Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

A new observation algorithm for nonlinear systems with unknown inputs

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Abstract— **This paper provides some new developments in the design of unknown input observers for nonlinear systems. This note is a generalization to nonlinear system of an algorithm for linear systems that states if it is possible to design or not an observer for systems subject to unknown inputs. The algorithm for nonlinear systems with unknown inputs is derived and a step by step sliding mode observer is built. An example is given in order to highlight the efficiency of the proposed method.**

I. INTRODUCTION

It is of importance to design observers for multivariable linear or nonlinear system partially driven by unknown inputs. Such a problem arises in systems subject to disturbances or with inaccessible inputs and in many applications such as fault detection and isolation, parameter identification or cryptography.

Different approaches have been considered to design unknown input observers for linear systems, following the conventional Luenberger design procedure [6], [13], or using the sliding mode theory [8], [20]. It is worth noticing that the existence conditions are exactly the same for both approaches. In [9], [10], the authors proposed an observation algorithm that allows to recover the state and the unknown inputs in finite time under less restrictive conditions.

The design of observers for nonlinear systems is much more complicated, even for accurately known systems. In this case, many approaches deal with nonlinear systems for which observers with linearizable error dynamics can be designed (see, e.g., [15], [16], [22]). Other ones are concerned with the design of high gain [4], [5], [12] or sliding mode observers [7], [19].

Few works dealing with the design of unknown input observers for nonlinear systems are available in the literature. Some of them are based on sliding mode consideration [2], [3]. Among them, the works [11] and [18] are concerned with the nonlinear Fundamental Problem of Residual Generation (FPRG). In [23], the authors proposed a sliding mode observer for a class of uncertain systems where the distribution spanned by the unknown input channels has to be involutive.

The main contribution of this paper is the description of a constructive algorithm that provides a change of coordinates which puts the system in a set of block triangular observable forms, even when the distribution spanned by the unknown input channels is not involutive. This algorithm is an extension, to nonlinear systems, of the methodology given in [9] for linear systems. In the linear case, only linear combinations of available information were allowed. Here, fictitious outputs are designed (if possible) over the algebra of the known outputs. This provides more solutions but submanifold of singularities may appear. After transformation of the system, a step-by-step sliding mode observer is designed [2]. The analysis of the so-called *equivalent output injection* (which is the analogous to the well known equivalent control in sliding mode control [21]), allows to recover the part of the state that is directly affected by the disturbances. Furthermore, step-by-step observers achieve finite time convergence of the state components. This property is often desirable in the framework of observation.

It is assumed henceforth that the reader is familiar with the basic concepts and tools of the differential geometric approach [14] and the sliding mode theory [17], [21].

The outline of this paper is the following. The next Section gives the problem statement. Section III establishes some conditions under which it is possible to recover suitable information. In Section IV, the algorithm that provides a suitable change of coordinates for the observation of the system state is described. In Section V, an illustrative example with the observer design highlights the efficiency of the proposed methodology.

II. PROBLEM STATEMENT

Consider the following nonlinear system:

$$\left\{ \begin{array}{l} \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) w_i \\ y = (h_1(x), .., h_p(x))^T \end{array} \right. \tag{1}$$

where $x \in U$ an open set of $\mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the output vector and where $w = [w_1, ..., w_m] \in \mathbb{R}^m$ represents the unknown inputs.

The vector fields f and $g_1, ..., g_m$, and the functions $h_1, ..., h_p$, are assumed to be sufficiently smooth on U. Without loss of generality, it is assumed that, for all $x \in U$:

$$\text{rank}\, [g_1(x), .., g_m(x)] = m$$

and that $h_1(x), ..., h_p(x)$ are independent. It is also assumed that $p \geq m$.

