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Output Dependent Observability Linear Normal Form

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Abstract— **This paper gives the sufficient and necessary conditions which guarantee the existence of a diffeomorphism and an output injection in order to transform a nonlinear system in a 'canonical' normal form depending on its output. We propose two methods: one is based on the commutativity via the Lie brackets of a family of vector fields and the second one is the dual of the first one, based on the closure of a family of $1-$forms.**

I. Introduction

In control theory, the design of an observer for linear system is well-known. Motivated by this, the so-called problem of observability linearization by means of a diffeomorphism and an output injection of nonlinear system was proposed. Till now, the linearization problem has been extensively studied in [9] [10] [7] [12]. The Sufficient and necessary geometrical conditions, which guarantee the existence of a diffeomorphism and an output injection to transform a nonlinear system into the canonical linear form, were first addressed in [9] and [10]. As these conditions are too restrictive, an analytical approach is introduced in [13] and is generalized by Krener and Xiao [11]. Other approaches using quadratic normal forms were given in [2], [3], [4] [5]. All these approaches allow us to design an observer for a larger class of nonlinear systems.

Recently, [8] gave the sufficient and necessary geometrical conditions to transform a nonlinear system to the so-called output-dependent time scaling linear canonical form. In [6] the author gave independently the dual geometrical conditions of [8].

In this paper, we will propose two different methods to deduce the geometrical conditions which are sufficient and necessary to guarantee the existence of a local diffeomorphism and an output injection to transform a nonlinear system in a 'canonical' normal form depending on its output. This kind of linearization will be called Output Dependent Observability linear normal form (ODO linear normal form).

This paper is organized as follows. Notations, definition and problem statement are given in section 2. In Section 3, we present our first method to deduce the geometrical conditions for a nonlinear system to be transformed to ODO linear normal form. The dual result, based on the closure of a family of $1-$forms, is given in section 4. Throughout the paper, an example is discussed to highlight our propositions.

II. Notations and problem statement

Consider the following system:

$$\left\{ \begin{array}{l} \dot{x} = f(x) \\ y = h(x) \end{array} \right. \tag{1}$$

where $U \sqsubset \Re^n$ is the set of admissible state, $f : U \sqsubseteq \mathbb{R}^n \to \mathbb{R}^n$ and $h : U \sqsubseteq \mathbb{R}^n \to \mathbb{R}$ are analytic.

Assume that for all $x \in U$ the codistribution $span\left\{dh, dL_f h, ..., dL_f^{n-1} h\right\}$ is of rank n. Set $\theta_i = dL_f^{i-1} h$ for $1 \leq i \leq n$ and $\theta = \left(\begin{array}{ccc} \theta_1 & .... & \theta_n \end{array} \right)^T$.

We call the components of θ the observability $1-$forms, and they form a basis of the cotangent bundle T^*U of U.

Let g_1 be the vector field defined by:

$$\left\{ \begin{array}{lll} \theta_i (g_1) = 0 & \text{for} & 1 \leq i \leq n-1 \\ \theta_n (g_1) = 1 & & \end{array} \right.$$

and define $g_k = (-1)^{k-1} ad_f^{k-1} (g_1)$ for $2 \leq k \leq n$. It is clear that $\{g_1, ..., g_n\}$ form a basis of the tangent bundle TU of U. Moreover, we have:

$$\theta (g_1, ..., g_n) = \left(\begin{array}{ccccc} 0 & 0 & ... & 0 & 1 \\ 0 & \vdots & ... & 1 & l_{2,n} \\ \vdots & ... & 1 & ... & \vdots \\ \vdots & 1 & ... & ... & \vdots \\ 1 & l_{n,2} & ... & ... & l_{n,n} \end{array} \right) := \Lambda$$

where

$$l_{k,j} = \theta_k(g_j) \tag{2}$$

with $2 \leq k \leq n$ and $n-k+2 \leq j \leq n$

Definition 1: We say that a dynamical system is in Output Dependent Observability linear normal form (ODO linear normal form) along its output trajectory $y(t)$ if it is in the following form:

$$\left\{ \begin{array}{l} \dot{z} = A(y)z + \beta(y) \\ y = z_n = Cz \end{array} \right. \tag{3}$$

where

$$A(y) = \left(\begin{array}{ccccc} 0 & 0 & ... & 0 & 0 \\ \alpha_1(y) & 0 & ... & 0 & 0 \\ 0 & \alpha_2(y) & 0 & ... & \vdots \\ \vdots & .. & ... & ... & \vdots \\ 0 & ... & 0 & \alpha_{n-1}(y) & 0 \end{array} \right)$$

