Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Synthesis of Fault Tolerant Control with Random FDI Delay

Feng Tao[1] and Qing Zhao[1*]

Abstract— **This paper investigates the controller synthesis problem for the stochastic fault tolerant control systems (FTCS). The faults of random nature is modeled as a Markov process. Because the real system fault modes are not directly accessible, the controller is reconfigured based on the output of a Fault Detection and Identification (FDI) process, which is assumed to indicate the actual system modes only after an exponentially distributed random time delay. By considering the modeling uncertainty and the external disturbances, both state feedback and output feedback control are developed to achieve the Mean Exponential Stability (MES) and the H_∞ performance.**

I. INTRODUCTION

Due to the increasing demands for high reliability and survivability of the complex control systems, the fault tolerant control (FTC) has attracted extensive interests and attention from both industry and academia during the last two decades. Based on whether or not the controller needs to be reconfigured, the FTC methodologies can be classified into active and passive ones. Compared with the passive design, the active FTC can achieve superior fault tolerance capability and has less design constraints, hence is more desirable for practical applications. On the other hand, design of active FTC is more challenging because the two important ingredients, the fault detection and Identification (FDI) and the control reconfiguration have coupled inter-relationships in a closed-loop configuration, especially when the separation principle does not hold under the circumstances of modeling uncertainty and unknown disturbances, see [1], [2], [3] etc. for details.

In the previous work on the integrated FTC design, faults/failures were modeled by using a Markov chain, then the *open-loop* system was simply described as a Markovian jump linear system (MJLS) for it has been widely adopted for modeling structural and parametric changes. But unlike MJLS, where the control law is constructed based on real system mode, a second Markov chain was introduced to model a simple memoryless FDI decision process. The reasons to incorporate the FDI in the model are two folds: First, It is known that in an active fault tolerant control system (FTCS), the real system mode is not directly accessible to the controller, but is 'identified' by a FDI scheme implemented in the loop. The FDI is usually imperfect with possibilities of detection delays, false alarms and missing detections due to the model uncertainty and noises/disturbances. Secondly, in the MJLS context, the mode estimation problem is actually NP-hard, and the algorithm for the estimation of precise system mode which can be executed online in real-time is not available yet. See [4] and the reference therein.

Such a formulation offers a convenient framework for analysis and is useful for demonstrating effects of imperfect FDI decision [5]. By using this formulation, [6],[7],[8] have studied the closed-loop system stability, with or without the presence of noises. However the design problem in this framework is more complicated, particularly because the controller only depends on the FDI process mode although there are two Markov chains. It means that the number of controllers to be designed is less than the total number of the closed loop system modes by combining both fault and FDI Markov chains. The design process involves searching feasible solutions where there are more constraints than the variables to be solved. Generally speaking, there lacks a tractable design method for this stochastic FTC problem. In [9], the state feedback controller for H_∞ performance was synthesized based on the assumption that the controller must access both fault and FDI modes. And the same situation occurs in [10]. [11] relaxed this restriction by designing a controller based on cluster observation of Markov states. However a common Lyapunov function like approach is used, which means the information of FDI is at least partially neglected, conservative controllers are expected.

It is worthwhile to mention that this FTC formulation is different from Markovian Jump Linear Systems especially in the synthesis problems. The latter is equivalent to the former only if it is assumed that the real fault mode is immediately available for the controller design. Otherwise, the controller design for FTC system with two Markov chains is generally more challenging. To the authors knowledge, in this well accepted stochastic FTCS framework, the design of fault tolerant controllers, which only access FDI outputs, is still an open problem.

In this paper, we adopt a simpler yet more practical model, where the FDI is assumed to be able to indicate the real modes of the system, but after a random time delay. This random memoryless detection delay can be interpreted as computation time when using single sample in FDI algorithm. Similar assumption have been made by Mariton in [12] to study the stability. And analysis of effects of the detection delay have been addressed in [8]. In this paper, the system under consideration is assumed to have both modeling uncertainties and external disturbances, in addition to the system component and actuator failures. By considering the Mean Exponential Stability and H_∞ performance, both state feedback and output feedback controllers are designed via convex optimization.

This paper is organized as following. Section II contains

[1] Feng Tao and Qing Zhao are with Department of Electrical and Computer Engineering, University of Alberta, Canada, T6G 2V4, {fengtao,qingzhao}@ece.ualberta.ca
[*] Corresponding author: Tel: (780)492-5792; Fax: (780)492-1811

the modeling and the problem formulation. In section III, without considering modeling errors, sufficient conditions are given as preliminary results for the nominal system to achieve the desirable stability and the H_∞ performance. In section IV, we extend the results to the uncertain system. The synthesis of state feedback control is then achieved by solving the intersection of parametrization controllers. The result on output feedback is presented in Section V, while a numerical example is included in Section VI to demonstrate the proposed design method. And the conclusions are contained in section VII.

