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Structured H_2 Controller Synthesis via a dilated LMI Based Algorithm

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Abstract—**It is well known that important problems such as structured H_2 control synthesis or simultaneous stabilization have no polynomial time solution in general. The underlying BMI problem can't be reduced to an LMI. The recent description of dilated LMI's under the continuous time setting however, enables new characterizations for such problems allowing interesting additional degrees of freedom. Based on it, a novel coordinate-descent iterative algorithm is proposed as an efficient alternative to existing ones, in order to approach the solution of the problems considered. The relevance of the approach is illustrated with several numerical examples.**

I. INTRODUCTION

DILATED LMI's [1] are often regarded as a powerful tool to establish new characterizations for control analysis and synthesis problems, even in the continuous time setting [2]. Dilated LMI's presented in [3] are shown to be efficient when dealing with mutiobjective control problems since they enable to associate distinct Lyapunov matrices to different objectives. This paper aims to show some other advantages of dilated LMI's especially when dealing with the H_2 control under structural constraints.

An important problem in control arises when a specific structure on the overall control scheme is considered, especially when dealing with complex or distributed systems [4]. This structure often depends on the structure of the system itself, subdivided in subsystems. It also depends on the signals accessible for measure and the authority on actuation variables of each separated controller included in the global desired controller.

Finding an H_2 controller under structural constraints is a difficult optimization problem. Except in some particular cases, for example when the controller/plant structure satisfies the so called *quadratic invariance* property [5], [6] the problem is indeed non convex. Even if many sub-optimal numerical procedures (see for example [7]-[13]) have been developed to reduce the conservatism and the computational time for related problems, looking for more and more efficient algorithm remains an important and challenging problem.

Using new characterizations based on dilated LMI's for some structured control problems, a novel coordinate-descent iterative numerical procedure is proposed in this paper. In particular, the H_2 control under structural constraints problem is considered in the continuous time setting. The algorithm developed is used to compute local optimal solutions. A decomposition in consecutive sub-problems of the initial problem enable to improve the numerical results obtained.

The paper is organized as follows. Leaning on the dilation lemma, section 2 recalls the definitions and characterizations of the stability and H_2 structured control problems considered. In section 3, a new iterative procedure is proposed to approach the solution of such problems. In section 4, some examples broadly studied in the literature are studied. Finally numerical test-examples whose global optimum is known are constructed and used for test in section 5. The conclusion takes place in section 6.

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II. SOME DEFINITIONS AND PROBLEMS

The notations below will be used through the paper.

- $sym(A) = A + A^T$ and $\begin{bmatrix} A & B \\ \bullet & C \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$
- $\otimes$ denotes the direct product of matrices
- $1_{n_u \times n_y} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{n_u \times n_y}$

In this paper, we consider the continuous linear time-invariant (LTI) system with state-space representation (1)

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{1}$$

$$\Leftrightarrow \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \begin{bmatrix} x \\ w \\ u \end{bmatrix}$$

with $D_{11} = D_{22} = 0$.

T_{zw} denotes the closed-loop transfer matrix between the exogenous inputs w and the weighted output z.

Let us consider, without loss of generality, that

$$T_{zw} := \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & 0 \end{array}\right] \quad (2)$$

where A_{cl}, B_{cl} and C_{cl} depend affinelly on a static output feedback K.

The structure constraint on the controller is defined as

$$K = \Lambda_K \otimes K, \ \ \Lambda_K \in \{0,1\}^{n_u \times n_y} \quad (3)$$

where $\Lambda_K \in \{0,1\}^{n_u \times n_y}$.

$\Omega_{\Lambda_K} := \left\{ K \in \Re_p^{n_u \times n_y} \ / \ K = \Lambda_K \otimes K, \ \ \Lambda_K \in \{0,1\}^{n_u \times n_y} \right\}$ is a convex set [14].

The end of the section uses the matrix inequality framework to characterize the structured Static Output Feedback (S.O.F.) H_2 and the stability problems.

