Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Overlapping Resilient H_∞ Control for Uncertain Time-Delayed Systems

Lubomír Bakule, José Rodellar and Josep M. Rossell

Abstract— **The paper presents the expansion-contraction relations within the Inclusion Principle for a class of continuous-time state-delayed uncertain systems when considering H_∞ state memoryless control with additive controller uncertainty. All uncertainties are supposed to be time-varying norm bounded. The main contribution is the derivation of conditions under which a resilient H_∞ control law designed in the expanded space is contracted into the initial system preserving simultaneously both the robust quadratic stability of closed-loop systems and the value of the H_∞-norm disturbance attenuation bound. A LMI delay independent procedure is supplied for control design. The results are specialized into the overlapping decentralized control setting which enables to construct robust H_∞ resilient block tridiagonal state feedback controllers.**

I. INTRODUCTION

Standard assumption on designed controllers is that they can be implemented exactly into real world systems. In practice, control laws designed using theoretical methods and simulations are implemented imprecisely because of various reasons such as finite word length in any digital system, round-off errors in the imprecision inherent in analog systems or the need for additional tuning of parameters in the final controller implementation. The controller designed for uncertain plants may be sufficiently robust against system parameters, but the controller parameters itself may be sensitive to relatively small perturbations and could even destabilize a closed-loop system. The importance of fragility, i.e. high sensitivity of controller parameters on its very small changes, is underlined when considering large-scale complex systems controlled by low cost local controllers. Such control systems are generally characterized by uncertainties, information structure constraints, delays, and high dimensionality. This situation naturally motivates the development of new effective control design methods taking into account particular features of these systems including implementation aspects of controller parameter uncertainties.

A. Relevant References

Robustness against model parameter uncertainty has been intensively studied for many years. Recent papers starting with [1], [2], [3] point out the possible fragility of such robust controllers. To cope with this difficulty, several types of controller uncertainties have considered. Fragility within an additive uncertainty for state feedback controllers is considered in [4] while [5] extends this issue on H_∞ robust state feedback control for time-delayed systems. A multiplicative uncertainty for H_∞ controllers considers [6]. Guaranteed cost control approach presents [7]. LMI results for H_∞ state feedback controller for both types of controller uncertainties considers [8]. Digital controller implementation and fragility issues studies [9].

Three different practically important forms of the gain matrix are usually considered when considering information structure constraints, i.e. a block diagonal gain matrix, a block tridiagonal gain matrix, and a double block bordered gain matrix. A systematic way of the controller design with a block tridiagonal gain matrix lead to the concept of overlapping decompositions. A general mathematical framework for this approach has been called the Inclusion Principle [10], [11]. The Inclusion Principle has been applied to different classes of systems and problems as illustrated for instance in [12], [13], [14], [15].

H_∞ state control of time-delayed uncertain systems has been the subject of recent investigations. This problem belongs to a class of convex optimization problems which may be effectively solved using LMI formulations. Multiple state delays considers [16]. Reference [5] takes into account also additive controller uncertainty. This paper includes also some additional references.

One of recent research issues within the H_∞ resilient control of time-delayed uncertain systems is the extension into decentralized control setting. The present paper extends the results in [5] as well in [15] to the overlapping H_∞ resilient state feedback control design for a class of time-delayed uncertain systems.

To the authors knowledge, the expansion-contraction relations have not been extended up to now on the concept of H_∞ resilient control for the considered class of systems.

II. PROBLEM FORMULATION

Notation. In this paper $\|w(t)\|_2 = \sqrt{\int_0^\infty w^T(t)w(t)\,dt}$ denotes the 2-norm for $w:[0,\infty)\mapsto\mathbb{R}^p$ belonging to the space $L_2^p[0,\infty)$, provided that $\|w(t)\|_2<\infty$. Further, $\|W\|=\sigma_{\max}(W)$ will denote the matrix norm of W, the largest singular value of W.

This work was supported in part by the Academy of Sciences of the Czech Republic (AVCR) under Grant A2075304 and by the Committee for Science and Technology (CICYT) of Spain under Grant DPI2002-04018-C02-01

Lubomír Bakule is with the Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, 182 08 Prague 8, Czech Republic bakule@utia.cas.cz

José Rodellar is with the Department of Applied Mathematics III, Universitat Politècnica de Catalunya (UPC), Campus Nord, C-2, 08034-Barcelona, Spain jose.rodellar@upc.edu

Josep M. Rossell is with the Department of Applied Mathematics III, Universitat Politècnica de Catalunya (UPC), 08240-Manresa, Spain josep.maria.rossell@upc.edu

A. Feedback Systems

Consider a class of linear continuous-time time-delay uncertain systems described by the state equation

$$\begin{aligned} \mathbf{S}: \dot{x}(t) &= [A+\Delta A(t)]\,x(t)+[B+\Delta B(t)]\,u(t) \\ &\quad +[A_d+\Delta A_d(t)]\,x(t-d)+B_1w(t), \\ z(t) &= Cx(t)+Du(t), \\ x(t_0) &= \phi(t_0),\ -d \leqslant t_0 \leqslant 0, \end{aligned} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the control input, $d>0$ is the time delay, $w(t)\in L_2^p[0,\infty)$ is the disturbance input, $z(t)\in\mathbb{R}^q$ is the controlled output and $\phi(t_0)$ is a given continuous initial function. The matrices A, B, A_d, B_1, C, D are known real constant of appropriate dimensions. $\Delta A(t)$, $\Delta B(t)$ and $\Delta A_d(t)$ are real-valued matrices of uncertain parameters. Norm-bounded time-varying uncertainties have the form

$$\begin{aligned} \Delta A(t) &= H_A F(t) E_A, \quad \Delta A_d(t) = H_d F(t) E_d, \\ \Delta B(t) &= H_B F(t) E_B, \end{aligned} \tag{2}$$

where H_A, H_d, H_B, E_A, E_d, E_B are known real constant matrices of appropriate dimensions and $F(t)\in\mathbb{R}^{i\times j}$ are unknown real time-varying matrices with Lebesgue measurable elements satisfying the condition $F^T(t)F(t)\leqslant I$.
Denote

$$\begin{aligned} \bar{A}(t) &= A+\Delta A(t), \quad \bar{B}(t) = B+\Delta B(t), \\ \bar{A}_d(t) &= A_d+\Delta A_d(t). \end{aligned} \tag{3}$$