II. Modeling and Problem Formulation

The nominal system to be studied in this paper is given as follows:

$$\begin{cases} \dot{x}(t) = A(r_t)x(t) + B(r_t)u(t,l_t) + D(r_t)w(t) \\ y(t) = C(r_t)x(t) \end{cases} \tag{1}$$

where $w(t)$, $x(t)$, $y(t)$ are disturbance, state and output, respectively; it is assumed that $w(t) \in L_2[0,+\infty)$; all the matrices have corresponding compatible dimensions. $\{r_t, t \geq 0\}$ represents the fault process of the system, and is assumed to be a continuous-time homogeneous Markov chain taking values on a finite set $S = \{1,\ 2, \ldots,\ m\}$. Let its transition rate matrix be (α_{ij}), then it follows that

$$r_t : p_{ij}(\Delta t) = \begin{cases} \alpha_{ij}\Delta t + o(\Delta t), & i \neq j \\ 1 + \alpha_{ii}\Delta t + o(\Delta t), & i = j \end{cases}$$

In a practical FTC system, the FDI itself can be a dynamic sub-system, hence its output can not be synchronous with the changes of the system modes. A time delay is always expected. Such a time delay may have an undesirable impact on the overall system stability and performance. The existence of modeling uncertainties can even result in a longer and more serious time delay. To model the delayed FDI, another random process $\{l_t, t \geq 0\}, l_t \in S$ is used as a system mode indicator. For simplicity but without loss of generality, we make the following assumptions for the characteristics of FDI:

1) When system jumps from one mode to another, the FDI output can always follow and jump to the same mode after a time delay.
2) The possibilities of multiple transitions of $r(t)$ between two consecutive transitions of $l(t)$ are negligible.
3) The FDI delay is modeled by an independent exponentially distributed variable, whose mean value is given as $1/\beta_{ij}, j \neq i$, where i, j is the current mode of l_t and r_t respectively.

This formulation of FTCS was firstly adopted by Mariton in [12] when studying the stability.

For notational simplicity, in the remainder of the paper, for any matrix M whose values depend on the particular modes of r_t or l_t, e.g. when $r_t = i, l_t = j$, we denote: $M(r_t) = M_i$, $M(l_t) = M_j$ or $M(r_t, l_t) = M_{ij}$.

Given the system model in (1),and the state feedback control law $u(t,l_t) = K(l_t)x(t)$, the closed-loop system model can then be written as following forms, assuming $r_t = i, l_t = j$:

$$\begin{cases} \dot{x}(t) = (A_i + B_iK_j)x(t) + D_iw(t) \\ y(t) = C_ix(t) \end{cases} \tag{2}$$

we can simplify the notation for the closed-loop system as:

$$\mathscr{G}: \begin{cases} \dot{x} = \tilde{A}_{ij}x + D_iw \\ y = C_ix \end{cases} \tag{3}$$

Finally the design objectives of the proposed FTC are defined as follows:

- For the closed-loop system $\mathscr{G}$, design a state feedback controller so that: (1) the system is internally Mean Exponentially Stable (MES); and (2) the H_∞ performance is satisfied in the sense that $\mathscr{E}(\|y\|_2) \leq \gamma\|w\|_2$.

where $\mathscr{E}\{.\}$ stands for the mean value, and γ is predetermined or to optimize.

Remark 1: In some literatures, the design objective is to make the system Mean Square Stable (MSS). [13] proved that MES and MSS are equivalent, and each implies almost-sure stability.

III. Preliminary Results on Nominal Stability and Performance

The purpose of this section is to introduce some preliminary results on MES and H_∞ performance for the nominal system $\mathscr{G}$ given in (3). The results can then be readily extended to the case when the system is subject to inevitable modeling errors, and used for designing the controllers in the next section.

Theorem 3.1: The nominal system $\mathscr{G}$ in the absence of disturbance $w(t)$ is MES if and only if there exist $P_{ij} > 0$, such that

$$\begin{aligned} N_{ij} = \ & \tilde{A}_{ij}^T P_{ij} + P_{ij}\tilde{A}_{ij} + \mathbf{1}_{\{i=j\}}(\textstyle\sum_{k\in S}\alpha_{ik}P_{kj}) \\ & + \mathbf{1}_{\{i\neq j\}}\beta_{ji}(P_{ii} - P_{ij}) < 0, \qquad i,j \in S \end{aligned} \tag{4}$$

where $\mathbf{1}_{\{.\}}$ stands for a measure, such that $\mathbf{1}_{\{x\}}$ equals one only if x is true, otherwise it equals zero.