A cornerstone of the problem that is considered in this paper is the notion of relative degree, i.e. the number of times one has to differentiate the outputs in order to have at least one of the component of the unknown input w explicitly appearing. Assume, in this preliminary study, that the system

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(1) has a vector relative degree $\{\rho_1, ..., \rho_p\}$, that is to say $\rho_i = \min\{s \text{ such that } L_{g_k} L_f^{s-1} h_i \neq 0 \text{ for } k = 1 : m\}$[1], $i = 1 : p$ where all ρ_i's are finite [2] and where $L_X \psi$ denotes the classical Lie derivative of the function ψ along the vector field X.

Then, it is known that there exists a change of coordinates $(\xi, \eta) = \phi(x)$ such that the system (1) is locally transformed in the following form (see [14]):

$$\frac{d\xi^i}{dt} = A_i \xi^i + F_i(\xi, \eta) + \overline{g}_i(\xi, \eta, w) \text{ for } i = 1 : p \qquad (2)$$

$$\dot{\eta} = p(\xi, \eta) + q(\xi, \eta) w \qquad (3)$$

$$y_i = \xi_1^i$$

where,

$$\xi = \left[\left(\xi^1\right)^T, ..., \left(\xi^p\right)^T \right]^T$$

$$\xi^i \triangleq \left[\xi_1^i, ..., \xi_{\rho_i}^i\right]^T = \left[h_i(x), ..., L_f^{\rho_i - 1} h_i(x)\right]^T,$$

$$A_i = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ F_i(x) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ L_f^{\rho_i} h_i(x) \end{pmatrix}$$

and $\overline{g}_i(\xi, \eta, w) = \left[0, ..., 0, \sum_{j=1}^{m} L_{g_j} L_f^{\rho_i - 1} h_i(x) w_j \right]^T$.

The aim of this paper is to give an algorithm in order to observe the variables η, when it is possible. In the algorithm, η is recovered via nonlinear functions of the outputs and their time derivatives. The main idea of the construction of the components of η is the following. Consider the matrix:

$$\Gamma(x) = \begin{pmatrix} L_{g_1} L_f^{\rho_1 - 1} h_1(x) & ... & L_{g_m} L_f^{\rho_1 - 1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{\rho_p - 1} h_p(x) & ... & L_{g_m} L_f^{\rho_p - 1} h_p(x) \end{pmatrix}$$

and the commutative algebra of measured functions of the outputs:

$$\pounds = \operatorname{span}\{h_1, ..., L_f^{\rho_1 - 1} h_1, ..., h_p, ..., L_f^{\rho_p - 1} h_p\}.$$

Throughout the paper, $\pounds$ is considered as a ring (i.e. when the terminology "over the algebra $\pounds$ is used"). One also has that $\dim(\pounds) = \rho_1 + ... + \rho_p = \rho$, when $\pounds$ is considered as a vector space over the real numbers field $\mathbb{R}$.

Following the lines of [2], [23], a step-by-step sliding mode observer for system (2) can be designed such that:

- for $i = 1 : p$, ξ^i is estimated in finite time;
- moreover one can recover in a finite time the equivalent vector V_{eq}^i for $i = 1 : p$,

$$V_{eq}^i = L_f^{\rho_i} h_i(x) + \sum_{j=1}^{m} L_{g_j} L_f^{\rho_i - 1} h_i(x) w_j = y_i^{(\rho_i)}.$$

[1] $L_{g_k} L_f^{s-1} h_i$ can be equal to zero on a submanifold of U but not everywhere in U.

[2] The case of infinite relative degree is discussed in the next section.

Obviously, if $\rho = n$, the whole state vector can be observed despite the unknown inputs. Unfortunately, this condition is not fulfilled by many systems. In the next section is given a methodology that allows, when $\rho < n$, to design tractable fictitious outputs in order to recover the remaining state vector η.