The system (1) can be rewritten into the form

$$\begin{aligned} \mathbf{S}: \dot{x}(t) &= \bar{A}(t)x(t)+\bar{B}(t)u(t)+\bar{A}_d(t)x(t-d)+B_1w(t), \\ z(t) &= Cx(t)+Du(t), \\ x(t_0) &= \phi(t_0),\ -d \leqslant t_0 \leqslant 0. \end{aligned} \tag{4}$$

It is well known that the unique solution of (4) is given by

$$\begin{aligned} x(t;\phi(t_0),u(t),w(t)) &= \Phi(t,0)x(0)+\int_0^t \Phi(t,s)\bar{A}_d(s)x(s-d)\,ds \\ &\quad +\int_0^t \Phi(t,s)[\bar{B}(s)u(s)+B_1w(s)]\,ds, \end{aligned} \tag{5}$$

where $\Phi(t,s)$ is the *transition matrix* of $\bar{A}(t)$.

Now, consider a memoryless state feedback controller in the form

$$u(t) = [K+\Delta K(t)]\,x(t) = \bar{K}(t)x(t) \tag{6}$$

for the system (1), where $K\in\mathbb{R}^{m\times n}$. The uncertain gain matrix $\Delta K(t)$ satisfies

$$\Delta K(t) = H_K F_K(t) E_K, \tag{7}$$

where $F_K^T(t)F_K(t)\leqslant I$. $F_K\in\mathbb{R}^{i\times j}$ is an unknown real time-varying matrix with Lebesgue measurable elements. H_K, E_K are known constant matrices of appropriate dimensions.

Further, suppose a given system (1) with a state feedback controller (6). We get the resulting closed-loop system $\mathbf{S}_C$ in the form

$$\begin{aligned} \dot{x}(t) &= \big[A+BK+H_BF(t)E_BK+H_AF(t)E_A+BH_KF_k(t)E_k \\ &\quad +H_BF(t)E_BH_kF_k(t)E_k\big]x(t) \\ &\quad +[A_d+H_dF(t)E_d]\,x(t-d)+B_1w(t) \\ &= A_p(t)x(t)+A_q(t)x(t-d)+B_1w(t), \\ z(t) &= \big[C+DK+DH_kF_k(t)E_k\big]\,x(t). \end{aligned} \tag{8}$$

B. Robust Stability and H_∞ Performance

Introduce the following inequality which is related to the system (1)

$$\| z(t) \|_2 \leqslant \gamma \| w(t) \|_2 \,. \tag{9}$$

The robust stability as well as the H_∞ performance issues are derived using the Lyapunov function

$$V(x,t) = x^T(t)Px(t)+\int_{t-d}^t x^T(\sigma)[I+E_d^TE_d]x(\sigma)\,d\sigma, \tag{10}$$

where P, $\left(I+E_d^TE_d\right)\in\mathbb{R}^{n\times n}$ are positive-definite symmetric matrices. It follows the way of reasoning by [5], [16].

Definition 1: Consider the system (1) with $u(t)$=0, $w(t)$=0. This system is *robustly quadratically stable* if there exist a positive-definite symmetric matrix P such that

$$PA+A^TP+PZP+G<0, \tag{11}$$

where

$$\begin{aligned} Z &= H_AH_A^T+H_dH_d^T+A_dA_d^T, \\ G &= E_A^TE_A+E_d^TE_d+I, \end{aligned} \tag{12}$$

for all admissible uncertainties (2).

Definition 2: Consider the system (1). This system is *robustly quadratically stabilizable* if there exist a positive-definite symmetric matrix P and a gain matrix K by (6) such that

$$P(A+BK)+(A+BK)^TP+\varepsilon_1K^TE_B^TE_BK+PZP+G<0, \tag{13}$$

where

$$\begin{aligned} Z &= H_AH_A^T+H_dH_d^T+(1+\frac{1}{\varepsilon_1})H_BH_B^T+A_dA_d^T+BH_KH_K^TB^T, \\ G &= E_A^TE_A+E_d^TE_d+(1+\alpha_1)E_K^TE_K+I, \\ \alpha_1 &= \| H_K^TE_B^TE_BH_K \|, \end{aligned} \tag{14}$$

for all $\varepsilon_1>0$ and all admissible uncertainties (2) and (7).

Suppose the selected matrices P,K satisfy the inequality (13). Then the closed-loop system (8) with such K is robustly quadratically stable.

Definition 3: Given the system (1) and a scalar $\gamma>0$. This system is *robustly quadratically stable with H_∞-norm bound* γ if it is robustly quadratically stable and under zero initial conditions the relation (9) holds for any non-zero $w(t)$ and for all admissible uncertainties (2).

Definition 4: Given the system (1) and a scalar $\gamma>0$. This system is *robustly quadratically stabilizable with H_∞-norm* *bound* γ if there exist a positive-definite symmetric matrix P and a gain matrix K by (6) such that

$$\begin{aligned} &P(A+BK)+(A+BK)^T P+P\left(Z+\gamma^{-2}B_1B_1^T\right)P \\ &+\varepsilon_1 K^T E_B^T E_B K+\left(1+\frac{1}{\varepsilon_2}\right)(C+DK)^T(C+DK) \\ &+(1+\varepsilon_2)\,\alpha_2 E_k^T E_k+G<0, \end{aligned} \tag{15}$$

hold for all admissible uncertainties (2) and (7), where $\alpha_2 = \| H_K^T D^T D H_K \|$.