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and $\alpha_i(y) \neq 0$ for $y \in]-a, a[$ and $a > 0$.

Without lose of generality we can assume that $\alpha_i(y) > 0$ on $]-a, a[$ for all $i = 1 : n-1$. Indeed, if for a certain k we have $\alpha_k < 0$ then we take $-z_{k+1}$ instead of z_{k+1}.

Remark 1: We just consider the linear output in this normal form as the work in [9], but it's not necessary to set the output as a linear one. A function of the linear output such as $h(Cz)$, even a nonlinear output such as $h(z)$ in [13] would be considered in the future.

Before to state the problem which we will deal with in this paper, we first give some motivations to study systems in the ODO linear normal form.

The main motivation for ODO linear normal form is to generalize the linearization theorem in [9], [4] and [6]. Moreover, we also want to apply this normal form into cryptography since that it introduces the observability bifurcations into the matrix $A(y)$ which can be use to increase the robustness of the cryptosystem. In addition, it's obvious that we can design observes for this kind of normal form.

Indeed, for system (3), there exist many ways to design an observer. We can construct a step by step sliding mode observer (see [1]) for the system (3) as follows:

$$\dot{\hat{z}} = A(y)\hat{z} + \beta(y) + \begin{pmatrix} E_{n-1}\kappa_n sign\left(\tilde{z}_1 - \hat{z}_1\right) \\ E_{n-2}\kappa_{n-1} sign\left(\tilde{z}_2 - \hat{z}_2\right) \\ \vdots \\ E_1\kappa_2 sign\left(\tilde{z}_{n-1} - \hat{z}_{n-1}\right) \\ \kappa_1 sign\left(z_n - \hat{z}_n\right) \end{pmatrix}$$

with the auxiliary states:

$$\begin{cases} \tilde{z}_{n-1} = \hat{z}_{n-1} + E_1\kappa_1 \frac{sign(z_n - \hat{z}_n)}{\alpha_{n-1}(y)} \\ \tilde{z}_{n-2} = \hat{z}_{n-2} + E_2\kappa_2 \frac{sign(\tilde{z}_{n-1} - \hat{z}_{n-1})}{\alpha_{n-2}(y)} \\ \vdots \\ \tilde{z}_1 = \hat{z}_1 + E_{n-1}\kappa_{n-1} \frac{sign(\tilde{z}_2 - \hat{z}_2)}{\alpha_1(y)} \end{cases}$$

where

$$\begin{array}{l} \text{if } z_n = \hat{z}_n, E_1 = 1, \text{ else } E_1 = 0 \\ \text{if } \tilde{z}_{n-1} = \hat{z}_{n-1} \text{ and } E_1 = 1, E_2 = 1, \text{ else } E_2 = 0 \\ \vdots \\ \text{if } \tilde{z}_2 = \hat{z}_2 \text{ and } E_{n-2} = 1, E_{n-1} = 1, \text{ else } E_{n-1} = 0 \end{array}$$

So, if all observation errors are bounded and all $\alpha_i \neq 0$ then it is possible to find $\kappa_1, ..., \kappa_n$ such that the observation errors go to zero in finite time.

Obviously, there exist many other observers which work for system (3), but the problem which we consider here is the following.

Problem statement: When does there exist a local diffeomorphism and an output injection which transform system (1) to the ODO linear normal form?

We answer this question in two ways: in next section we give the sufficient and necessary conditions using a family vector fields associated with the nonlinear system and in the following section we give the dual of the first method which give us the diffeomorphism.

III. MAIN RESULT

In this section, we will use Lie brackets commutativity of a family of vector fields to give the sufficient and necessary conditions to solve our problem.