III. Existence conditions for suitable fictitious outputs

The main idea is to get extra information from the henceforth known variables, that is to say the previously estimated state ξ^i and the equivalent vectors V_{eq}^i. To this end, let us note

$$V_{eq}(x) = \begin{pmatrix} V_{eq}^1 \\ V_{eq}^2 \\ \vdots \\ V_{eq}^p \end{pmatrix}(x) = \begin{pmatrix} L_f^{\rho_1} h_1(x) \\ L_f^{\rho_2} h_2(x) \\ \vdots \\ L_f^{\rho_p} h_2(x) \end{pmatrix} + \Gamma(x) w.$$

Then, let us assume that there exists a $1 \times m$ vector $K(x) = (k_1(x), ..., k_p(x)) \neq 0$ where $k_i \in \pounds$ for $1 \leq i \leq p$ such that:

$$K(x)\Gamma(x) = 0 \text{ for all } x \in U \qquad (4)$$

and let us define:

$$\bar{y} = \bar{h}(x) = K V_{eq} = \sum_{i=1}^{p} k_i V_{eq}^i = \sum_{i=1}^{p} k_i L_f^{(\rho_i)} h_i \qquad (5)$$

If $\bar{y}$ does not belong to $\pounds$, it can be chosen as a suitable fictitious output since it is a new available information that is independent from the original outputs and since it is not affected by the unknown inputs.

Remark 1: In general, $\bar{y}$ is not linearly independent from the already available state variable ξ for every point of U. Thus, it may exist a submanifold of singularities $S = \left\{ x \in U \text{ such that } \bar{h}(x) \in \pounds(x) \right\}$.

Before giving conditions which guarantee the existence of a solution to equation (4), let us define the following sets:

- G is the smallest involutive distribution containing $\{g_1(x), .., g_m(x)\}$.
- $G^\perp = \operatorname{span}\{\alpha_1, ..., \alpha_{n-m}\}$ is the annihilator of G: α_i are one-forms such that: $\iota_{g_k} \alpha_i = 0$ for $i = 1 : n - m$, where $\iota_f \alpha = \alpha(f)$ is the inner product of f and α.
- Noting $d\psi$ the differential of the function ψ, $\Omega_\pounds$ is the module spanned by $\left\{ dh_1, ..., dL_f^{\rho_1 - 1} h_1, ..., dh_p, ..., dL_f^{\rho_p - 1} h_p \right\}$ over $\pounds$ and $\Omega_\pounds^1$ is the submodule spanned by $\left\{ dh_1, ..., dL_f^{\rho_1 - 2} h_1, ..., dh_p, ..., dL_f^{\rho_p - 2} h_p \right\}$ where $L_f^{-1} h_j = 0$ if $\rho_i = 1$.

The following proposition gives some equivalent conditions for the existence of a proper fictitious output.

Proposition 2: The following conditions are equivalent:

i) Equation (4) has a non trivial solution α,

ii) $\frac{G^\perp \cap \Omega_\pounds}{G^\perp \cap \Omega_\pounds^1} \neq \{0\}$,

iii) $\Xi = \left\{ \alpha \in G^\perp \cap \Omega_\pounds \text{ such that } \iota_f \alpha \notin \pounds \right\} \neq \varnothing$.

Proof: Let $\overline{\alpha} \in \frac{G^{\perp}\cap\Omega_{\pounds}}{G^{\perp}\cap\Omega^{1}_{\pounds}}$. Then there exists a one-form $\alpha \in G^{\perp} \cap \Omega_{\pounds}$ such that $\overline{\alpha} = \alpha + G^{\perp} \cap \Omega^{1}_{\pounds}$. Thus α is a representative element of the class $\overline{\alpha}$ modulo $G^{\perp} \cap \Omega_{\pounds}$. Let any $\beta \in G^{\perp} \cap \Omega^{1}_{\pounds}$ then $\iota_f(\alpha+\beta)$ is a solution of (4). Moreover, $\iota_f(\alpha+\beta) \notin \pounds$. ■

Remark 3:
a. Point (iii) gives $\iota_f\alpha$ as a new fictitious output.
b. Point (ii) means that a solution α is defined modulo $G^{\perp} \cap \Omega^{1}_{\pounds}$.
c. By construction, for all $\tau \in \Omega_{\pounds}$ the inner product $\iota_{f+gu}\tau$ can be decomposed as $\iota_f\tau + \iota_{gu}\tau$. Condition (iii) implies; first that $\iota_f\tau$ is not only composed of the previously estimated state and this gives directly a new state component; secondly that $\iota_{gu}\tau = 0$ and this implicitly solves the equation (4).