Suppose the selected matrices P, K satisfy the inequality (15). Then the closed-loop system (8) with such K is robustly quadratically stable with H_∞-norm bound γ.

C. LMI Control Design

We select a well known LMI control design time independent procedure for the considered class of systems by [5]. It is included here as an available control design tool to complete the presentation.

Theorem 1: Suppose given an uncertain linear time-delay system (1) and positive constant $\gamma>0$. If for some positive real values ε_1, ε_2, there exist positive-definite symmetric matrix Q and matrix Y such that the following LMI

$$\begin{bmatrix} W_1 & QW_2^T & QE_K^T & Y^T E_B^T & W_3^T \\ W_2 Q & -I & 0 & 0 & 0 \\ E_K Q & 0 & -\left[\frac{\alpha_2^{-1}}{1+\varepsilon_2}\right]I & 0 & 0 \\ E_B Y & 0 & 0 & -\left[\frac{1}{\varepsilon_1}\right]I & 0 \\ W_3 & 0 & 0 & 0 & -\left[\frac{\varepsilon_2}{1+\varepsilon_2}\right]I \end{bmatrix} < 0 \tag{16}$$

holds, where

$$\begin{aligned} W_1 &= AQ+QA^T BY+Y^T B^T+Z+\gamma^{-2}B_1B_1^T, \\ W_2^T &= \left[E_A^T \quad E_d^T \quad (1+\alpha_2)^{1/2}E_K^T \quad I\right], \\ W_3^T &= QC^T+Y^T D^T \end{aligned} \tag{17}$$

then, there exists a memoryless state feedback controller (6) such that the resulting closed-loop system (8) is robustly quadratically stable stable with H_∞-norm bound γ. Moreover, the gain matrix K is given by $K = YQ^{-1}$.

D. Inclusion Principle

Similarly to system (1), consider a new bigger dimension system in the form

$$\begin{aligned} \tilde{\mathbf{S}}: \dot{\tilde{x}}(t) &= \left[\tilde{A}+\Delta\tilde{A}(t)\right]\tilde{x}(t)+\left[\tilde{B}+\Delta\tilde{B}(t)\right]u(t) \\ &\quad+\left[\tilde{A}_d+\Delta\tilde{A}_d(t)\right]\tilde{x}(t-d)+\tilde{B}_1 w(t), \\ \tilde{z}(t) &= \tilde{C}\tilde{x}(t)+Du(t), \\ \tilde{x}(t_0) &= \tilde{\phi}(t_0),\ -d \leqslant t_0 \leqslant 0, \end{aligned} \tag{18}$$

where $\tilde{x}(t)\in\mathbb{R}^{\tilde{n}}$ is the state, $u(t)\in\mathbb{R}^m$ is the control input, $d>0$ is the delay time, $w(t)\in L_2^p[0,\infty)$ is the disturbance input, $\tilde{z}(t)\in\mathbb{R}^q$ is the controlled output and $\tilde{\phi}(t_0)$ is a continuous initial function. Suppose $n \leqslant \tilde{n}$, $q \leqslant \tilde{q}$.

The conditions (2) and the notation (3) for the system $\mathbf{S}$ are analogous for the system $\tilde{\mathbf{S}}$, but considering now all matrices with tilde ($\sim$). The unique solution of (18) has the following form

$$\begin{aligned} \tilde{x}(t;\tilde{\phi}(t_0),u(t),w(t)) &= \tilde{\Phi}(t,0)\tilde{x}(0)+\int_0^t \tilde{\Phi}(t,s)\tilde{\bar{A}}_d(s)\tilde{x}(s-d)\,ds \\ &\quad+\int_0^t \tilde{\Phi}(t,s)\left[\tilde{\bar{B}}(s)u(s)+\tilde{B}_1 w(s)\right]ds. \end{aligned} \tag{19}$$

The corresponding state feedback controller for the system $\tilde{\mathbf{S}}$ is given by

$$u(t) = \left[\tilde{K}+\Delta\tilde{K}(t)\right]\tilde{x}(t), \tag{20}$$

where $\tilde{K}\in\mathbb{R}^{m\times\tilde{n}}$. The uncertain gain matrix $\Delta\tilde{K}(t)$ satisfies

$$\Delta\tilde{K}(t) = \tilde{H}_K \tilde{F}_K(t)\tilde{E}_K, \tag{21}$$

where $\tilde{F}_K^T(t)\tilde{F}_K(t)\leqslant I$. $\tilde{F}_K\in\mathbb{R}^{i\times j}$ is an unknown real time-varying matrix with Lebesgue measurable elements. $\tilde{H}_K$, $\tilde{E}_K$ are known constant matrices of appropriate dimensions.

Introduce the inequality which is related with the expanded system $\tilde{\mathbf{S}}$ by

$$\| \tilde{z}(t) \|_2 \leqslant \tilde{\gamma} \| w(t) \|_2 \tag{22}$$

supposing that the corresponding expanded closed-loop system is robustly quadratically stable with zero initial condition.

Let $x(t)=x(t;\phi(t_0),u(t),w(t))$, $\tilde{x}(t)=\tilde{x}(t;\tilde{\phi}(t_0),u(t),w(t))$ be the solutions of (1) and (18) for initial functions $\phi(t_0)$ and $\tilde{\phi}(t_0)$, given inputs $u(t)$ and disturbance inputs $w(t)$, respectively. Consider the standard relations between the states within the Inclusion Principle. It means that the systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are related by the following linear transformations:

$$\tilde{x}(t) = Vx(t), \qquad x(t) = U\tilde{x}(t), \tag{23}$$

where V and its pseudoinverse matrix $U=(V^T V)^{-1}V^T$ are constant full-rank matrices of appropriate dimensions [11].