Theorem 1: System (1) can be transformed to ODO linear normal form (3) by a diffeomorphism and an output injection if and only if there exist a family of functions $(\alpha_i(y))_{1 \leq i \leq n-1}$ such that the following family of vector fields: $\widetilde{g}_1 = \pi_1 g$ and $\widetilde{g}_2 = \frac{1}{\alpha_1}[\widetilde{g}_1, f], ..., \widetilde{g}_n = \frac{1}{\alpha_{n-1}}[\widetilde{g}_{n-1}, f]$ satisfy to the commutativity conditions:

$$[\widetilde{g}_i, \widetilde{g}_j] = 0 \text{ for } 1 \leq i, j \leq n$$

where $\pi_1 = \alpha_1 ... \alpha_{n-1}$.

Remarks 1: 1) If α_i for all $1 \leq i \leq n-1$ are constants, so their derivatives are zero, then we get $[g_i, g_j] = 0$ for $1 \leq i, j \leq n$, so we obtain the linearization theorem of ([9]).

2) If $\alpha_i = s(y)$ for all $1 \leq i \leq n-1$ then we obtain the result stated in [8] and [6].

Proof: Necessary condition:

For system in the form (3), it is easy to show that $g_1 = \frac{1}{\pi_1}\frac{\partial}{\partial z_1}$ which yields that $\widetilde{g}_1 = \frac{\partial}{\partial z_1}$ and then by construction we obtain $\widetilde{g}_i = \frac{\partial}{\partial z_i}$ for all $2 \leq i \leq n$.

Sufficient condition:

Assume that there exist $(\alpha_i(y))_{1 \leq i \leq n-1}$ such that:

$$[\widetilde{g}_i, \widetilde{g}_j] = 0 \text{ for } 1 \leq i, j \leq n$$

then there exists a local diffeomorphism $\phi = z$ such that

$$d\phi(\widetilde{g}_i) = \frac{\partial}{\partial z_i}$$

As $d\phi$ is a multiclosed $1-$forms, and $d\phi(\widetilde{g}_i) = \frac{\partial}{\partial z_i}$ is constant, so $\frac{\partial}{\partial z_i}d\phi(f) = d\phi\left([\widetilde{g}_i, f]\right) = \alpha_i d\phi(\widetilde{g}_{i+1}) = \alpha_i \frac{\partial}{\partial z_{i+1}}$, thus $\frac{\partial}{\partial z_i}d\phi(f) = \alpha_i \frac{\partial}{\partial z_{i+1}}$ for $1 \leq i \leq n-1$. Thus, by integration we obtain: $d\phi(f) = A(y)z + \beta(y)$. ■

It's natural to ask how to compute such functions $(\alpha_i(y))_{1 \leq i \leq n-1}$? The following criterion, in many cases, yields us to determine functions $(\alpha_i(y))_{1 \leq i \leq n-1}$.

Criterion 1: If system (1) can be transformed to ODO linear normal form then there exist functions $\lambda_1, ..., \lambda_{n-1}$ such that:

$$\begin{array}{l} [g_1, g_n] = \lambda_1 g_1 \\ [g_k, g_n] = \lambda_k g_k \ mod\ [g_1, ..., g_{k-1}] \\ \text{for all } 2 \leq k \leq n-1 \end{array}$$

where

$$\begin{array}{rcl} \lambda_{n-k} & = & (-1)^{k+1}\left(C_{k-1}^{k}\frac{\pi'_{n-k-1}}{\pi_{n-k-1}} + C_{k-1}^{k+1}\frac{\pi'_{n-k-2}}{\pi_{n-k-2}} \right. \\ & & \left. +.... + C_{k-1}^{n-1}\frac{\pi'_1}{\pi_1}\right) + \left(\frac{\pi'_{n-k}}{\pi_{n-k}} + \right. \\ & & \left. (-1)^{k+1}\frac{\pi'_{n-k}}{\pi_{n-k}}\right) \quad \text{for } 1 \leq k \leq n-2 \\ \lambda_1 & = & (-1)^n\frac{\pi'_1}{\pi_1} + \frac{\pi'_1}{\pi_1} \end{array} \qquad (4)$$

and $\pi_k = \prod_{i=1}^{n-k} \alpha_{n-i}$ for all $1 \le k \le n-1$, and $\pi'_k = \frac{d\pi_k}{dy}$.