If equation (4) has a non trivial solution, one can design $\bar{y}$ as defined in (5). Then, note $\bar{\rho}$ the relative degree of (1) with respect to the output $(y^T, \bar{y}^T)^T = (h^T, \bar{h}(x)^T)^T$.
If $\bar{\rho} = n$, the η-dynamics can be put in a triangular observable form and it is then possible to design a step by step sliding mode observer for the system (2-3).

Here, we give a simple example in order to highlight the Proposition 2 and Remark 3.

Example 4: Consider the following system with two outputs and one unknown input:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 + x_4^2 w \\ \dot{x}_3 = -x_2 \\ \dot{x}_4 = -x_4 + x_2^3 w \end{cases} \quad y_1 = x_1 \text{ and } y_2 = x_4$$

It can be computed that: $\pounds = \operatorname{span}\{h_1, L_f h_1, h_2\}$, $\Gamma = \begin{pmatrix} x_4^2 \\ x_2^3 \end{pmatrix}$, and that $K = \left((L_f h_1)^3, -h_2^2\right)$.

In this case, one gets that

$$\bar{y} = (L_f h_1)^3 \ddot{y}_1 - h_2^2 \dot{y}_2 = x_2^3 x_3 + x_4^3.$$

This new output does not belong to $\pounds$ only for $x_2 \neq 0$. Then the submanifold of singularities is

$$S = \left\{x \in \mathbb{R}^4; x_2 = 0\right\}.$$

Thus y_1, y_2, $y_3 = x_2$ and $y_4 = \bar{y}$ are new coordinates which allow to observe the state everywhere except on the submanifold of singularities.

If $\bar{\rho} < n$, the same procedure can be done again in order to get a new fictitious output.

This is generalized, in the next section. When the system with unknown input is observable and under conditions of Proposition 2, an algorithm that gives directly a suitable change of coordinate is described. Then, it is possible to recover all the state with a step-by-step sliding mode observer.

IV. Nonlinear output information algorithm

The preliminary step of the algorithm is to compute G and its annihilator $G^{\perp}$.

Consider the output vector $y \triangleq h(x) \in \mathbb{R}^p$.

[a.] Reorder the components of y as following:

$$y = (h_1, ..., h_l, h_{l+1}, ..., h_p)$$

such that for $1 \leq j \leq l$ and for all $x \in U$

$$\forall i \in [1, \dots, m] \qquad L_{g_i} L_f^k h_j = 0 \qquad \text{and} \qquad \forall k \in \mathbb{N}$$

0and for $1 \leq j \leq p-l$, there exists an integer r_j such that:

$$\begin{array}{lll} \forall i \in [1, \dots, m] & L_{g_i} L_f^k h_j = 0 & \forall k < r_j - 1 \\ \exists i \in [1, \dots, m] & L_{g_i} L_f^{r_j-1} h_j \neq 0 & \end{array}$$

[b.] Compute the codistribution:

$$\Phi = \operatorname{span}\{dh_1, ..., dL_f^{n-1}h_1, ..., dh_l, ..., dL_f^{n-1}h_l\}.$$

In this paper, it is assumed that there is no observability singularities. Therefore, no derivative outputs greater or equal to the system dimension are considered. Note φ the dimension of Φ over $\mathbb{R}$.

Definition 5: The integers φ_j, $1 \leq j \leq l$ are defined such that

$$I = \{dh_1, ., dL_f^{\varphi_1 - 1}h_1, ., dh_l, ., dL_f^{\varphi_l - 1}h_l\}$$

is a basis of Φ. One has $\varphi = \sum_{i=1}^{l} \varphi_i$.