Proof: For $r_t = i, l_t = j$, we define the Lyapunov function of the joint Markov process x, r, l as $V(x, r_t, l_t, t) = x^T P_{ij} x$. For such a Lyapunov function candidate, a weak infinitesimal operator $\mathscr{A}V$ can be defined as follows:

$$\begin{aligned} & \mathscr{A}V(x, r_t, l_t) \\ = \ & \lim_{\Delta\to 0}\tfrac{1}{\Delta}\Big(\mathscr{E}\{V(x(t+\Delta), r_t+\Delta, l_t+\Delta | x(t), r_t, l_t)\} \\ & -V(x(t), r_t, l_t)\Big) \end{aligned}$$

Given $r_t = i$, $l_t = j$, $\mathscr{A}V$ is calculated differently in the following two cases:

case 1: if $i = j$, then $\mathscr{A}V$ can be calculated as:

$$\begin{aligned} \mathscr{A}V = \ & x^T[\tilde{A}_{ij}^T P_{ij} + P_{ij}\tilde{A}_{ij} + \textstyle\sum_{k\in S}\alpha_{ik}P_{kj}]x \\ & + w^T D_i^T P_{ij} x + x^T P_{ij} D_i w \end{aligned} \tag{5}$$

case 2: if $i \neq j$, then the weak infinitesimal operator of $V(x, r_t, l_t)$ becomes:

$$\begin{aligned} \mathscr{A}V = \ & x^T[\tilde{A}_{ij}^T P_{ij} + P_{ij}\tilde{A}_{ij} + \beta_{ji}(P_{ii} - P_{ij})]x \\ & + w^T D_i^T P_{ij} x + x^T P_{ij} D_i w \end{aligned} \tag{6}$$

By using the notation N_{ij} defined in the theorem, for both cases, the weak infinitesimal operator can be expressed in a unified form as:

$$\mathscr{A}V = x^T N_{ij} x + w^T D_i^T P_{ij} x + x^T P_{ij} D_i w \quad (7)$$

By setting $w(t) = 0$ in the system $\mathscr{G}$, it is known that the system is mean exponentially stable if and only if the weak infinitesimal operator $\mathscr{A}V < 0$ [6]. It is equivalent to that

$$N_{ij} < 0, \qquad i, j \in S \quad (8)$$

This completes the proof. □

Remark 2: From Remark 1, it is clear that $\dot{x} = \tilde{A}_{ij}x$ is an ordinary MJLS. It has been shown that for a stable system, when choosing the Lyapunov function as $x^T P(r_t, l_t, t)x$, the matrix $P(r_t, l_t, t)$ will converge to a constant matrix $P(r_t, l_t)$ as $t \to \infty$ ([6]. Hence using a constant P_{ij} herein will not introduce any conservatism in the design.

Although the stability analysis of the stochastic FTCS considered in this paper can be similarly done as that of an ordinary MJLS, it can be seen later that the existence of the FDI delay in the FTCS makes the controller synthesis much more difficult and complex than that of MJLS.

In addition to the critical stability criterion, other important performance, such as disturbance/noise attenuation is also desirable. By considering the system H_∞ performance as defined in the design objective, the sufficient condition is given in the following theorem.

Theorem 3.2: By assuming that the system $\mathscr{G}$ is internally MES and the disturbance $w \in L_2[0, +\infty)$, given $\gamma > 0$, the system H_∞ performance is achieved if the following inequalities hold,

$$N_{ij} + C_i^T C_i + \gamma^{-2} P_{ij} D_i D_i^T P_{ij} < 0 \quad (9)$$

where the expression of N_{ij} is given in (4)

Proof: For H_∞ performance, we can write:

$$\begin{aligned} J &= \mathscr{E}\{\|y\|_2\} - \gamma^2 \|w\|_2 \\ &= \mathscr{E}\left\{ \int_0^\infty (y^T y - \gamma^2 w^T w) dt \right\} \\ &= \mathscr{E}\left\{ \int_0^\infty (y^T y - \gamma^2 w^T w + \mathscr{A}V) dt \right\} - \mathscr{E}\left\{ \int_0^\infty \mathscr{A}V dt \right\} \\ &= \mathscr{E}\left\{ \int_0^\infty (y^T y - \gamma^2 w^T w + \mathscr{A}V) dt \right\} \\ &\quad - \mathscr{E}\{V(\infty)\} + V(0) \end{aligned} \quad (10)$$

Since $w \in L_2[0, +\infty)$ and system is internally MES, so $\mathscr{E}(V(\infty)) = 0$ [14], and if we assume zero initial conditions, then the equation above will be:

$$J = \mathscr{E}\left\{ \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} N(r_{t_k}, l_{t_k}) + C(r_{t_k})^T C(r_{t_k}) & P(r_{t_k}, l_{t_k}) D(r_{t_k}) \\ D(r_{t_k})^T P(r_{t_k}, l_{t_k}) & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} dt \right\} < 0 \quad (11)$$

where t_k is the k-th transition time of two processes r_t and l_t. The inequality will hold if for any states $i, j \in S$:

$$\begin{bmatrix} N_{ij} + C_i^T C_i & P_{ij} D_i \\ D_i^T P_{ij} & -\gamma^2 I \end{bmatrix} < 0 \quad (12)$$

From Schur Complement, it is the equivalent expression of (9), and this completes the proof. □

IV. Design of State Feedback Fault Tolerant Control for Stability and Performance

For practical systems, exact mathematical models are extremely hard or even impossible to obtain. In FTCS, modeling errors and unknown disturbances are the major causes of an imperfect FDI result, which in turn affect the control performance. In this section, the FTC design problem is considered for the same stochastic system treated in the previous section, but with modeling uncertainties. In this case, the system and input matrices in system $\mathscr{G}$ are assumed to be of the following form:

$$A_i = A_{0i} + A_{1i}\Delta_{1i}A_{2i}, \quad B_i = B_{0i} + B_{1i}\Delta_{2i}B_{2i}$$

where $\|\Delta_{1i}\| \le 1$ and $\|\Delta_{2i}\| \le 1$. Furthermore, the FDI parameter may change as well, because while the system has time-varying model uncertainties, the threshold and decision rules in a FDI scheme are usually fixed. Hence, we further assume that the FDI delay parameter, β_{ij} is uncertain but bounded, i.e. $\beta_{ij}^- \le \beta_{ij} \le \beta_{ij}^+$.