A standard bilinear matrix inequality formulation is recalled here.

The H_2 optimal structured S.O.F. problem :

It consists in finding the optimal controller K^* such that

$$K^* = \arg\min_{K \in \Omega_{\Lambda_K}} \|T_{zw}\|_2$$

K^* may be obtained by solving the BMI optimization problem (4).

$$\begin{cases} \min\limits_{X_2,Y,K} trace(Y) \\ \begin{bmatrix} A_{cl}X_2 + X_2A_{cl}^T & X_2C_{cl}^T & & \\ \bullet & -I & & 0 \\ & & -Y & B_{cl}^T \\ 0 & & \bullet & -X_2 \end{bmatrix} < 0 \\ K = \Lambda_K \otimes K, \ \Lambda_K \in \{0,1\}^{n_u \times n_y} \end{cases} \quad (4)$$

In the following, some fundamental results related to the dilated LMI's are recalled.

A. Stability condition

In the theorem given below the equivalence between the conditions immediately follow from the well-known dilation lemma [2].

Theorem 1 [2]

The following conditions are equivalent, where $b = a^{-1} > 0$ is an arbitrary prescribed number.

(i) The matrix A is stable.

(ii) There exists a matrix $X > 0$ such that

$$AX + XA^T < 0 \quad (5)$$

(iii) There exist matrices $\tilde{X} > 0$ and G such that

$$\begin{bmatrix} 0 & -\tilde{X} \\ \bullet & 0 \end{bmatrix} + sym\left\{ \begin{bmatrix} A \\ I \end{bmatrix} G \begin{bmatrix} I & -bI \end{bmatrix} \right\} < 0 \quad (6)$$

Moreover, for every solution $X > 0$ of (5), $\begin{bmatrix} \tilde{X} & G \end{bmatrix} = \begin{bmatrix} X & -a(A-aI)^{-1}X \end{bmatrix}$ is a solution of (6). Conversely, of every matrix $\tilde{X} > 0$ such that (6) holds for some G, $X = \tilde{X}$ satisfies (5).

The introduction of the extra variable G enables to associate different Lyapunov functions to different objectives or even system in the case of multimodel approach. The following example illustrates the way to take advantage of this additional degree of freedom. A solution exists to the dilated LMI associated to this simultaneous static state feedback problem when the standard LMI formulation fails.

Example : Simultaneous static state feedback synthesis.

Let us consider the three systems $\dot{x}_i = A_i x_i + B_i u_i$, $i \in \{1,2,3\}$ with :

$$A_1 = \begin{bmatrix} 1 & 10 \\ -2 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 1 & 8 \\ -2 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \ B_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } B_3 = \begin{bmatrix} -4 & -1 \\ 1 & 1 \end{bmatrix}$$

and the single state-feedback $u = -Kx$.

The LMI condition (7) (based on (5)) is a well known sufficient condition to simultaneous stabilization as a single Lyapunov X is used to show the stability of the three closed-loop matrices A_{cli}, $i \in \{1,2,3\}$. Unfortunately, it is too conservative and therefore infeasible in this case.

$$\begin{cases} X_i > 0, \\ A_iX + XA_i^T + B_iW + WB_i^T < 0, \ i = \{1,2,3\} \\ K = WX^{-1} \end{cases} \quad (7)$$

On the contrary, condition (8), derived from the dilated condition (6) and using different Lyapunov matrices is less conservative and succeed to find the solution :

$$K = \begin{bmatrix} -0.1713 & -7.4925 \\ 1.9370 & 2.0570 \end{bmatrix} \text{ with } a = 0{,}1.$$

$$\begin{cases} \tilde{X}_i > 0, \\ \begin{bmatrix} 0 & -\tilde{X}_i \\ \bullet & 0 \end{bmatrix} + sym\left\{ \begin{bmatrix} A_i \\ I \end{bmatrix} G \begin{bmatrix} I & -bI \end{bmatrix} \right\} \\ +sym\left\{ \begin{bmatrix} B_i \\ 0 \end{bmatrix} W \begin{bmatrix} I & -bI \end{bmatrix} \right\} < 0 \qquad , \ i = \{1,2,3\} \\ K = WG^{-1} \end{cases} \quad (8)$$

Note that G is obviously non singular (see [2]).