If $\varphi = n$, the l-first output components allows to observe all the state and **the algorithm stops**.

[c.] Consider the outputs affected by the unknown inputs, and define:

$$\Upsilon = \{dh_{l+1}, ., dL_f^{r_1-1}h_{l+1}, ., dh_p, ., dL_f^{r_{p-l}-1}h_p\}.$$

Compute the codistribution $\Omega := \operatorname{span}\{I \bigcup \Upsilon\}$. Ω may be written as follows:

$$\begin{aligned} \Omega &= \operatorname{span}\{dh_1, ., dL_f^{\varphi_1-1}h_1, ., dh_l, ., dL_f^{\varphi_l-1}h_l, \\ &\qquad dh_{l+1}, ., dL_f^{\kappa_1-1}h_1, ., dh_p, ., dL_f^{\kappa_{p-l}-1}h_p\} \\ \Omega &= \operatorname{span}\{dz_1, ..., dz_\mu\} \end{aligned}$$

where $\mu :=\dim\Omega = \varphi + \kappa$ with $\kappa = \sum_{i=1}^{p-l}\kappa_i$.
If $\mu = n$, it is not necessary to find a fictitious output in order to observe all the state and **the algorithm ends**.
[d.] If $\mu < n$, note $\pounds := \operatorname{span}\{z_1, ..., z_\mu\}$ the algebra spanned by the measurable variables $\{z_1, ..., z_\mu\}$ and the $\pounds$ module $\Omega_{\pounds} = \pounds.\Omega = \{\sum_{i=1}^{\mu}\phi_i dz_i,\ \phi_i \in \pounds\}$. Define $\Xi = \left\{\alpha \in G^{\perp} \cap \Omega_{\pounds} \text{ such that } \iota_f\alpha \notin \pounds\right\}$.

If Ξ is empty, the state variables of the system (1) can not be recover with the method described in this paper and **the algorithm ends**.

If Ξ is not empty, a new output is computed.
Denoting $\dim \Xi = \bar{p}$, there exist $\bar{p}$ one-forms α_i such that $\Xi = \operatorname{span}\{\alpha_1, ., \alpha_{\bar{p}}\}$ and one can define the new fictitious outputs as follows:

$$\bar{y} = (\iota_f\alpha_1, ..., \iota_f\alpha_{\bar{p}})^T \triangleq \left(h_1^k, ..., h_{p^k}^k\right).$$

Then return to [a.]. Set as output vector

$$y := \left[\begin{array}{c} y \\ \bar{y} \end{array} \right] \tag{6}$$

and note

$$\bar{y} \triangleq \left(h_1^k, ..., h_{p^k}^k \right),$$

where k stands for the iteration number of the algorithm. If the algorithm ends such that $\mu = n$ after k^* iterations, the change of coordinate ψ given by

$$\begin{aligned} z &= \psi(x) \\ &= (h_1, .., L_f^{\rho_1 - 1} h_1, .., h_l, .., L_f^{\rho_l - 1} h_l, \\ &\quad h_{l+1}, .., L_f^{\kappa_1 - 1} h_{l+1}, .., h_p, .., L_f^{\kappa_{p-l} - 1} h_p, ..., \\ &\quad h_1^{k^*}, .., L_f^{\rho_1^{k^*} - 1} h_1^{k^*}, .., h_{l^{k^*}}^{k^*}, .., L_f^{\rho_{l^{k^*}}^{k^*} - 1} h_{l^{k^*}}^{k^*}, \\ &\quad h_{l^{k^*}+1}^{k^*}, .., L_f^{\kappa_1^{k^*} - 1} h_{l^{k^*}+1}^{k^*}, .., h_{p^{k^*}}^{k^*}, .., L_f^{\kappa_{p^{k^*} - l^{k^*}}^{k^*} - 1} h_{p^{k^*}}^{k^*})^T \end{aligned}$$

is well defined and the system is transformed in a set of triangular observable forms. Then, a step-by-step sliding mode observer can be designed. For a sake of simplicity and place, the observer design is given on an example in the next section.