Definition 5: A system $\tilde{\mathbf{S}}$ *includes* the system $\mathbf{S}$, denoted by $\tilde{\mathbf{S}}\supset\mathbf{S}$, if there exists a pair of constant matrices (U,V) such that $UV=I_n$, and for any initial function $\phi(t_0)$, any fixed input $u(t)$ and any disturbance input $w(t)$ of $\mathbf{S}$, $x(t;\phi(t_0),u(t),w(t))=U\tilde{x}(t;V\phi(t_0),u(t),w(t))$ for all t.

Definition 6: A pair $(\tilde{\mathbf{S}},\tilde{\gamma})$ is an *expansion* of $(\mathbf{S},\gamma)$, denoted by $(\tilde{\mathbf{S}},\tilde{\gamma})\supset(\mathbf{S},\gamma)$, if $\tilde{\mathbf{S}}\supset\mathbf{S}$ and $\gamma=\tilde{\gamma}$.

Definition 7: A controller $u(t)=\left[\tilde{K}+\Delta\tilde{K}(t)\right]\tilde{x}(t)$ for $\tilde{\mathbf{S}}$ is *contractible* to $u(t)=[K+\Delta K(t)]x(t)$ for $\mathbf{S}$ if the choice $\tilde{\phi}(t_0)=V\phi(t_0)$ implies $[K+\Delta K(t)]x(t;\phi(t_0),u(t),w(t))=\left[\tilde{K}+\Delta\tilde{K}(t)\right]\tilde{x}(t;V\phi(t_0),u(t),w(t))$ for all t, any initial function $\phi(t_0)$, any fixed input $u(t)$ and any disturbance input $w(t)$ of $\mathbf{S}$.

Remark. The Inclusion Principle can be used for the analysis and control design of different classes of dynamic systems with a variety of objectives. In general, a dynamic system is expanded to obtain an another bigger dimension system containing all information about the initial system. Then, the controller design is usually performed for the expanded system and consequently contracted into the original system. This approach is effective mainly when considering decentralized controller design for the expanded system without shared parts. This paper considers that the gain matrix $\tilde{K}$

appearing in the controller $u(t)=\left[\tilde{K}+\Delta\tilde{K}(t)\right]\tilde{x}(t)$ is not the expanded gain matrix of K but a "free" matrix designed for the system $\tilde{\mathbf{S}}$ with given $\tilde{\gamma}$.

Suppose given a pair of matrices (U,V). Then, the matrices $\tilde{A}$, $\Delta\tilde{A}(t)$, $\tilde{B}$, $\Delta\tilde{B}(t)$, $\tilde{A}_d$, $\Delta\tilde{A}_d(t)$, $\tilde{B}_1$ and $\tilde{C}$ can be described as follows:

$$\begin{aligned}\tilde{A} &= VAU+M, & \Delta\tilde{A}(t) &= V\Delta A(t)U,\\ \tilde{B} &= VB+N, & \Delta\tilde{B}(t) &= V\Delta B(t),\\ \tilde{A}_d &= VA_dU+M_d, & \Delta\tilde{A}_d(t) &= V\Delta A_d(t)U,\\ \tilde{B}_1 &= VB_1+M_1, & \tilde{C} &= CU+L,\end{aligned} \tag{24}$$

where M, N, M_d, M_1 and L are called *complementary matrices*. Usually, the transformations (U,V) are selected a priori to define structural relations between the state variables in both systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$. Given these transformations, the choice of the complementary matrices provides degrees of freedom to obtain different expanded spaces with desirable properties [12], [13].

E. The Problem

Suppose given a linear continuous-time uncertain delayed system $\mathbf{S}$ by (1) and positive numbers $\gamma,\tilde{\gamma}$. Consider an expanded system $\tilde{\mathbf{S}}$ by (18) which is expanded using (24). Suppose that the relation $\tilde{\mathbf{S}}\supset\mathbf{S}$ holds. Then, the specific goals are as follows:

- Derive conditions under which $(\tilde{\mathbf{S}}_\mathbf{C},\tilde{\gamma})\supset(\mathbf{S}_\mathbf{C},\gamma)$. Use the concept of robust quadratic stability with H_∞-norm bound γ.
- Specialize the global system results into decentralized control design setting.
- Derive all the above results in terms of complementary matrices.

III. SOLUTION

The necessity of overlapping structure is given by structural constraints on the system. However, there is in fact no principal need to expand also the controlled output. We can put $z(t)=\tilde{z}(t)$ without any restriction. We also do not expand the disturbance input. The consequence of this approach is that we can introduce the equality on the H_∞-norm bound γ, i.e. $\gamma=\tilde{\gamma}$ in (9) and (22). Thus, the solution is presented in this section under the equality requirement on γ bounds. Further, some conditions on the complementary matrices (24) must be imposed on $(\tilde{\mathbf{S}},\tilde{\gamma})$ to be an expansion of $(\mathbf{S},\gamma)$ by Definition 6. This is provided by the following theorem.

Theorem 2: Consider the systems (1) with (9) and (18) with (22). A pair $(\tilde{\mathbf{S}},\tilde{\gamma})$ includes the pair $(\mathbf{S},\gamma)$ if and only if

$$\begin{aligned}U\tilde{\Phi}(t,0)V &= \Phi(t,0), & U\tilde{\Phi}(t,s)M_dV &= 0,\\ U\tilde{\Phi}(t,s)N &= 0, & U\tilde{\Phi}(t,s)M_1 &= 0,\\ LV &= 0\end{aligned} \tag{25}$$

hold for all t and s.