B. H_2 performance

The dilated LMI characterization of the H_2 performance is recalled in what follows.

Theorem 2 [2]

Let us consider the system described by $T(s) := \{A, B, C, 0\}$. For an arbitrary prescribed number $b = a^{-1} > 0$, the following conditions are equivalent.

(i) There exist $X > 0$ and $Y > 0$ such that

$$\begin{cases} AX + XA^T + XC^T CX < 0, \\ Y - B^T XB >, \quad trace(Y) < \gamma^2 \end{cases} \tag{9}$$

(ii) There exist matrices $\tilde{X} > 0$, $\tilde{Y} > 0$ and G such that

$$\begin{cases} \begin{bmatrix} 0 & -\tilde{X} & 0 \\ \bullet & 0 & 0 \\ \bullet & \bullet & -I \end{bmatrix} + sym\left\{ \begin{bmatrix} A \\ I \\ C \end{bmatrix} G \begin{bmatrix} I & -bI & 0 \end{bmatrix} \right\} < 0 \\ \tilde{Y} - B^T \tilde{X} B > 0, \quad trace(\tilde{Y}) < \gamma^2 \end{cases} \tag{10}$$

Moreover, for every solution $X > 0, Y > 0$ of (9), $\begin{bmatrix} \tilde{X} & \tilde{Y} & G \end{bmatrix} = \begin{bmatrix} X & Y & -a(A - aI)^{-1} X \end{bmatrix}$ is a solution of (10). Conversely, of every pair of matrices $\tilde{X} > 0, \tilde{Y} > 0$ such that (10) holds for some G, $X = \tilde{X}$ and $Y = \tilde{Y}$ satisfy (9).

III. An Iterative Procedure for the Structured H_2 Control via Static Output Feedback

Solving the H_2 optimal structured S.O.F. problem implies to solve a bilinear matrix inequality (4) or (10). This section introduces a coordinate-descent based iterative procedure, taking advantage of the extra variable G in the dilated LMI formulation (10). A local solution to the H_2 optimal structured S.O.F. problem may be obtained by proceeding iteratively [12], by solving first the S.O.F. stabilization problem and the structured S.O.F. stabilization one. These problems involve a BMI also. Algorithms based on the corresponding dilated LMI formulation (see Theorems 1 and 2) are proposed next to solve each of them. A major drawback from which suffer a large number of iterative algorithms is the dependence on the initialization. The idea of decomposing the problem aims mainly to overcome this problem.

In this section will be given successively three algorithms associated to each sub-problem stated above.

The statement "for an arbitrary prescribed number $b = a^{-1} > 0$" will be omitted for brevity.

Algorithm 1 : Stabilizing S.O.F. synthesis

Step1- Set $K = 0_{n_u \times n_y}$.

Step2- Solve the following optimization problem for $\tilde{X}$, G and β.

$$\min_{\beta, \tilde{X}, G} \beta$$

$$\begin{cases} \tilde{X} > 0, \\ \begin{bmatrix} -\beta \tilde{X} & -\tilde{X} \\ \bullet & 0 \end{bmatrix} + sym\left\{ \begin{bmatrix} A_{cl} \\ I \end{bmatrix} G \begin{bmatrix} I & -bI \end{bmatrix} \right\} < 0 \end{cases} \tag{11}$$

Step3- If $\beta \le 0$, a feasible solution is found, stop ; else fix β and G, go to Step4.