Remark 6: The change of coordinate has the same singularities than the successive Ξ (i.e. successive singularity submanifold S). In some particular cases, there exist methods which allow to observe despite the existence of singularity submanifold (see [1]).

V. Example

Let us consider the following nonlinear system subject to one unknown input w

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^3 \\ \dot{x}_2 &= x_3 + x_2^2 - x_2^3 + x_4^2 x_3 w \\ \dot{x}_3 &= x_5 \\ \dot{x}_4 &= -x_4 + x_2^2 + x_2^3 x_3 w \\ \dot{x}_5 &= -x_3 \end{aligned} \tag{7}$$

with outputs $y_1 = x_1$ and $y_2 = x_4$. It could be computed here that

$$S = \left\{ x \in \mathbb{R}^5; x_2 = 0 \right\}$$

is a singularity manifold. Thus, it is assumed that $x \in U$, an open set of $\mathbb{R}^5$, such that $U \cap S = \emptyset$.
The annihilator of

$$G = span\{g\} = span\{(0, x_4^2 x_3, 0, x_2^3 x_3, 0)^T\}$$

is given by:

$$G^\perp = \text{span}\{dx_1, dx_3, dx_5, x_2^3 dx_2 - x_4^2 dx_4\}.$$

For the first iteration of the algorithm, steps **[a.]** and **[b.]** are not relevant, because both outputs are influenced by the perturbation w, then I is empty.

[c.] One has

$$\begin{aligned} \pounds &= \text{span}\{h_1, L_f h_1, h_2\} = \text{span}\{x_1, x_2 - x_1^3, x_4\} \\ &= \text{span}\{x_1, x_2, x_4\} \end{aligned}$$

and $\Gamma(x) = \begin{pmatrix} x_4^2 x_3 \\ x_2^3 x_3 \end{pmatrix}$. The codistribution Ω is given by:

$$\begin{aligned} \Omega &= \text{span}\{dh_1, dL_f h_1, dh_2, \} \\ &= \text{span}\{dx_1, -3x_1^2 dx_1 + dx_2, dx_4\} \end{aligned}$$

It can be computed that dim$\Omega = 3$ and that $x_2^3 dx_2 - x_4^2 dx_4 \in \Omega_\pounds$.
Then

$$G^\perp \bigcap \Omega_\pounds = \text{span}\{dx_1, x_2^3 dx_2 - x_4^2 dx_4\}.$$

Under the definition

$$\Xi = \left\{ \alpha \in G^\perp \cap \Omega_\pounds \text{ such that } \iota_f \alpha \notin \pounds \right\}$$

and since

$$\iota_f dx_1 = x_2 - x_1^3 \in \pounds$$

and

$$\iota_f (x_2^3 dx_2 - x_4^2 dx_4) = (x_3 + x_2^2 - x_2^3) x_2^3 - x_4^2 (-x_4 + x_2^2) \notin \pounds,$$

one gets:

$$\Xi = \text{span}\{x_2^3 dx_2 - x_4^2 dx_4\}$$

Since Ξ is a non empty set, a fictitious output can be given as:

$$\begin{aligned} y_{fic} &= \iota_f (x_2^3 dx_2 - x_4^2 dx_4) \\ &= x_2^3 (x_3 + x_2^2 - x_2^3) - x_4^2 (-x_4 + x_2^2) \\ &= x_2^3 \ddot{y}_1 - x_4^2 \dot{y}_2. \end{aligned}$$

Moreover, this new output can be defined up to some functions of $\pounds$:

$$\bar{y} = x_2^3 \ddot{y}_1 - x_4^2 \dot{y}_2 - x_2^5 + x_2^6 - x_4^3 - x_4^2 x_2^2 = x_2^3 x_3$$