Proof: The proof follows an analogous way of reasoning as that one of Theorem 3.1 in [17]. ∎

Remark. It is well-known that to obtain an explicit solution of a time-varying system is a difficult problem. An attempt has been made to approximate the solutions using transition matrices. However, even to compute such approximation via Peano-Baker series can be a complicated task excluding trivial cases [14]. Therefore, we focus on conditions under which $(\tilde{\mathbf{S}},\tilde{\gamma})\supset(\mathbf{S},\gamma)$ holds but without any necessity to compute the transition matrices.

The following theorem presents equivalent conditions as those ones by Theorem 2 but expressed in terms of complementary matrices. Thus avoiding the computational problem.

Theorem 3: Consider the systems (1) with (9) and (18) with (22). A pair $(\tilde{\mathbf{S}},\tilde{\gamma})$ includes the pair $(\mathbf{S},\gamma)$ if and only if

$$\begin{aligned}UM^iV &= 0, & UM^{i-1}M_dV &= 0,\\ UM^{i-1}N &= 0, & UM^{i-1}M_1 &= 0,\\ LV &= 0\end{aligned} \tag{26}$$

hold for all $i=1,2,...,\tilde{n}$.

Proof: Consider the transition matrix $\tilde{\Phi}(t,0)$ of $\tilde{\tilde{A}}$. Following the notation given in (3), $\tilde{\tilde{A}}=\tilde{A}+\Delta\tilde{A}(t)$ represents the state matrix of the expanded space $\tilde{\mathbf{S}}$ as a function of two variables defined by the Peano-Baker series [17], [18]

$$\begin{aligned}\tilde{\Phi}(t,0) = I &+ \int_0^t \tilde{\tilde{A}}(\sigma_1)\,d\sigma_1 + \int_0^t \tilde{\tilde{A}}(\sigma_1)\int_0^{\sigma_1}\tilde{\tilde{A}}(\sigma_2)\,d\sigma_2 d\sigma_1\\ &+ \int_0^t \tilde{\tilde{A}}(\sigma_1)\int_0^{\sigma_1}\tilde{\tilde{A}}(\sigma_2)\int_0^{\sigma_2}\tilde{\tilde{A}}(\sigma_3)\,d\sigma_3 d\sigma_2 d\sigma_1 + \cdots\end{aligned} \tag{27}$$

where according to (24), $\tilde{\tilde{A}}(\sigma_i)=\tilde{A}+\Delta\tilde{A}(\sigma_i)=VAU+M+V\Delta A(\sigma_i)U$ for all $i=1,2,...$ From Theorem 2, pre and post-multiplying both sides of $\tilde{\Phi}(t,0)$ by U and V, respectively, we can prove that $U\tilde{\Phi}(t,0)V=\Phi(t,0)$ is equivalent to $UM^iV=0$, for $i=1,2,...,\tilde{n}$. Following a similar reasoning, $U\tilde{\Phi}(t,s)M_dV=0$ is equivalent to $UM^{i-1}M_dV=0$ for $i=1,2,...,\tilde{n}$. The condition $U\tilde{\Phi}(t,s)N=0$ is equivalent to $UM^{i-1}N=0$ and $U\tilde{\Phi}(t,s)M_1=0$ is equivalent to $UM^{i-1}M_1=0$, for all $i=1,2,...,\tilde{n}$. The requirement $LV=0$ is obtained by imposing $\gamma=\tilde{\gamma}$ in (9) and (22). ∎

Thus, Theorem 3 allows to obtain expanded systems satisfying the Inclusion Principle with the same H_∞ performance attenuation bound γ without an exact knowledge of transition matrices.

Proposition 1: Consider the systems (1) with (9) and (18) with (22). A pair $(\tilde{\mathbf{S}},\tilde{\gamma})$ includes the pair $(\mathbf{S},\gamma)$ if $LV=0$ and

$$\begin{aligned}&a)\quad MV=0,\quad M_dV=0,\quad N=0,\quad M_1=0\ \text{ or}\\ &b)\quad UM=0,\quad UM_d=0,\quad UN=0,\quad UM_1=0.\end{aligned} \tag{28}$$

Proof: The proof is straightforward when applying Theorem 3. ∎

Remark. If $M_d=0$, $M_1=0$ in (28), then *a)* and *b)* correspond to particular cases within the Inclusion Principle called *restrictions* and *aggregations*, respectively, [11].

Definition 7 presents the conditions under which a control law designed in the expanded system $\tilde{\mathbf{S}}$ can be contracted and implemented into the initial system $\mathbf{S}$. However, these requirements do not guarantee that the closed-loop system $\tilde{\mathbf{S}}_\mathbf{C}$ includes the closed-loop system $\mathbf{S}_\mathbf{C}$ in the sense of the Inclusion Principle, i.e. $\tilde{\mathbf{S}}_\mathbf{C}\supset\mathbf{S}_\mathbf{C}$. Now, consider conditions which include also H_∞ performance attenuation bounds γ.

They are presented by the following theorem using complementary matrices.

Theorem 4: Consider the systems (1) with (9) and (18) with (22) such that $\tilde{\mathbf{S}}\supset\mathbf{S}$. Suppose that $u(t)=\left[\tilde{K}+\Delta\tilde{K}(t)\right]\tilde{x}(t)$ is a contractible control law designed in $\tilde{\mathbf{S}}$. If $MV=0$, $M_dV=0$, $N=0$, $M_1=0$ and $LV=0$ then $(\tilde{\mathbf{S}}_\mathbf{C},\tilde{\gamma})\supset(\mathbf{S}_\mathbf{C},\gamma)$.