Step4- Solve the following optimization problem for $\tilde{X}$ and K.

$$\min_{\tilde{X}, K} trace(\tilde{X})$$

$$\begin{cases} \tilde{X} > 0, \\ \begin{bmatrix} -\beta \tilde{X} & -\tilde{X} \\ \bullet & 0 \end{bmatrix} + sym\left\{ \begin{bmatrix} A_{cl} \\ I \end{bmatrix} G \begin{bmatrix} I & -bI \end{bmatrix} \right\} < 0 \end{cases} \tag{12}$$

fix K and go to step2.

It's obvious that a stopping criterion must be provided to stop the algorithm in the case it could not get a feasible solution. In the sequel K_s will denote the solution obtained by Algorithm 1.

Remarks

- Algorithm 1 gives a stabilizing static output feedback. This solution will be used as an initial point for Algorithm 2 which consists in finding a structured stabilizing static output feedback controller.
- Optimization problem (11) consists in a generalized eigenvalue minimization problem.
- This algorithm and results in [8] may be brought together. However, the dilated LMI based algorithm seems to be much more efficient.

Let us consider now the problem that consists in finding a structured stabilizing static output feedback controller. The main idea underlying Algorithm 2 comes from an obvious fact : any stabilizing output feedback gain K_s may be decomposed as

$$\begin{aligned} K_s &= \Lambda_K \otimes K_s + \tilde{\Lambda}_K \otimes K_s \\ &:= K_1 + K_2, \end{aligned} \tag{13}$$

where $\tilde{\Lambda}_K = 1_{n_u \times n_y} - \Lambda_K$. A structured static feedback (associated to the desired structure represented by Λ_K) may then be obtained in annulling K_2.

Algorithm 2 : Structured stabilizing S.O.F. synthesis

Step1- Set $K = K_s$ and find matrices $\tilde{X} > 0$ and G such that constraint (6) holds. Fix G and go to Step2.

Step2- Solve the following optimization problem for $\tilde{X}$, K_1 and K_2.

$$\min_{K_1, K_2, \tilde{X}} \beta$$

$$\begin{cases} \tilde{X} > 0, \\ \begin{bmatrix} 0 & -\tilde{X} \\ \bullet & 0 \end{bmatrix} + sym\left\{ \begin{bmatrix} A_{cl} \\ I \end{bmatrix} G \begin{bmatrix} I & -bI \end{bmatrix} \right\} < 0 \\ trace(K_2^T K_2) < \beta, \; K_1 = \Lambda_K \otimes K_1, K_2 = \tilde{\Lambda}_K \otimes K_2 \end{cases} \tag{14}$$

Step3- If $(A + B_2 K_1 C_2)$ is stable a feasible solution is found, stop ; else fix $K = K_1 + K_2$ and go to Step4.

Step4- Solve the following optimization problem for $(\tilde{X}, G)$

$$\min_{\tilde{X},G} \; trace(\tilde{X})$$

$$\begin{cases} \tilde{X} > 0, \\ \begin{bmatrix} 0 & -\tilde{X} \\ \bullet & 0 \end{bmatrix} + sym\left\{ \begin{bmatrix} A_{cl} \\ I \end{bmatrix} G \begin{bmatrix} I & -bI \end{bmatrix} \right\} < 0 \end{cases} \quad (15)$$

fix G and go to step2.

In the sequel, K_{ss} will denote the solution obtained by Algorithm 2.

Remark

- Algorithm 2 gives a stabilizing static output feedback. This solution will be used as an initial point for Algorithm 3 which consists in finding a sub-optimal structured output feedback controller that minimizes the H_2 norm of the closed-loop transfer.

Let us consider now the problem that consists in finding a sub-optimal structured static output feedback controller that minimizes the H_2 norm of the closed-loop transfer (2). Algorithm 3, that is proposed next, is a coordinate-descent numerical procedure based on a the "dilated" LMI formulation in Theorem 2.

