler matrix, then, for any $r \in \mathbb{R}$, $\sigma_{\max}(\Theta) < r$, there exists $\gamma \in \mathbb{R}^n$, $\gamma > \mathbf{0}$, such that $\Theta\gamma < r\gamma$. ■

Lemma A3: Let $\Theta \in \mathbb{R}^{n\times n}$ be a Metzler matrix, $\alpha, \beta \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}^n$, $\gamma > \mathbf{0}$. If $\alpha\gamma \le \Theta\gamma \le \beta\gamma$ then $\alpha \le \sigma_{\max}(\Theta)$ and $\sigma_{\max}(\Theta) \le \beta$. If $\alpha\gamma < \Theta\gamma$ then $\alpha < \sigma_{\max}(\Theta)$. If $\Theta\gamma < \beta\gamma$ then $\sigma_{\max}(\Theta) < \beta$. ■

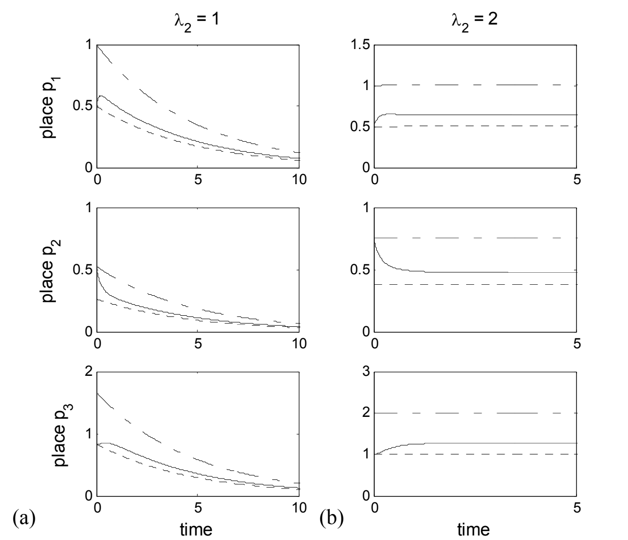

Fig. 3. Bounds of a flow-invariant set (dashed lines) and the evolutions of the markings (solid line) remaining inside the flow-invariant set for the JFTCPNs (a) $\langle \mathcal{N}, \boldsymbol{\lambda}_1 \rangle$ and (b) $\langle \mathcal{N}, \boldsymbol{\lambda}_2 \rangle$.

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