Under the assumption that all states are accessible, the state feedback control strategy can be adopted. It can be seen from (2) and (3), that the closed-loop system matrix is then written as:

$$\tilde{A}_{ij} = (A_{0i} + A_{1i}\Delta_{1i}A_{2i}) + (B_{0i} + B_{1i}\Delta_{2i}B_{2i})K_j \quad (13)$$

In this case, it can be easily shown that the matrix inequality (9) in Theorem 3.2 becomes:

$$\begin{aligned} &A_{0i}^T P_{ij} + P_{ij} A_{0i} + A_{2i}^T \Delta_{1i}^T A_{1i}^T P_{ij} + P_{ij} A_{1i} \Delta_{1i} A_{2i} + K_j^T B_{0i}^T P_{ij} \\ &+ P_{ij} B_{0i} K_j + P_{ij} B_{1i} \Delta_{2i} B_{2i} K_j + K_j^T B_{2i}^T \Delta_{2i}^T B_{1i}^T P_{ij} + C_i^T C_i \\ &+ \gamma^{-2} P_{ij} D_i D_i^T P_{ij} + \mathbf{1}_{\{i=j\}} \Big(\sum_{k \in S} \alpha_{ik} P_{kj} \Big) \\ &+ \mathbf{1}_{\{i \ne j\}} \beta_{ji} (P_{ii} - P_{ij}) < 0 \end{aligned} \quad (14)$$

Furthermore, it is true that, $\forall \varepsilon_{ij}, \delta_{ij} > 0$,

$$\begin{aligned} P_{ij} A_{1i} \Delta_{1i} A_{2i} + A_{2i}^T \Delta_{1i}^T A_{1i}^T P_{ij} &\le \varepsilon_{ij}^{-1} A_{2i}^T A_{2i} + \varepsilon_{ij} P_{ij} A_{1i} A_{1i}^T P_{ij} \\ P_{ij} B_{1i} \Delta_{2i} B_{2i} K_j + K_j^T B_{2i}^T \Delta_{2i}^T B_{1i}^T P_{ij} &\le \delta_{ij} P_{ij} B_{1i} B_{1i}^T P_{ij} \\ &\quad + \delta_{ij}^{-1} K_j^T B_{2i}^T B_{2i} K_j \end{aligned} \quad (15)$$

The proof is obvious from several useful lemmas in the robust control literature, hence is omitted.

By substituting these two inequalities into (14), following the similar steps in the proof of Theorems 3.1 and 3.2, the following lemma can be obtained:

Lemma 4.1: For the uncertain system $\mathscr{G}$ with the system matrix shown in (13), given $\gamma > 0$, the MES and H_∞ performance (i.e. disturbance attenuation by γ) will hold if there exist matrices K_j, $P_{ij} > 0$, and scalars $\varepsilon_{ij} > 0, \delta_{ij} > 0$, such that the following matrix inequality holds for any $i, j \in S$,

$$A_{0i}^TP_{ij}+P_{ij}A_{0i}+\varepsilon_{ij}^{-1}A_{2i}^TA_{2i}+\varepsilon_{ij}P_{ij}A_{1i}A_{1i}^TP_{ij}+K_j^TB_{0i}^TP_{ij} +P_{ij}B_{0i}K_j+\delta_{ij}P_{ij}B_{1i}B_{1i}^TP_{ij}+\delta_{ij}^{-1}K_j^TB_{2i}^TB_{2i}K_j+C_i^TC_i +\gamma^{-2}P_{ij}D_iD_i^TP_{ij}+\mathbf{1}_{\{i=j\}}(\sum_{k\in S}\alpha_{ik}P_{kj}) +\mathbf{1}_{\{i\neq j\}}\beta_{ji}(t)(P_{ii}-P_{ij})<0 \tag{16}$$

Proof: the proof is omitted. □

Remark 3: Since we assume that the parameter of FDI delay, β_{ji}, $i,j\in S$ are uncertain but bounded, they are contained in a polytope in the parameter space. It can also be seen that the left hand side of (16) is affine with respect to β_{ji}. In the following sections, when solving the matrix inequalities, the vertices formed by β_{ij}^- and β_{ij}^+ will be substituted to replace β_{ij}. If all these inequalities with β_{ji}^+ or β_{ji}^- hold, then for any β_{ij} in the polytope, the corresponding inequalities will also hold.