Proof: Suppose $\tilde{\mathbf{S}}\supset\mathbf{S}$ and consider $u(t)=\left[\tilde{K}+\Delta\tilde{K}(t)\right]\tilde{x}(t)$ a contractible control law designed in $\tilde{\mathbf{S}}$. The corresponding state matrix of the closed-loop expanded system is given as follows

$$\begin{aligned}\tilde{\mathbf{S}}_\mathbf{C}: \dot{\tilde{x}}(t) &= \left[\tilde{A}+\tilde{B}\tilde{K}+\Delta\tilde{A}(t)+\Delta\tilde{B}(t)\tilde{K}+\tilde{B}\Delta\tilde{K}(t)\right.\\ &\quad\left.+\Delta\tilde{B}(t)\Delta\tilde{K}(t)\right]\tilde{x}(t)+\left[\tilde{A}_d+\Delta\tilde{A}_d(t)\right]\tilde{x}(t-d)\\ &\quad+\tilde{B}_1 w(t) \\ &=\tilde{A}_p(t)\tilde{x}(t)+\tilde{A}_q(t)\tilde{x}(t-d)+\tilde{B}_1 w(t),\\ \tilde{z}(t) &= \left[\tilde{C}+D[\tilde{K}+\Delta\tilde{K}(t)]\right]\tilde{x}(t).\end{aligned} \tag{29}$$

It follows from Definition 7 the contracted state feedback controller $u(t)=[K+\Delta K(t)]x(t)$ has the form $u(t)=\left[\tilde{K}+\Delta\tilde{K}(t)\right]Vx(t)$. Consider the relation between the state matrices of the closed-loop systems $\mathbf{S}_\mathbf{C}$ and $\tilde{\mathbf{S}}_\mathbf{C}$ given in (8) and (29), respectively. The relation $\tilde{A}_p(t)=VA_p(t)U+M_p$ implies

$$\begin{aligned}M_p=M+N\left[\tilde{K}+\Delta\tilde{K}(t)\right]+VB\left[\tilde{K}+\Delta\tilde{K}(t)\right]-VB\left[\tilde{K}+\Delta\tilde{K}(t)\right]VU\\ +V\Delta B(t)\left[\tilde{K}+\Delta\tilde{K}(t)\right](I-VU),\end{aligned}$$

where M_p is a complementary matrix to be determined. Since $\tilde{\mathbf{S}}_\mathbf{C}\supset\mathbf{S}_\mathbf{C}$ is desired, the condition $UM_p^iV=0$, $i=1,2,\ldots,\tilde{n}$, must be satisfied. Imposing this requirement and using equations (24), we can prove that $MV=0$, $N=0$ is a sufficient condition so that $UM_p^iV=0$ holds for all $i=1,2,\ldots,\tilde{n}$. Analogously, the relation $\tilde{A}_q(t)=VA_q(t)U+M_q$ implies $M_q=M_d$. Imposing $UM_d^iV=0$ in order to verify the Inclusion Principle, $M_dV=0$ is a sufficient condition under which $UM_d^iV=0$ holds for all $i=1,2,\ldots,\tilde{n}$. The condition $U\left[\tilde{B}_1w(t)\right]=B_1w(t)$ is equivalent to $UM_1=0$ in terms of complementary matrices and consequently $M_1=0$ is a sufficient condition. Finally, $LV=0$ is the same requirement as given in Theorem 3 when imposing $\gamma=\tilde{\gamma}$. ■

The particular case *a)* in Proposition 1 satisfies Theorem 4 under which $(\tilde{\mathbf{S}}_\mathbf{C},\tilde{\gamma})\supset(\mathbf{S}_\mathbf{C},\gamma)$. This case is used in the following text.

Now, suppose that a controller $u(t)=\left[\tilde{K}+\Delta\tilde{K}(t)\right]\tilde{x}(t)$ designed in the expanded system $\tilde{\mathbf{S}}$ satisfies the inequality in Definition 4. It means that the robust quadratic stability with H_∞-norm bound γ is guaranteed for the closed-loop system (29). Note that the controller design is performed with a given $\tilde{\gamma}$.

Then, it is necessary to prove if the condition $\tilde{\gamma}=\gamma$ is satisfied for the closed-loop system (8) with the contracted controller $u(t)=[K+\Delta K(t)]x(t)$ implemented into the initial system $\mathbf{S}$. This condition presents the following theorem.

Theorem 5: Consider the systems (1) with (9) and (18) with (22). Suppose that $MV=0$, $M_dV=0$, $N=0$, $M_1=0$ and $LV=0$ hold. Suppose that there exist a positive-definite symmetric matrix $\tilde{P}$ and a gain matrix $\tilde{K}$ such that the inequality

$$\begin{aligned}&\tilde{P}(\tilde{A}+\tilde{B}\tilde{K})+(\tilde{A}+\tilde{B}\tilde{K})^T\tilde{P}+\tilde{P}\left(\tilde{Z}+\gamma^{-2}\tilde{B}_1\tilde{B}_1^T\right)\tilde{P}\\ &+\varepsilon_1\tilde{K}^T\tilde{E}_B^T\tilde{E}_B\tilde{K}+\left(1+\frac{1}{\varepsilon_2}\right)(\tilde{C}+\tilde{D}\tilde{K})^T(\tilde{C}+\tilde{D}\tilde{K})\\ &+(1+\varepsilon_2)\tilde{\alpha}_2\tilde{E}_k^T\tilde{E}_k+\tilde{G}<0\end{aligned} \tag{30}$$

holds for $\varepsilon_1>0$, $\varepsilon_2>0$ and all admissible uncertainties, where $\tilde{\alpha}_2=\|\tilde{H}_K^TD^TD\tilde{H}_K\|$. Then, there exist a positive-definite symmetric matrix $P=V^T\tilde{P}V$ and a gain matrix $K=\tilde{K}V$ such that the robust stability with H_∞ performance attenuation bound γ is guaranteed for closed-loop system (8). Moreover, $\gamma=\tilde{\gamma}$.