Before, to proof this criterion, we will give some remarks on the family $(\lambda_i)_{1 \le i \le n-1}$.

Remarks 2: 1) Until the compilation of this paper we don't know if equation (4) is sufficient to find the family $(\alpha_i(y))_{1 \le i \le n-1}$. Indeed in (4), we can have the same equations for different λ_i. If n is odd then $\lambda_1 = 0$, thus for $n = 2k+1$ we have at most $n-2$ equations, and as we will show it in our example for $n = 4$, we have $\lambda_3 = -\lambda_2$. However, in many cases, by identification, the writing of equation (4) allows us to give $(\alpha_i(y))_{1 \le i \le n-1}$.

2) Lie brackets $[g_i, g_j]$ for $1 \le i, j \le n-1$ don't give us more equations than those obtained in (4). Indeed, if we set $B = (b_{i,j})_{1 \le i,j \le n} = ([g_i, g_j])_{1 \le i,j \le n}$, then B is an antisymmetric matrix and

$$b_{i,i} = b_{i,j} = 0 \text{ for all } i \in [1, n-1] \text{ and } j \in [1, n-i]$$

and by using the Jacobi's identity we have:

$$\begin{aligned} [g_2, g_{n-1}] &= [[g_1, f], g_{n-1}] \\ &= (-[[g_{n-1}, g_1], f] \\ &\quad + [g_1, g_n]) \end{aligned}$$

As $[g_{n-1}, g_1] = 0$, we have

$$[g_2, g_{n-1}] = [g_1, g_n]$$

By the same argument we obtain:

$$b_{1+k,n-k} = \lambda_1 g_1 \quad \text{for } k = 1 : p$$

where $p = \frac{n+1}{2}$ if n is odd else $p = \frac{n}{2}$.

To show that the other $b_{i,j}$ provide equations which are combinations of λ_k, we use over and over the Jacobi's identity. For example for $b_{3,n-1}$, we have:

$$\begin{aligned} b_{3,n-1} = [g_3, g_{n-1}] &= [[g_2, f], g_{n-1}] \\ &= -[[g_{n-1}, g_2], f] + [g_2, g_n] \\ &= -[[g_1, g_n], f] + \lambda_2 g_2 \ mod\ [g_1] \\ &= \lambda_2 g_2 \ mod\ [g_1] - [\lambda_1 g_1, f] \\ &= \lambda_2 g_2 \ mod\ [g_1] - \lambda_1 [g_1, f] \\ &\quad + (L_f \lambda_1) g_1 \\ &= (\lambda_2 - \lambda_1) g_2 \ mod\ [g_1]. \end{aligned}$$

Proof: For system (3) we compute g_1 and g_2 and we get:

$$\begin{aligned} g_1 &= \frac{1}{\pi_1} \frac{\partial}{\partial z_1} \\ g_2 &= \frac{1}{\pi_2} \frac{\partial}{\partial z_2} + \frac{\pi'_1}{\pi_1^2} \alpha_{n-1} z_{n-1} \frac{\partial}{\partial z_1} \end{aligned}$$

Now, as for $3 \le k \le n$ we have $g_k = [g_{k-1}, f]$ then by induction we obtain:

$$\begin{aligned} g_k &= \tfrac{1}{\pi_k} \tfrac{\partial}{\partial z_k} \\ &+ \begin{pmatrix} \frac{\pi'_{k-1}}{\pi_{k-1}^2} + \frac{\pi'_{k-2}}{\pi_{k-2}^2} \alpha_{k-2} + \\ \frac{\pi'_{k-3}}{\pi_{k-3}^2} \alpha_{k-1} \alpha_{k-2} + ... \\ + \frac{\pi'_1}{\pi_1^2} \alpha_1 \alpha_2 ... \alpha_{k-2} \end{pmatrix} \pi_{n-1} z_{n-1} \tfrac{\partial}{\partial z_{k-1}} \\ &- \begin{pmatrix} \frac{\pi'_{k-2}}{\pi_{k-2}^2} + C_1^2 \frac{\pi'_{k-3}}{\pi_{k-3}^2} \alpha_{k-2} + \\ C_1^3 \frac{\pi'_{k-4}}{\pi_{k-4}^2} \alpha_{k-2} \alpha_{k-1} + ... \\ + C_1^{(k-2)} \frac{\pi'_1}{\pi_1^2} \alpha_1 \alpha_2 ... \alpha_{k-2} \end{pmatrix} \pi_{n-2} z_{n-2} \tfrac{\partial}{\partial z_{k-2}} \\ &+ \begin{pmatrix} \frac{\pi'_{k-3}}{\pi_{k-3}^2} + C_2^3 \frac{\pi'_{k-4}}{\pi_{k-4}^2} \alpha_{k-4} \\ + ... \\ + C_2^{(k-2)} \frac{\pi'_1}{\pi_1^2} \alpha_1 \alpha_2 ... \alpha_{k-2} \end{pmatrix} \pi_{n-3} z_{n-3} \tfrac{\partial}{\partial z_{k-3}} \\ &\cdots \\ &+ (-1)^k \tfrac{\pi'_1}{\pi_1^2} \pi_{n-k+1} z_{n-k+1} \tfrac{\partial}{\partial x_1} \\ &+ R_{k,1}(z_n, z_{n-1}, .., z_{n-k+2}) g_1 + ... \\ &+ R_{k,k-2}(z_n, z_{n-1}^2) g_{k-2} \end{aligned} \tag{5}$$

From this it is easy to compute all Lie Brackets $[g_k, g_n]$ for $1 \le k \le n-1$ and deduce equation (4). ■

Here, a simple example is studied in order to illustrate the previous theorem. Moreover, it allows us to highlight our concern stressed in Remarks 2.

Example 1: Consider the following system:

$$\begin{cases} \dot{x}_1 = \frac{\gamma(y)}{1+x_4} x_1 x_3 \\ \dot{x}_2 = \frac{\beta(y)}{1+x_4} x_1 \\ \dot{x}_3 = \mu(y) x_2 \\ \dot{x}_4 = \gamma(y) x_3 \end{cases} \quad y = x_4 \tag{6}$$

Its family vector fields is:

$$\begin{aligned} g_1 &= \frac{1+x_4}{\gamma\mu\beta} \frac{\partial}{\partial x_1} \\ g_2 &= \frac{1}{\gamma\mu} \frac{\partial}{\partial x_2} + \left(\frac{x_3}{\mu\beta} - \gamma x_3 \left(\frac{1+x_4}{\gamma\mu\beta} \right)' \right) \frac{\partial}{\partial x_1} \\ g_3 &= \frac{1}{\gamma} \frac{\partial}{\partial x_3} + \left(\frac{(\gamma\mu)'}{(\gamma\mu)^2} + \beta \frac{(\gamma\mu\beta)'}{(\gamma\mu\beta)^2} \right) \gamma x_3 \frac{\partial}{\partial x_2} \\ &\quad -\gamma\mu x_2 (1+x_4) \frac{(\gamma\mu\beta)'}{(\gamma\mu\beta)^2} \frac{\partial}{\partial x_1} + R_{1,3} g_1 \\ g_4 &= \left(\frac{\gamma'}{\gamma^2} + \left(\frac{(\gamma\mu)'}{(\gamma\mu)^2} + \beta \frac{(\gamma\mu\beta)'}{(\gamma\mu\beta)^2} \right) \mu \right) \gamma x_3 \frac{\partial}{\partial x_3} \\ &\quad + \frac{\partial}{\partial x_4} - \left(\frac{(\gamma\mu)'}{(\gamma\mu)^2} + 2\beta \frac{(\gamma\mu\beta)'}{(\gamma\mu\beta)^2} \right) \gamma\mu x_2 \frac{\partial}{\partial x_2} \\ &\quad + \left(\frac{1}{1+x_4} + \frac{(\gamma\mu\beta)'}{\gamma\mu\beta} \right) x_1 \frac{\partial}{\partial x_1} \\ &\quad + R_{1,4}(z_3, z_2) g_1 + R_{2,3}(z_3^2) g_2 \end{aligned}$$