Algorithm 3 : H_2 structured stabilizing S.O.F. synthesis

Step1- Set $K = K_{ss}$ and solve the optimization problem (16) for $\tilde{X} > 0$ and G

$$\min_{\tilde{X},G} \; \gamma^2$$

$$\begin{cases} \begin{bmatrix} 0 & -\tilde{X} & 0 \\ \bullet & 0 & 0 \\ \bullet & \bullet & -I \end{bmatrix} + sym\left\{ \begin{bmatrix} A_{cl} \\ I \\ C_{cl} \end{bmatrix} G \begin{bmatrix} I & -bI & 0 \end{bmatrix} \right\} < 0 \\ \tilde{Y} - B_{cl}^T \tilde{X} B_{cl} > 0, \; trace(\tilde{Y}) < \gamma^2 \end{cases} \quad (16)$$

fix G and go to step2.

Step2- Solve the following optimization problem for $\tilde{X}$ and K.

$$\min_{\tilde{X},K} \; \gamma^2$$

$$\begin{cases} \begin{bmatrix} 0 & -\tilde{X} & 0 \\ \bullet & 0 & 0 \\ \bullet & \bullet & -I \end{bmatrix} + sym\left\{ \begin{bmatrix} A_{cl} \\ I \\ C_{cl} \end{bmatrix} G \begin{bmatrix} I & -bI & 0 \end{bmatrix} \right\} < 0 \\ \tilde{Y} - B_{cl}^T \tilde{X} B_{cl} > 0, \; trace(\tilde{Y}) < \gamma^2, K = \Lambda_K \otimes K \end{cases} \quad (17)$$

fix K and go to step3.

Step3- Solve the following optimization problem for $\tilde{X}$ and G.

$$\min_{\tilde{X},G} \; \gamma^2$$

$$\begin{cases} \begin{bmatrix} 0 & -\tilde{X} & 0 \\ \bullet & 0 & 0 \\ \bullet & \bullet & -I \end{bmatrix} + sym\left\{ \begin{bmatrix} A_{cl} \\ I \\ C_{cl} \end{bmatrix} G \begin{bmatrix} I & -bI & 0 \end{bmatrix} \right\} < 0 \\ \tilde{Y} - B_{cl}^T \tilde{X} B_{cl} > 0, \; trace(\tilde{Y}) < \gamma^2, K = \Lambda_K \otimes K \end{cases} \quad (18)$$

Step4- If the decreasing rate of γ is less than a predetermined tolerance, a sub-optimal solution is found, stop ; else fix G and go to step2.

K_{ss}^* will denote the sub-optimal solution obtained by Algorithm 3.

IV. Application

In order to evaluate the efficiency of these algorithms, a the COnstrained Matrix-optimization Problems Library ($Compl_e ib$) [9] is used first, specially the set of systems named "Aircraft models" (AC). Table I summarizes the results obtained when applying Algorithm 1 to problems (AC1,AC3,AC6,AC12) (see [9]).

Table I. Results of Algorithm 1

Examples	Order n	Iterations number	β^*	K_s
AC1	5	3	-0.0377	K_s^1
AC3	5	1	-0,0183	K_s^2
AC6	7	1	-0.0157	K_s^3
AC12	4	3	-1.9860	K_s^4

where

$$K_s^1 = \begin{bmatrix} -0.4683 & -5.4924 & 0.2689 \\ -0.1876 & -2.1269 & 0.0701 \\ -1.5231 & -16.7352 & 0.2116 \end{bmatrix}, K_s^2 = \begin{bmatrix} -0.0072 & -0.2101 & 2.0426 & -0.2177 \\ 0.0130 & 0.3800 & -3.6528 & 0.3937 \end{bmatrix},$$

$$K_s^3 = \begin{bmatrix} -0.0106 & -0.0017 & 0.0022 & 0.0036 \\ -0.0071 & -0.0012 & 0.0014 & 0.0017 \end{bmatrix} \text{ and}$$

$$K_s^4 = 10^4 . \begin{bmatrix} 0.0001 & 0.0073 & -0.0080 & -1.7707 \\ -0.0004 & -0.0003 & 0.0003 & 0.0052 \\ -0.0000 & 0.0003 & -0.0003 & -0.0699 \end{bmatrix}$$

$\left(K_s^i\right)$ were used as initial points for Algorithm 2. Table II summarizes the results obtained when applying Algorithm 2 to the same set of problems.