Proof: Consider $u(t)=\left[\tilde{K}+\Delta\tilde{K}(t)\right]\tilde{x}(t)$ a contractible controller designed for the system $\tilde{\mathbf{S}}$. Suppose the gain matrix $\tilde{K}$ together with a given positive-definite symmetric matrix $\tilde{P}$ satisfy (30). Substitute the relations (24) into (30), where the matrices $\tilde{Z}$ and $\tilde{G}$ have the same meaning as Z and G in (13) but for the expanded system $\tilde{\mathbf{S}}$. Then, the inequality (30) implies inequality (15) by Definition 4 when considering a positive-definite symmetric matrix $P>0$ in the form $P=V^T\tilde{P}V$, $E_B=\tilde{E}_B$, $E_K=\tilde{E}_KV$ and $H_K=\tilde{H}_KV$. Thus, the robust quadratic stability with H_∞-norm bound γ is guaranteed for closed-loop system (8) by using the contracted controller $u(t)=[K+\Delta K(t)]x(t)=\left[\tilde{K}+\Delta\tilde{K}(t)\right]Vx(t)$. By the hypothesis of this theorem $\gamma=\tilde{\gamma}$. It satisfies Theorem 4. ■

A. Overlapping State Feedback

Information structure constraints on the state feedback gain matrices includes different practically important structures corresponding with the sparsity forms of gain matrices well known in the theory of sparse matrices. These particular forms are a block diagonal form, a block tridiagonal form, and a double bordered block diagonal form corresponding with decentralized, overlapping, and DBBD gain matrices, respectively. Generally, their control design may be effectively performed using LMI approach [19].

Standard way of reasoning supposes two overlapping subsystems with the structure of matrices A, $\Delta A(t)$, A_d, $\Delta A_d(t)$ and B, $\Delta B(t)$, B_1, respectively, in the form:

$$\left[\begin{array}{cc|c} * & * & * \\ \hline * & * & * \\ \hline * & * & * \end{array}\right], \qquad \left[\begin{array}{c|c} * & * \\ \hline * & * \\ \hline * & * \end{array}\right], \tag{31}$$

where A_{ii}, $\Delta A_{ii}(t)$, $A_{d_{ii}}$, $\Delta A_{d_{ii}}(t)$ for $i=1,2,3$ are $n_i\times n_i$ dimensional matrices. B_{ij}, $\Delta B_{ij}(t)$ and $B_{1_{ij}}$ for $i=1,2,3$ as well as $j=1,2$ are $n_i\times m_j$ and $n_i\times p_j$ dimensional matrices, respectively. The dimensions of the components of the vector $x^T(t)=\left[x_1^T(t),x_2^T(t),x_3^T(t)\right]$ are n_1, n_2, n_3, respectively, and satisfy $n_1+n_2+n_3=n$. The partition of $u^T(t)=\left[u_1^T(t),u_2^T(t)\right]$ has two components of dimensions m_1, m_2 such that $m_1+m_2=m$. A standard particular selection of the matrix V is selected as follows:

$$V=\begin{bmatrix} I_{n_1} & 0 & 0\\ 0 & I_{n_2} & 0\\ 0 & I_{n_2} & 0\\ 0 & 0 & I_{n_3}\end{bmatrix}. \tag{32}$$

This transformation leads in a simple natural way to an expanded system where the state vector x_2 appears repeated in $\tilde{x}^T(t)=\left[x_1^T(t), x_2^T(t), x_2^T(t), x_3^T(t)\right]$, [11]. The expanded controller has a block diagonal form with two subblocks of dimensions $m_1 \times (n_1+n_2)$ and $m_2 \times (n_2+n_3)$ as follows:

$$\tilde{K}_D = \left[\begin{array}{cc|cc} \tilde{K}_{11} & \tilde{K}_{12} & 0 & 0 \\ \hline 0 & 0 & \tilde{K}_{23} & \tilde{K}_{24} \end{array}\right]. \qquad (33)$$

The corresponding contracted gain matrix has a block tridiagonal form as follows:

$$K_{TD} = \left[\begin{array}{c|c|c} \tilde{K}_{11} & \tilde{K}_{12} & 0 \\ \hline 0 & \tilde{K}_{23} & \tilde{K}_{24} \end{array}\right]. \qquad (34)$$

However, the design of overlapping controllers depends on the structure of matrices B and $\Delta_B(t)$. Type I corresponds with all nonzero element of all input matrices in (31), while Type II corresponds with all elements $(*)_{21}=0$ and $(*)_{22}=0$. The LMI control design for Type I can be performed directly on the original system. Type II requires to perform the LMI control design in the expanded space because the direct design usually leads to infeasibility [19].

To simplify the control design for Type II case, let us introduce the following concept. Denote P_{TD} a block tridiagonal matrix. Suppose that the dimensions of its blocks correspond with the dimensions of overlapping subsystems.

Definition 8: Consider the system (1) with (9). The controller $u_{TD}x(t)=[K_{TD}+\Delta K_{TD}(t)]\,x(t)$ with block tridiagonal matrices K_{TD} and $\Delta K_{TD}(t)$ is a *td-resilient H_∞ state feedback controller* if there exist a positive-definite symmetric matrix P_{TD} and a gain matrix K_{TD} such that the inequality

$$\begin{aligned} &P_{TD}(A+BK_{TD})+(A+BK_{TD})^T P_{TD}+P_{TD}\left(Z+\gamma^{-2}B_1B_1^T\right)P_{TD} \\ &+\varepsilon_1 K_{TD}^T E_B^T E_B K_{TD}+\left(1+\frac{1}{\varepsilon_2}\right)(C+DK_{TD})^T(C+DK_{TD}) \\ &+(1+\varepsilon_2)\,\alpha_2 E_K^T E_K+G<0 \end{aligned} \qquad (35)$$

holds for all admissible uncertainties (2) and (7), where $\alpha_2 = \| H_K^T D^T D H_K \|$.