Hence the design problem can be tackled by solving (16) for K_j, $j\in S$, the state feedback control gains. Unfortunately, (16) is in a complex form of nonlinear matrix inequalities, and cannot be solved directly. To handle this difficult problem, an effective procedure is introduced, in which this nonlinear matrix inequality is transformed into its two-step equivalence, and each step only involves solving the LMIs. First of all, the following projection lemma is introduced.

Lemma 4.2: Projection Lemma ([15]): Given Ψ,U,V, there exists X such that

$$\Psi+U^TX^TV+V^TXU<0$$

if and only if

$$N_U^T\Psi N_U<0,\qquad N_V^T\Psi N_V<0$$

holds, where N_U and N_V are bases of null spaces of U and V respectively.

Using this lemma, the state feedback control gains can be calculated by solving the two-step LMIs. The first step give out the necessary conditions for existence of H_∞ controller, which also lead to the controller K_{ij}, controller accesses both fault mode and FDI output. Then the controller K_j, if exists under given set of parameters, lies in the intersection of controllers K_{ij}.

Theorem 4.1: In the case of state feedback control, if the nonlinear matrix inequality (16) has feasible solutions, then there exist positive definite matrices X_{ij} and positive scalars $\varepsilon_{ij},\delta_{ij}$,$i,j\in S$, such that the following LMIs are feasible for $\forall i,j\in S$:

$$\begin{bmatrix} M_{2ij} & W_{1i}^TX_{ij}A_{2i}^T & W_{1i}^TX_{ij}C_i^T & \varepsilon_{ij}W_{1i}^TA_{1i} & \delta_{ij}W_{1i}^TB_{1i} & W_{1i}^TD_i & M_{3ij}\\ A_{2i}X_{ij}W_{1i} & -\varepsilon_{ij}I & 0 & 0 & 0 & 0 & 0\\ C_iX_{ij}W_{1i} & 0 & -I & 0 & 0 & 0 & 0\\ \varepsilon_{ij}A_{1i}^TW_{1i} & 0 & 0 & -\varepsilon_{ij}I & 0 & 0 & 0\\ \delta_{ij}B_{1i}^TW_{1i} & 0 & 0 & 0 & -\delta_{ij}I & 0 & 0\\ D_i^TW_{1i} & 0 & 0 & 0 & 0 & -\gamma^2I & 0\\ M_{3ij}^T & 0 & 0 & 0 & 0 & 0 & M_{4ij} \end{bmatrix}<0 \tag{17}$$

where

$$\begin{bmatrix}W_{1i}^T & W_{2i}^T\end{bmatrix}^T=Ker(\begin{bmatrix}B_{0i}^T & B_{2i}^T\end{bmatrix})$$
$$M_{2ij}=W_{1i}^T[X_{ij}A_{0i}^T+A_{0i}X_{ij}+(\mathbf{1}_{\{i=j\}}\alpha_{ii}-\mathbf{1}_{\{i\neq j\}}\beta_{ji})X_{ij}]W_{1i}-\delta_{ij}W_{2i}^TW_{2i}$$
$$M_{3ij}=W_{1i}^T\left[\sqrt{\mathbf{1}_{\{i\neq j\}}\beta_{ji}},\sqrt{\mathbf{1}_{\{i=j\}}\alpha_{i1}},\cdots,\sqrt{\mathbf{1}_{\{i=j\}}\alpha_{i,i-1}},\sqrt{\mathbf{1}_{\{i=j\}}\alpha_{i,i+1}},\cdots\right]X_{ij}$$
$$M_{4ij}=-\text{diag}\ \{X_{ii},X_{1j},\cdots,X_{i-1,j},X_{i+1,j},\cdots\}$$

Proof:

First re-write the matrix inequality (16) by using Schur Complement:

$$\begin{bmatrix} G_{11ij}+P_{ij}B_{0i}K_j+K_j^TB_{0i}^TP_{ij} & \varepsilon_{ij}P_{ij}A_{1i} & \delta_{ij}P_{ij}B_{1i} & P_{ij}D_i & K_j^TB_{2i}^T\\ \varepsilon_{ij}A_{1i}^TP_{ij} & -\varepsilon_{ij}I & 0 & 0 & 0\\ \delta_{ij}B_{1i}^TP_{ij} & 0 & -\delta_{ij}I & 0 & 0\\ D_i^TP_{ij} & 0 & 0 & -\gamma^2I & 0\\ B_{2i}K_j & 0 & 0 & 0 & -\delta_{ij}I \end{bmatrix}<0 \tag{18}$$

where

$$G_{11ij}=A_{0i}^TP_{ij}+P_{ij}A_{0i}+\varepsilon_{ij}^{-1}A_{2i}^TA_{2i}+C_i^TC_i +\mathbf{1}_{\{i=j\}}(\textstyle\sum_{k\in S}\alpha_{ik}P_{kj})+\mathbf{1}_{\{i\neq j\}}\beta_{ji}(P_{ii}-P_{ij})$$