Table II. Results of Algorithm 2

Examples	Iterations number	K_{ss}
AC1	4	$K_{ss}^1 = \begin{bmatrix} 0.0754 & 0 & 0 \\ 0 & -0.1758 & 0 \\ 0 & 0 & 0.6533 \end{bmatrix}$
AC3	4	$K_{ss}^2 = \begin{bmatrix} 0.2951 & 0.0989 & 0 & 0 \\ 0 & 0 & -3.0185 & 0.0430 \end{bmatrix}$
AC6	1	$K_{ss}^3 = \begin{bmatrix} -0.0206 & 0.0009 & 0 & 0 \\ 0 & 0 & 0.0006 & 0.0064 \end{bmatrix}$
AC12	4	$K_{ss}^4 = 10^4 . \begin{bmatrix} 0.0157 & 0.0006 & 0 & 0 \\ 0 & 0 & 0.0001 & -0.7602 \\ 0 & 0 & -0.0003 & 2.4698 \end{bmatrix}$

To show the efficiency of the proposed procedure (section III) an example is treated in order to find a sub-optimal H_2 structured static output feedback controller by applying successively Algorithm 1, Algorithm 2 and Algorithm 3.

The example considered here consists in a decentralized state feedback control problem borrowed from [10] and shared with [11]. The system (a linearized model of a double inverted pendulum) is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 9.8 & 0 & -9.8 & 0 \\ 0 & 0 & 0 & 1 \\ -9.8 & 0 & 2.94 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 0 \\ -2 & 5 \end{bmatrix} u$$

$$z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u.$$

Table III summarizes the results obtained when applying successively the proposed three algorithms.

Table III. A sub-optimal H_2 state feedback design

Algorithm	Iterations number	The obtained state feedback
Alg.1	2	$K_s = 10^4 \cdot \begin{bmatrix} -2.6223 & -0.5575 & -1.1529 & -0.1622 \\ -1.0606 & -0.2230 & -0.5599 & -0.0850 \end{bmatrix}$
Alg.2	1	$K_{ss} = \begin{bmatrix} -715.8935 & -182.0585 & 0 & 0 \\ 0 & 0 & -319.9573 & -76.1548 \end{bmatrix}$
Alg.3	61	$K_{ss}^{*} = \begin{bmatrix} -15.1519 & -4.1767 & 0 & 0 \\ 0 & 0 & -5.2052 & -0.2039 \end{bmatrix}$

Fig. 1 shows the graph of the H_2 norm versus the number of iterations. The optimal value of the closed-loop H_2 norm is $\gamma^* = 11{,}52$.

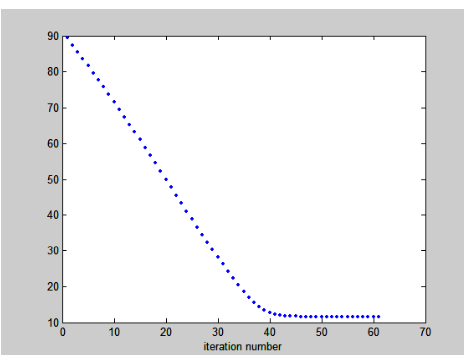

Fig. 1 H_2 norm of the structured state feedback

The difficulty with this kind of problems is that the global optimal is not known. To remedy to this, the next section will propose numerical test-examples allowing to test the initialization step sensitivity of the algorithm (1,2,3) proposed.