From which a straight computation gives:

$$\begin{aligned}\lambda_1 &= \frac{(\gamma\mu\beta)'}{\gamma\mu\beta} + \frac{(\gamma\mu)'}{\gamma\mu} + 2\frac{\gamma'}{\gamma} \\ \lambda_2 &= -2\frac{(\gamma\mu\beta)'}{\gamma\mu\beta} \\ \lambda_3 &= 2\frac{(\gamma\mu\beta)'}{\gamma\mu\beta}\end{aligned}$$

which by identification with equation (4) yields to $\alpha_1 = \beta, \alpha_2 = \mu$ and $\alpha_3 = \gamma$, so

$$\begin{aligned}\widetilde{g}_1 &= (1+x_4)\frac{\partial}{\partial x_1} \\ \widetilde{g}_2 &= \frac{\partial}{\partial x_2} \\ \widetilde{g}_3 &= \frac{\partial}{\partial x_3} \\ \widetilde{g}_4 &= \frac{\partial}{\partial x_4} + \frac{x_1}{1+x_4}\frac{\partial}{\partial x_1}\end{aligned}$$

It is clear that $[\widetilde{g}_i, \widetilde{g}_j] = 0$ for all $1 \leq i, j \leq 4$. Therefore the two conditions of theorem 1 are fulfilled, thus system (6) can be transformed to ODO linear normal form.

Theorem 1 allows us to know whether a nonlinear system can be or not transformed to ODO linear normal form. However, it doesn't give the diffeomorphism for transforming system (1) to ODO linear normal form. The following section deals with this question.

IV. The dual version and the diffeomorphism

In this section, we propose another method, based on the closure of a family of 1−forms, to give the diffeomorphism which transforms system (1) to ODO linear normal form.

Considering the family of vector fields $(\widetilde{g}_i)_{1\leq i\leq n}$ defined in theorem 1 it is easy to show that:

$$\theta(\widetilde{g}_1, ..., \widetilde{g}_n) = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ 0 & \vdots & ... & \pi_{n-2} & l_{2,n} \\ \vdots & ... & .. & ... & \vdots \\ \vdots & \pi_2 & ... & ... & \vdots \\ \pi_1 & l_{n,2} & ... & ... & l_{n,n} \end{pmatrix} := \widetilde{\Lambda}$$

where

$\widetilde{l}_{k,j} = \theta_k(\widetilde{g}_j)$
with $2 \leq k \leq n$ and $n-k+2 \leq j \leq n$

Set

$$\omega = \widetilde{\Lambda}^{-1}\theta := (\omega_1, \omega_2, ..., \omega_n)^T \tag{7}$$

where, for all $1 \leq s \leq n$, we have:

$$\omega_s = \sum_{m=1}^{n} r_{s,m}\theta_m \tag{8}$$

Then the following algorithm gives all components of ω:

Algorithm 1:

for all $1 \leq j \leq n$
$r_{n,j} = ... = r_{n-j+2,j} = 0$ and $r_{n-j+1,j} = 1$
for $2 \leq k \leq n-1$ and $1 \leq j \leq n$
$r_{n-k,j} = -\sum_{i=2}^{k} \widetilde{l}_{k,n-k+i-(j-1)} r_{n-k+i-(j-1),j}$

and then (8) become: $\omega_s = \sum_{m=1}^{n-s+1} r_{s,m}\theta_m$.

Theorem 2: System (1) can be transformed to ODO linear normal form (3) by a diffeomorphism and an output injection if and only if there exist a family functions $(\alpha_i(y))_{1\leq i\leq n-1}$ such that the multi 1−form ω given in (7) is such that $d\omega = 0$.