Then we rearrange the inequality above in the same form as appeared in the projection lemma 4.2 as:

$$G_{ij}+U^TK_j^TV_{ij}+V_{ij}^TK_jU<0$$

where

$$G_{ij}=\begin{bmatrix} G_{11ij} & \varepsilon_{ij}P_{ij}A_{1i} & \delta_{ij}P_{ij}B_{1i} & P_{ij}D_i & 0\\ \varepsilon_{ij}A_{1i}^TP_{ij} & -\varepsilon_{ij}I & 0 & 0 & 0\\ \delta_{ij}B_{1i}^TP_{ij} & 0 & -\delta_{ij}I & 0 & 0\\ D_i^TP_{ij} & 0 & 0 & -\gamma^2I & 0\\ 0 & 0 & 0 & 0 & -\delta_{ij}I \end{bmatrix}$$
$$U=\begin{bmatrix}I & 0 & 0 & 0 & 0\end{bmatrix},V_{ij}=\begin{bmatrix}B_{0i}^TP_{ij} & 0 & 0 & 0 & B_{2i}^T\end{bmatrix}$$

Using the projection lemma, we conclude that this nonlinear matrix inequality has solution if and only if

$$N_U^TG_{ij}N_U<0\ \text{ and }\ N_V^TG_{ij}N_V<0$$

Notice that for positive scalars ε_{ij}, δ_{ij},

$$N_U^T G_{ij} N_U = \begin{bmatrix} -\varepsilon_{ij} I & 0 & 0 & 0 \\ 0 & -\delta_{ij} I & 0 & 0 \\ 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & -\delta_{ij} I \end{bmatrix} < 0$$

always holds. Therefore the only constraint left is

$$N_V^T G_{ij} N_V < 0 \tag{19}$$

We have

$$N_V = diag\{P_{ij}^{-1}, I, I, I, I\} \cdot \begin{bmatrix} W_{1i} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ W_{2i} & 0 & 0 & 0 \end{bmatrix}$$

Substitute this equation into the inequality (19) to obtain,

$$\begin{bmatrix} W_{1i}^T M_{1ij} W_{1i} - \delta_{ij} W_{2i}^T W_{2i} & \varepsilon_{ij} W_{1i}^T A_{1i} & \delta_{ij} W_{1i}^T B_{1i} & W_{1i}^T D_i \\ \varepsilon_{ij} A_{1i}^T W_{1i} & -\varepsilon_{ij} I & 0 & 0 \\ \delta_{ij} B_{1i}^T W_{1i} & 0 & -\delta_{ij} I & 0 \\ D_i^T W_{1i} & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{20}$$

where

$$\begin{aligned} M_{1ij} = \; & A_{0i} P_{ij}^{-1} + P_{ij}^{-1} A_{0i}^T + P_{ij}^{-1} [\varepsilon_{ij}^{-1} A_{2i}^T A_{2i} + C_i^T C_i \\ & + \mathbf{1}_{\{i=j\}} (\textstyle\sum_{k \in S} \alpha_{ik} P_{kj}) + \mathbf{1}_{\{i \neq j\}} \beta_{ji} (P_{ii} - P_{ij})] P_{ij}^{-1} \end{aligned}$$

With M_{1ij} substituted in, the block matrix $W_{1i}^T M_{1ij} W_{1i} - \delta_{ij} W_{2i}^T W_{2i}$ in the left-hand-side of the above inequality can be expanded into the following equivalent matrix by using Schur complements:

$$\begin{bmatrix} M_{2ij} & W_{1i}^T P_{ij}^{-1} A_{2i}^T & W_{1i}^T P_{ij}^{-1} C_i^T & M_{3ij} \\ A_{2i} P_{ij}^{-1} W_{1i} & -\varepsilon_{ij} I & 0 & 0 \\ C_i P_{ij}^{-1} W_{1i} & 0 & -I & 0 \\ M_{3ij}^T & 0 & 0 & M_{4ij} \end{bmatrix}$$

where M_{2ij}, M_{3ij}, M_{4ij} are defined in the theorem.

Finally with simple matrix rearrangement, and definition $X_{ij} = P_{ij}^{-1}$, the LMI (20) is made equivalent to the LMI (17) given in the theorem. In conclusion, based on the projection lemma, the original nonlinear matrix inequality (18) holds if and only if the LMI (17) has feasible solutions. Hence the proof. □

Based on the result of Theorem 4.1, the controller only accessing FDI mode, can be obtained, by substituting the values of $P_{ij}, \varepsilon_{ij}, \delta_{ij}$ into the nonlinear matrix inequality (18). Obviously, it becomes an LMI with respect to K_j, and the feasible controllers K_j can be solved.