V. A TEST FOR STRUCTURED H_2 CONTROL ASSOCIATED BMI ALGORITHMS

Let us introduce the following augmented model

$$(P_a) \left[\begin{array}{cc|ccc} A & 0 & B_1 & B_2 & 0 \\ 0 & A & 0 & B_2 & I \\ \hline C_1 & 0 & 0 & D_{12} & 0 \\ 0 & -I & 0 & 0 & 0 \\ C_2 & -C_2 & D_{21} & 0 & 0 \end{array}\right] := \begin{bmatrix} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \bar{C}_1 & 0 & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & 0 \end{bmatrix} \quad (19)$$

and the structured static output feedback

$$K_{ss} = \begin{bmatrix} K_{(n_u \times n)} & 0 \\ 0 & L_{(n \times n_y)} \end{bmatrix} \quad (20)$$

The main idea is summarized in the Theorem 3.

Theorem 3

The following problems are equivalent:

i) Find an optimal H_2 controller for the standard system P.

ii) Find $K_{ss} = \begin{bmatrix} K_{(n_u \times n)} & 0 \\ 0 & L_{(n \times n_y)} \end{bmatrix}$ such that it internally stabilizes $F_l(P_a, K_{ss})$ and minimizes $\|F_l(P_a, K_{ss})\|_2$ if (P_a) is related to (P) by relation (19).

iii) Solve the following optimization problem for $\tilde{X}$, G and K_{ss} for a prescribed number $b = a^{-1} > 0$.

$$\min_{\tilde{X}, K_{ss}, G} \gamma^2$$

$$\begin{cases} \begin{bmatrix} 0 & -\tilde{X} & 0 \\ \bullet & 0 & 0 \\ \bullet & \bullet & -I \end{bmatrix} + sym\left\{ \begin{bmatrix} \bar{A} + \bar{B}_2 K_{ss} \bar{C}_2 \\ I \\ \bar{C}_1 + \bar{D}_{12} K_{ss} \bar{C}_2 \end{bmatrix} G \begin{bmatrix} I & -bI & 0 \end{bmatrix} \right\} < 0 \\ \tilde{Y} - \bar{B}_1^T \tilde{X} \bar{B}_1 > 0, \ trace(\tilde{Y}) < \gamma^2, K_{ss} = \begin{bmatrix} 1_{(n_u \times n)} & 0 \\ 0 & 1_{(n \times n_y)} \end{bmatrix} \otimes K_{ss} \end{cases} \quad (21)$$

where $1_{n \times m}$ denotes a n by m ones matrix.

Proof

The equivalence between i) and ii) follows immediately by construction of the augmented model (19) and the superposition principle. The equivalent between i) and iii) is straightforwardly derived from the dilation lemma from [2].

Remarks

- The optimization problem (21) is a BMI problem whose (global) solution may be obtained trivially thanks to the separation principle (the standard H_2 control problem). It can be used extensively to build structured S.O.F. H_2 problem and test the efficiency of the algorithms proposed.

The results obtained when applying successively the proposed three algorithms to solve the test proposed in this section to some academic problems borrowed from [9] (NN2,NN4) (see [9]) are given in what follows. (NN1 and NN2 were chosen for brevity reason)

Table III. The results obtained for problem NN2

Algorithm	Iterations number	The obtained state feedback
Alg.1	3	$K_s = 10^4 . \begin{bmatrix} 0.0069 & 0.6769 & 0.0100 \\ -5.1744 & 0.0501 & -0.0000 \\ -0.0582 & -5.6927 & -0.0100 \end{bmatrix}$
Alg.2	2	$K_{ss} = \begin{bmatrix} -6.3154 & -772.9716 & 0 \\ 0 & 0 & 0.0001 \\ 0 & 0 & -81.1619 \end{bmatrix}$
Alg.3	141	$K^*_{ss} = \begin{bmatrix} -1.2364 & -9.7673 & 0 \\ 0 & 0 & 0.1864 \\ 0 & 0 & -0.8302 \end{bmatrix}$

Fig. 2 shows the graph of the H_2 norm versus the number of iterations. The optimal value of the closed-loop H_2 norm is $\gamma^* = 1.49$ and the suboptimal value obtained is $\gamma = 1.53$ (Algorithm 3 was stopped due to slow progression).