Moreover, the diffeomorphism which transforms (1) to (3) is given locally by $\phi(x) = z$ where

$$z_i = \phi_i(x) = \int_\gamma \omega_i + \phi_i(0) \quad \text{for all } 1 \leq i \leq n$$

where γ is a smooth path from 0 to x lying in a neighborhood $V_0 \subseteq U$ of 0.

Proof: To prove this theorem we will show that the two following conditions are equivalent:

$i)\ [\widetilde{g}_i, \widetilde{g}_j] = 0, \forall i, j \in [1, n]$
$ii) d\omega = 0$

Recall that for any two vector fields X, Y we have:

$$d\omega(X, Y) = L_X(\omega(Y)) - L_Y(\omega(X)) - \omega([X, Y])$$

Now set $X = \widetilde{g}_i$ and $Y = \widetilde{g}_j$, we obtain:

$$d\omega(\widetilde{g}_i, \widetilde{g}_j) = L_{g_i}\omega(\widetilde{g}_j) - L_{g_j}\omega(\widetilde{g}_i) - \omega([\widetilde{g}_i, \widetilde{g}_j])$$

As $\omega(\widetilde{g}_j)$ and $\omega(\widetilde{g}_i)$ are constants, then we have

$$d\omega(\widetilde{g}_i, \widetilde{g}_j) = -\omega([\widetilde{g}_i, \widetilde{g}_j])$$

As ω is an isomorphism and $(\widetilde{g}_i)_{1\leq i\leq n}$ is a basis of TU, then condition $i)$ is equivalent to condition $ii)$. ■

The following gives another way to find equations in $(\alpha_i)_{1\leq i\leq n-1}$.

Criterion 2: If system (1) can be transformed to ODO linear normal form (3) by means of a diffeomorphism and an output injection then there exist functions $\sigma_1, ..., \sigma_{n-1}$ such that:

$d[\theta_n(g_k)] = \sigma_{k-1}\theta_k \ mod\ [\theta_1, ..., \theta_{k-1}]$
for all $2 \leq k \leq n-1$

Remarks 3: $i)$ The multi 1−form ω can be viewed as an isomorphism $TU^n \to U \times \mathbb{R}^n$ which sends each $\widetilde{g}_i$ to the canonical vector basis $\frac{\partial}{\partial z_{n-i+1}}$. Moreover, $d\omega = 0$ means that locally there is $\phi : U \to U$ such that ω is the tangent map of ϕ.

$ii)$ The family of $(\sigma_i)_{1\leq i\leq n-1}$ is linearly dependent of the family of $(\lambda_i)_{1\leq i\leq n-1}$.

$iii)$ For a given $k = 2 : n-1$ we have:

$$d\theta_n(g_k) = dl_{n,k}$$

where $l_{k,j}$ for $0 \leq i \leq n-2$ are given in matrix Λ (2). The other components of Λ don't yield other equations than those we obtain in (4). Indeed,

$$\begin{aligned} d\theta_n\left(g_k\right) &= d\theta_{n-i}\left(g_{k+i}\right) \\ &= dl_{n-i,k+i} \\ &= \sigma_{k-1}\theta_k mod[\theta_1, ..., \theta_{k-1}] \end{aligned}$$

$\theta_{n-1}\left(g_3\right) = \theta_{n-1}\left([g_2, f]\right) = \theta_n\left(g_2\right) - L_f\theta_{n-1}\left(g_2\right) = \theta_n\left(g_2\right)$ because $\theta_{n-1}\left(g_2\right) = 1$ is constant. By using inductively the same argument over and over we obtain the above formula.

Now, $l_{n-1,4} = \theta_{n-1}\left(g_4\right) = \theta_{n-1}\left([g_3, f]\right) = \theta_n\left(g_3\right) - L_f\theta_{n-1}\left(g_3\right) = \theta_n\left(g_3\right) - \theta_n\left(g_2\right)$. So, $dl_{n-1,4} = \sigma_2\theta_3 mod[\theta_1, \theta_2] - L_f\left(\sigma_2\theta_2 mod[\theta_1]\right) = \sigma_2\theta_3 mod[\theta_1, \theta_2] - \left(\left(\left(L_f\sigma_2\right)\theta_2 + \sigma_1 L_f\theta_2\right) mod[\theta_1]\right)$. Thus we have:

$$dl_{n-1,4} = \left(\sigma_2 - \sigma_1\right)\theta_3 \ mod \ [\theta_1, \theta_2].$$

In the same way we can show that the other components of $d\Lambda$ don't give us more equations than those given by $\left(dl_{n,i}\right)_{2 \leq i \leq n-1}$ in σ_i.