V. Output Feedback Controller Synthesis

In this section, the output feedback design will be briefly discussed. First we make following definitions:

$$K_j = \begin{bmatrix} \hat{A}_j & \hat{B}_j \\ \hat{C}_j & \hat{D}_j \end{bmatrix}, \bar{A}_{0i} = \begin{bmatrix} A_{0i} & 0 \\ 0 & 0 \end{bmatrix}, \bar{B}_{0i} = \begin{bmatrix} 0 & B_{0i} \\ I & 0 \end{bmatrix},$$

$$\bar{C}_i = \begin{bmatrix} 0 & I \\ C_i & 0 \end{bmatrix}, \bar{\Lambda}_{1i} = \begin{bmatrix} A_{1i} & B_{1i} \\ 0 & 0 \end{bmatrix}, \bar{A}_{2i} = \begin{bmatrix} A_{2i} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bar{B}_{2i} = \begin{bmatrix} 0 & 0 \\ 0 & B_{2i} \end{bmatrix}, \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \bar{C}_{1i} = \begin{bmatrix} C_i & 0 \end{bmatrix}, \bar{D}_i = \begin{bmatrix} D_i \\ 0 \end{bmatrix}$$

For output feedback case, we now have,

$$\begin{aligned} & P_{ij} \bar{A}_{0i} + \bar{A}_{0i}^T P_{ij} + P_{ij} \bar{B}_{0i} K_j \bar{C}_i + \bar{C}_i^T K_j^T \bar{B}_{0i}^T P_{ij} + P_{ij} \bar{\Lambda}_{1i} \Delta (\bar{A}_{2i} + \\ & \bar{B}_{2i} K_j \bar{C}_i) + (\bar{A}_{2i} + \bar{B}_{2i} K_j \bar{C}_i)^T \Delta^T \bar{\Lambda}_{1i}^T P_{ij} + \mathbf{1}_{\{i=j\}} (\textstyle\sum_{k \in S} \alpha_{ik} P_{kj}) \\ & + \mathbf{1}_{\{i \neq j\}} \beta_{ji} (P_{ii} - P_{ij}) \bar{C}_{1i}^T \bar{C}_{1i} + \gamma^{-2} P_{ij} \bar{D}_i \bar{D}_i^T P_{ij} < 0 \end{aligned}$$

Which is equivalent to: there exist scalars $\varepsilon_{ij} > 0$, such that the following matrix inequality holds for any $i, j \in S$,

$$\begin{aligned} & P_{ij} \bar{A}_{0i} + \bar{A}_{0i}^T P_{ij} + P_{ij} \bar{B}_{0i} K_j \bar{C}_i + \bar{C}_i^T K_j^T \bar{B}_{0i}^T P_{ij} + \varepsilon_{ij}^{-1} (\bar{A}_{2i} \\ & + \bar{B}_{2i} K_j \bar{C}_i)^T (\bar{A}_{2i} + \bar{B}_{2i} K_j \bar{C}_i) + \varepsilon_{ij} P_{ij} \bar{\Lambda}_{1i} \bar{\Lambda}_{1i}^T P_{ij} + \mathbf{1}_{\{i=j\}} (\textstyle\sum_{k \in S} \alpha_{ik} P_{kj}) \\ & + \mathbf{1}_{\{i \neq j\}} \beta_{ji} (P_{ii} - P_{ij}) + \bar{C}_{1i}^T \bar{C}_{1i} + \gamma^{-2} P_{ij} \bar{D}_i \bar{D}_i^T P_{ij} < 0 \end{aligned} \tag{21}$$

Similar to state-feedback situation, after using the Projection lemma, we have the following result.

Theorem 5.1: The necessary condition for the existence of output feedback controller is that there exist $\varepsilon_{ij} > 0$, $P_{ij} > 0$, $X_{ij} = P_{ij}^{-1}$ such that the following two matrix inequalities hold:

$$\begin{bmatrix} W_{3i}^T M_{5ij} W_{3i} & W_{3i}^T \bar{A}_{2i}^T & W_{3i}^T P_{ij} \bar{\Lambda}_{1i} & W_{3i}^T P_{ij} \bar{D}_i \\ \bar{A}_{2i} W_{3i} & -\varepsilon_{ij} I & 0 & 0 \\ \bar{\Lambda}_{1i}^T P_{ij} W_{3i} & 0 & -\varepsilon_{ij}^{-1} I & 0 \\ \bar{D}_i^T P_{ij} W_{3i} & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{22}$$

$$\begin{bmatrix} M_{6ij} & \varepsilon_{ij} W_{1i}^T \bar{\Lambda}_{1i} & W_{1i}^T \bar{D}_i & M_{3ij} \\ \varepsilon_{ij} \bar{\Lambda}_{1i}^T W_{1i} & -\varepsilon_{ij} I & 0 & 0 \\ \bar{D}_i^T W_{1i} & 0 & -\gamma^2 I & 0 \\ M_{3ij}^T & 0 & 0 & M_{4ij} \end{bmatrix} < 0 \tag{23}$$

where

$$\begin{aligned} \begin{bmatrix} W_{1i} \\ W_{2i} \end{bmatrix} & = ker(\begin{bmatrix} \bar{B}_{0i}^T & \bar{B}_{2i}^T \end{bmatrix}), \; W_{3i} = ker(\bar{C}_i) \\ M_{5ij} = \; & P_{ij} \bar{A}_{0i} + \bar{A}_{0i}^T P_{ij} + \mathbf{1}_{\{i=j\}} (\sum_{k \in S} \alpha_{ik} P_{kj}) \\ & + \mathbf{1}_{\{i \neq j\}} \beta_{ji} (P_{ii} - P_{ij}) + \bar{C}_{1i}^T \bar{C}_{1i} \\ M_{6ij} = \; & W_{1i}^T [X_{ij} \bar{A}_{0i}^T + \bar{A}_{0i} X_{ij} + (\mathbf{1}_{\{i=j\}} \alpha_{ii} - \mathbf{1}_{\{i \neq j\}} \beta_{ji}) \\ & X_{ij}] W_{1i} + W_{2i}^T \bar{A}_{2i} X_{ij} W_{1i} + W_{1i}^T X_{ij} \bar{A}_{2i}^T W_{2i} \\ & - \varepsilon_{ij} W_{2i}^T W_{2i} \end{aligned}$$