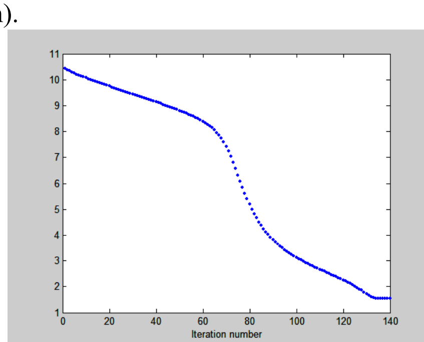

Fig. 2 H_2 norm versus iteration number

Table III. The results obtained for problem NN4

Alg.	Iterations number	The obtained state feedback
Alg.1	2	$K_s = 10^4 . \begin{bmatrix} 0.0065 & 0.0010 & -0.0044 & 0.0051 & 0.0020 & 0.0018 & 0.0004 \\ -0.0089 & -0.0014 & 0.0060 & -0.0070 & -0.0027 & -0.0025 & -0.0005 \\ -9.4089 & -3.1704 & 4.4615 & -5.6952 & -0.0000 & -0.0000 & -0.0000 \\ -3.2886 & -1.1127 & 1.5685 & -1.9990 & -0.0105 & -0.0099 & -0.0020 \\ 4.5370 & 1.5304 & -2.1636 & 2.7492 & -0.0001 & -0.0001 & -0.0000 \\ -5.6917 & -1.9234 & 2.6977 & -3.4553 & 0.0058 & 0.0054 & 0.0011 \end{bmatrix}$
Alg.2	2	$K_{ss} = 10^3 . \begin{bmatrix} 1.9501 & 0.6551 & -0.9274 & 1.1856 & 0 & 0 & 0 \\ -0.8426 & -0.2806 & 0.4017 & -0.5146 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0064 & -0.0055 & -0.0015 \\ 0 & 0 & 0 & 0 & -0.0273 & -0.0236 & -0.0015 \\ 0 & 0 & 0 & 0 & -0.0137 & -0.0125 & -0.0032 \\ 0 & 0 & 0 & 0 & 0.0218 & 0.0193 & 0.0028 \end{bmatrix}$
Alg.3	135	$K^*_{ss} = \begin{bmatrix} 577.4048 & 194.7855 & -273.6447 & 350.3615 & 0 & 0 & 0 \\ 234.8373 & 80.3860 & -110.4816 & 141.5532 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 39.8286 & 40.5473 & 4.3140 \\ 0 & 0 & 0 & 0 & -53.1702 & -52.0482 & -11.2586 \\ 0 & 0 & 0 & 0 & 0.0402 & 0.9228 & -2.6935 \\ 0 & 0 & 0 & 0 & -38.8766 & -39.7031 & -0.6580 \end{bmatrix}$

The optimal value of the closed-loop H_2 norm is $\gamma^* = 1.53$ and the suboptimal value obtained is $\gamma = 1.7$

VI. CONCLUSION

The structured H_2 control synthesis or simultaneous stabilization problems have no polynomial time solution in general. The underlying BMI problem can't be reduced to an LMI. The dilation lemma however, enables new characterizations for such problems allowing interesting additional degrees of freedom. Based on it, a novel coordinate-descent iterative algorithm has been proposed as an efficient alternative to existing one. The relevance of the approach has been illustrated with several numerical examples. The numerical results obtained as well as the computational effort to obtain them (in terms of number of LMI iterations) have been presented and analyzed. They are promising and make the author confident concerning their applicability to concrete multivariable structured H_2 control problem. In particular, the problem of an overlapped controller design of a three motors web transport system is at present under consideration using this technique.

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