Example 2: (example 1 continued)

The observability $1-$forms of (6):

$$\begin{aligned} \theta_1 &= dx_4 \\ \theta_2 &= \gamma dx_3 + \gamma' x_3 dx_4 \\ \theta_3 &= \gamma\mu dx_2 + 2\gamma'\gamma x_3 dx_3 \\ &\quad + \left(\left(\gamma\mu\right)' x_2 + 2\left(\gamma'\gamma\right)' x_3^2\right) dx_4 \\ \theta_4 &= \gamma\mu \ \frac{\beta}{1+x_4} dx_1 + \left(2\gamma'\mu + \left(\gamma\mu\right)'\right)\gamma x_3 dx_2 \\ &\quad + \left(2\gamma'\gamma\mu x_2 + \gamma\left(\gamma\mu\right)' x_2 + 3\gamma\left(\gamma'\gamma\right)' x_3^2\right) dx_3 \\ &\quad + r\theta_1 \end{aligned}$$

According to criterion 2, we get:

$$\begin{aligned} \sigma_1 &= \frac{\left(\gamma\mu\beta\right)'}{\gamma\mu\beta} + \frac{\left(\gamma\mu\right)'}{\gamma\mu} + 2\frac{\gamma'}{\gamma} \\ \sigma_2 &= -\frac{\left(\gamma\mu\beta\right)'}{\gamma\mu\beta} + \frac{\left(\gamma\mu\right)'}{\gamma\mu} + 2\frac{\gamma'}{\gamma} \\ \sigma_3 &= 2\frac{\left(\gamma\mu\beta\right)'}{\gamma\mu\beta} \end{aligned}$$

It is clear that $\lambda_1 = \sigma_1$, $\lambda_1 + \lambda_2 = \sigma_2$ and $\lambda_3 = \sigma_3$.

After some computations we have:

$$\widetilde{\Lambda} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \gamma & \gamma' x_3 \\ 0 & \gamma\mu & 2\gamma'\gamma x_3 & \left(\gamma\mu\right)' x_2 + 2\left(\gamma'\gamma\right)' x_3^2 \\ \gamma\mu\beta & \left(2\gamma'\gamma\mu x_3 + \left(\gamma\mu\right)'\gamma x_3\right) & \begin{pmatrix} 2\gamma'\gamma\mu x_2 + \gamma\left(\gamma\mu\right)' x_2 \\ +3\gamma\left(\gamma'\gamma\right)' x_3^2 \end{pmatrix} & \gamma\mu\frac{x_1\beta}{\left(1+x_4\right)^2} + r \end{pmatrix}$$

A straight computation gives

$$\omega = \widetilde{\Lambda}^{-1}\theta = \begin{pmatrix} d\left(\frac{x_1}{1+x_4}\right) \\ dx_2 \\ dx_3 \\ dx_4 \end{pmatrix}$$

Thus $\omega = d\phi$ where $\phi(x) = z = \left(\frac{x_1}{1+x_4}, x_2, x_3, x_4\right)^T$, and so we obtain the diffeomorphism coordinate.

V. Conclusion

In this paper we propose the geometrical conditions which allow us to determine whether a nonlinear system can be or not transformed locally to ODO linear normal form by means of a diffeomorphism and an output injection. In addition, two equivalent results are given. In the first one, we use Lie brackets commutativity and the second one is based on the closure of a family of $1-$forms. Moreover the last solution gives explicitly the diffeomorphism coordinate.

However, one question remained to solve is to find more equations to determine functions $\left(\alpha_i\right)_{1 \leq i \leq n-1}$. Finally, this paper only deals with the system with only single input single output, and the case of multi-inputs multi-outputs is under development.

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