We notice that in the inequalities above, both P_{ij}, ε_{ij} and their inverse appeared. This difficult situation appears in the static output feedback (SOF) problem as well. Many algorithms have been proposed to solve this nonconvex problem, such as alternating projection, XY-centering, Min-Max algorithm and cone complementarity Linearization (CCL). where the comparison of these algorithms can be found in [16] and the an improved CCL algorithm is presented in [17]. And the objective function for minimization is $\sum_i \sum_j \{\text{trace}(P_{ij} X_{ij} - I) + \varepsilon_{ij} \bar{\varepsilon}_{ij} - 1\}$, where $\bar{\varepsilon}_{ij} = \varepsilon_{ij}^{-1}$. Based on this objective function, the corresponding SLPMM algorithm can be constructed similarly as in [17]. Due to the page limit, the details of the algorithm is omitted here.

After solving these two matrix inequalities, controllers can be solved in exactly the same way as the state feedback case.

VI. Numerical Example

Consider a second-order system, $S = \{1, 2\}$. (A_1, B_1, C_1) is assumed to be the normal system model and (A_2, B_2, C_2)

is a faulty one:

$$A_1 = A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 \\ -0.25 & 0.25 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0.2 \\ -0.25 & 0.05 \end{bmatrix}, C_1 = C_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

The weighting matrices for the disturbance, and the bounds for $\Delta B_i,\ i = 1,\ 2$, the model uncertainties on B matrices, are given as:

$$D_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, D_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, A_{ij} = B_{ij} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},\ i, j \in S$$

The transition rate matrix for failure Markov chain is chosen as: $\alpha = \begin{bmatrix} -0.5 & 0.5 \\ 1 & -1 \end{bmatrix}$, $2 \leq \beta_{12} \leq 3,\ 2.5 \leq \beta_{21} \leq 3.5$.

For this system, a state feedback control is designed by solving the LMIs developed in the previous section. By pre-setting the disturbance attenuation level as $\gamma = 0.8$, the solutions are obtained as:

$$K_1 = \begin{bmatrix} 2.4563 & 29.4954 \\ -21.0010 & -7.5161 \end{bmatrix}, K_2 = \begin{bmatrix} 1.9998 & 24.5445 \\ -12.4093 & -3.2219 \end{bmatrix}$$

By using the first set of controllers, a single sample path simulation is performed, and the results are shown in Fig. 1. The disturbance is modeled as $w = e^{-0.1t}\sin t$. It is calculated that $\|y\|_2 = 0.3202,\ \|w\|_2 = 1.5767$. Obviously, the desirable disturbance attenuation is achieved.

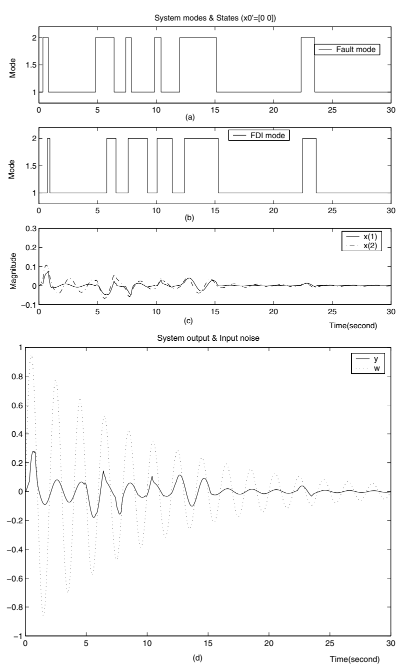

Fig. 1. Single sample path simulation: (a) system modes; (b) FDI modes; (c) system state response; (d) the system output and disturbance

VII. Conclusion

In this paper, we focus our discussions on the design of stochastic Fault Tolerant Control in the presence of random FDI delays. The fault/failure of the system is described by a continuous-time Markov chain, and the FDI delay is represented by another exponentially distributed random variable. The main difficulty in the design lies in the fact that the mode of controller is solely dependent on the mode of the FDI process, whose transitions depend on those of the system failure mode as well as the FDI delay. The sufficient conditions for the desirable Mean Exponential Stability and H_∞ performance are obtained for the system with modeling uncertainties and external disturbances. By transforming the given nonlinear matrix inequalities into the tractable two-step LMI formulations, state feedback controllers can be solved conveniently by using the available commercial software packages. An illustrative example is given at last and simulation results are shown to demonstrate the effectiveness of the proposed design algorithm.

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