It guarantees the robust quadratic stability with H_∞-norm bound γ of the closed-loop system (8).

Theorem 6: Consider the systems (1) with (9) and (18) with (22). Suppose that MV=0, M_dV=0, N=0, M_1=0 and LV=0 hold. Consider the subsystem structure (31) and the transformation matrix (32). Suppose there exist a matrix $\tilde{P}_D>0$ and a gain matrix $\tilde{K}_D$ satisfying (30). Then $u_{TD}(t)=[K_{TD}+\Delta K_{TD}(t)]\,x(t)=\left[\tilde{K}_D+\Delta\tilde{K}_D(t)\right]Vx(t)$ is a td-resilient H_∞ state feedback controller with the matrix $P_{TD}=V^t\tilde{P}_D V>0$ for the system $\mathbf{S}$ satisfying $\gamma=\tilde{\gamma}$.

Proof: It is straightforward because this theorem is a particular case of Theorem 5. ■

IV. CONCLUSIONS

The paper contributes to the solution of the overlapping H_∞ resilient state feedback control design for a class of nonlinear continuous-time uncertain state delayed nominally linear systems. Time-varying unknown norm bounded parameter uncertainties are considered in the system and the controller. Conditions preserving closed-loop systems expansion-contraction relations guaranteeing the H_∞ disturbance attenuation bounds have been proved. They are derived in terms of conditions on complementary matrices. An LMI delay independent procedure has been supplied as a control design tool. The results have been specialized into overlapping decentralized control setting. It means that the presented method leads to the control design of robust H_∞ resilient state tridiagonal feedback controllers.

REFERENCES

[1] L. Keel and S. Bhattacharyya, "Robust, fragile, or optimal?" *IEEE Transactions on Automatic Control*, vol. 42, no. 8, pp. 1098–1105, August 1997.

[2] P. Dorado, "Non-fragile controller design: an overview," in *Proceedings of the American Control Conference*, Philadelphia, Pennsylvania, USA, June 1998, pp. 2829–2831.

[3] J. Corrado and W. Haddad, "Static output feedback controllers for systems with parametric uncertainty and controller gain variations," in *Proceedings of the American Control Conference*, San Diego, California, USA, June 1999, pp. 915–919.

[4] R. Takahashi, D. Dutra, R. Palhares, and P. Peres, "On robust non-fragile static state-feedback controller synthesis," in *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, December 2000, pp. 4909–4914.

[5] F. Liu, P. Jiang, H. Su, and J. Chu, "Robust H_∞ control for time-delay systems with additive controller uncertainty," in *Proceedings of the 4th World Congress on Intelligent Control and Automation*, Shanghai, P.R. China, 2002, pp. 1718–1722.

[6] G.-H. Yang and J. Wang, "Non-fragile H_∞ control for linear systems with multiplicative controller gain variations," *Automatica*, vol. 37, pp. 727–737, 2001.

[7] J.-S. Yee, G.-H. Wang, and J. Wang, "Non-fragile guaranteed cost control for discrete-time uncertain linear systems," *International Journal of Systems Science*, vol. 32, no. 7, pp. 845–853, 2001.

[8] D. Peaucelle, D. Arzelier, and C. Farges, "LMI results for resilient state-feedback with H_∞ performance," in *Proceedings of the 43rd IEEE Conference on Decision and Control*, Paradise Island, Bahamas, December 2004, pp. 400–405.

[9] R. Istepanian and J. Whidborne, Eds., *Digital Controller Implementation and Fragility*. Springer Verlag, 2001.

[10] M. Ikeda and D. Šiljak, "Overlapping decompositions, expansions and contractions of dynamic systems," *Large Scale Systems*, vol. 1, no. 1, pp. 29–38, 1980.

[11] D. Šiljak, *Decentralized Control of Complex Systems*. New York, USA: Academic Press, 1991.

[12] L. Bakule, J. Rodellar, and J. Rossell, "Generalized selection of complementary matrices in the inclusion principle," *IEEE Transactions on Automatic Control*, vol. AC–45, no. 6, pp. 1237–1243, 2000.

[13] ——, "Structure of expansion-contraction matrices in the inclusion principle for dynamic systems," *SIAM Journal on Matrix Analysis and Applications*, vol. 21, no. 4, pp. 1136–1155, 2000.

[14] S. Stanković and D. Šiljak, "Inclusion Principle for linear time-varying systems," *SIAM Journal on Control and Optimization*, vol. 42, no. 1, pp. 321–341, 2003.

[15] L. Bakule, J. Rodellar, and J. Rossell, "Overlapping guaranteed cost control for uncertain continuous-time delayed systems," in *Proceedings of the 16th IFAC World Congress*, Prague, Czech Republic, July 2005.

[16] Y.-Y. Cao, Y.-X. Sun, and J. Lam, "Delay-dependent robust H_∞ control for uncertain systems with time delays," *IEE Proceedings: Control Theory and Applications*, vol. 145, no. 3, pp. 338–344, 1998.

[17] L. Bakule, J. Rodellar, and J. Rossell, "Overlapping quadratic optimal control of linear time-varying commutative systems," *SIAM Journal on Control and Optimization*, vol. 40, no. 5, pp. 1611–1627, 2002.

[18] S. Stanković and D. Šiljak, "Contractibility of overlapping decentralized control," *Systems & Control Letters*, vol. 44, pp. 189–199, 2001.

[19] D. Šiljak and A. Zečević, "Control of large-scale systems: beyond decentralized feedback," in *Preprints of the 10th IFAC/IFORS/IMACS/-IFIP Symposium on Large Scale Systems: Theory and Applications*, vol. 1, Osaka, Japan, 2004, pp. 1–10.