The second Iteration starts by defining the output:

$$y = \left(x_1, x_2, x_4, x_2^3 x_3 \right)^T$$

Again, Steps **[a.]** and **[b.]** are not relevant because all the outputs are affected by w.
[c.] One has

$$\pounds = \text{span}\{y_1, L_f h_1, y_2, \bar{y}\} = \text{span}\{x_1, x_2, x_4, x_2^3 x_3\}$$

and

$$\Gamma = (x_4^2 x_3, x_2^3 x_3, 3 x_2^2 x_3^2 x_4^2)^T.$$

The new codistribution Ω is given by:

$$\Omega = \text{span}\{dx_1, dx_2, dx_4, d\bar{y}\}.$$

One gets:

$$G^\perp \bigcap \Omega_\pounds = \text{span} \left\{ dx_1, x_2^3 dx_2 - x_4^2 dx_4, d\bar{y} - 3 \frac{\bar{y}}{x_2} dx_2 \right\}$$

and

$$\begin{aligned} \iota_f dx_1 &= x_2 - x_1^3 \in \pounds \\ \iota_f (x_2^3 dx_2 - x_4^2 dx_4) &= \bar{y} \in \pounds \\ \iota_f (d\bar{y} - 3 x_3 x_2^2 dx_2) &= x_2^3 x_5 \notin \pounds. \end{aligned}$$

Thus

$$\Xi = \text{span}\{d\bar{y} - 3x_3x_2^2dx_2\}$$

One can choose as new fictitious output:

$$\bar{\bar{y}} = \iota_f(d\bar{y} - 3x_3x_2^2dx_2) = x_2^3x_5$$

The third iteration starts by defining the output:

$$y = (x_1, x_2, x_4, x_2^3x_3, x_2^3x_5)^T$$

As for the previous iterations, **[a.]** and **[b.]** are not relevant again . The codistribution Ω is equal to:

$$\Omega = \text{span}\{dx_1, dx_2, dx_4, d\bar{y}, d\bar{\bar{y}}\}$$

and since dim $\Omega = 5$, it is possible to design a sliding mode observer for the system (7) in spite of the unknown input. From Ω, one defines the change of coordinates:

$$z = \psi(x) = (y_1, L_fy_1, y_2, \bar{y}, \bar{\bar{y}})^T$$

In the new coordinates, the system is rewritten as:

$$\begin{array}{rcl} \dot{z}_1 &=& z_2 - z_1^3 \\ \dot{z}_2 &=& x_3 + x_2^2 - x_2^3 + x_4^2x_3w \\ \dot{z}_3 &=& -x_4 + x_2^2 + x_2^3x_3w \\ \dot{z}_4 &=& 3x_2^2x_3\left(x_3 + x_2^2 - x_2^3 + x_4^2x_3w\right) + x_2^3x_5 \\ \dot{z}_5 &=& -x_3 \end{array} \tag{8}$$

The first part of the observer involves the available information, i.e. $y_1 = z_1$ and $y_2 = z_3$, that is the outputs of system (7):

$$\begin{array}{rcl} \dot{\hat{z}}_1 &=& -y_1^3 + k_1\text{sign}\left(y_1 - \hat{z}_1\right) \\ \dot{\hat{z}}_2 &=& k_2E\text{sign}\left(\tilde{z}_2 - \hat{z}_2\right) \\ \dot{\hat{z}}_3 &=& k_3\text{sign}\left(y_2 - \hat{z}_3\right) \end{array} \tag{9}$$

where

$$\tilde{z}_2 = \left(k_1\text{sign}\left(y_1 - \hat{z}_1\right)\right)_{\text{eq}} \tag{10}$$

and $k_i > 0$.The function E is defined by

$$E = \begin{cases} 0 \text{ if } y_1 - \hat{z}_1 \neq 0 \\ 1 \text{ otherwise} \end{cases}$$

In (10), the term sign_{eq} is the equivalent information injection (by analogy with the well known equivalent control). It actually represents the mean value of the sign function in sliding mode and can be obtained in finite time via a low pass filter [21].

Then the dynamics of the observation error is given by:

$$\begin{array}{rcl} \dot{e}_1 &=& z_2 - k_1\text{sign}\left(e_1\right) \\ \dot{e}_2 &=& x_3 + x_2^2 - x_2^3 + x_4^2x_3w - k_2E\text{sign}\left(\tilde{z}_2 - \hat{z}_2\right) \\ \dot{e}_3 &=& -x_4 + x_2^2 + x_2^3x_3w - k_3\text{sign}\left(e_3\right) \end{array}$$

Under sliding mode consideration, it is known that, if k_1 and k_3 are large enough, the sliding manifolds $e_1 = 0$ and $e_3 = 0$ are reached in a finite time t_1 and t_3, respectively. Moreover, one recovers the following equivalent output injections (by writing $\dot{e}_1 = 0$ and $\dot{e}_3 = 0$):

$$\left(k_1\text{sign}\left(e_1\right)\right)_{\text{eq}} = z_2, \tag{11}$$

$$\left(k_3\text{sign}\left(e_3\right)\right)_{\text{eq}} = -x_4 + x_2^2 + x_2^3x_3w. \tag{12}$$

Thus, after t_1, one gets:

$$\dot{e}_2 = x_3 + x_2^2 - x_2^3 + x_4^2x_3w - k_2\text{sign}\left(e_2\right).$$

Using the same arguments, one has, after a finite time, $e_2 = 0$ and

$$\left(k_2\text{sign}\left(e_2\right)\right)_{\text{eq}} = x_3 + x_2^2 - x_2^3 + x_4^2x_3w.$$

Then

$$\begin{array}{rcl} y_{fic} &=& \iota_f(x_2^3dx_2 - x_4^2dx_4) \\ &=& x_2^3\left(-x_4 + x_2^2 + x_2^3x_3w\right) \\ && -x_4^2\left(x_3 + x_2^2 - x_2^3 + x_4^2x_3w\right) \\ &=& x_2^3\left(k_3\text{sign}\left(e_3\right)\right)_{\text{eq}} - x_4^2\left(k_2\text{sign}\left(e_2\right)\right)_{\text{eq}} \end{array}$$

Hence

$$\begin{array}{rcl} z_4 &=& \bar{y} = x_2^3\left(k_3\text{sign}\left(e_3\right)\right)_{\text{eq}} - x_4^2\left(k_2\text{sign}\left(e_2\right)\right)_{\text{eq}} \\ && -x_2^5 + x_2^6 - x_4^3 - x_4^2x_2^2 \end{array}$$

is an available information. The following observer for the z_4 dynamics can be built:

$$\dot{\hat{z}}_4 = k_4\text{sign}\left(\bar{y} - \hat{z}_4\right)$$

and the error dynamics is given by:

$$\dot{e}_4 = 3x_2^2x_3\left(x_3 + x_2^2 - x_2^3 + x_4^2x_3w\right) + x_2^3x_5 - k_4\text{sign}\left(e_4\right)$$

If is large enough, the manifold is reached in finite time and one gets the equivalent output injection:

$$\left(k_4\text{sign}\left(e_4\right)\right)_{\text{eq}} = 3x_2^2x_3\left(x_3 + x_2^2 - x_2^3 + x_4^2x_3w\right) + x_2^3x_5.$$

Then the last state variable is recovered, also in finite time:

$$\begin{array}{rcl} z_5 &=& \iota_f\left(d\bar{y} - 3\dfrac{\bar{y}}{x_2}dx_2\right) \\ &=& \left(k_4\text{sign}\left(e_4\right)\right)_{\text{eq}} - 3\dfrac{\bar{y}}{x_2}\left(k_2\text{sign}\left(e_2\right)\right)_{\text{eq}}. \end{array}$$

VI. Conclusion

In this paper has been given a new observation algorithm that allows to state if it is possible to recover or not the state of a nonlinear system subject to unknown inputs. The observer is designed under sliding mode considerations and the convergence is obtained in finite time.

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