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Relative Sensing Networks: Observability, Estimation, and the Control Structure

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Abstract— **In this paper we consider various structural aspects of estimation and control over relative sensing networks. We first expand on our earlier characterization of equivalent sensing and control topologies that incorporates the presence of measurement noise. Next, we consider the interplay between system minimality, network observability, and estimation problems. We conclude our presentation by pointing out a reciprocity between the relative sensing geometry on one hand, and the control structure for distributed dynamic systems on the other. Simulations and examples are incorporated throughout the paper to further complement the theoretical analysis.**

Index Terms— **Relative sensing networks, algebraic graph theory, estimation over networks, networked dynamic systems**

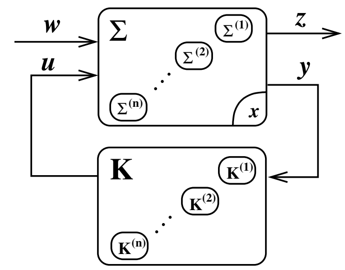

Fig. 1. The RSN feedback configuration

I. Introduction

Our goal in this work is to gain a deeper understanding of how the geometry of an underlying relative sensing network (RSN) influences the observability, state estimation, and the control system structure for a networked dynamic system. The control configuration of interest is shown in Fig. 1 where the signal z captures the coordination state among multiple dynamic systems; signals x, w, y, and u, denote respectively, the system state- comprised of states of the individual dynamic elements- the exogenous signal, the information vector available to the controller (sensed or communicated), and finally, the control input. As the control objective is the coordination of relative states among dynamic units, it is assumed that $z(t)$ in Fig. 1 consists of components that are functions of vector differences $x_i(t) - x_j(t)$ ($i \neq j$). Likewise, it is natural to assume that the information available to the controller, $y(t)$, consists of a subset of these relative states. We refer to such a feedback system setup, and the resulting host of control issues, as the problem of control over RSNs. Such systems have recently been considered by Smith and Hadaegh [14]; this reference has in fact motivated our studies on RSNs in [13]. Other related works include [1], [4], [9], [10], [16], and [17].

In the present work, we study the structural aspects of estimation and control over RSNs to gain insight into the problem of controlling a networked dynamic system. In this direction, we first expand on our earlier characterization of equivalent sensing and control topologies that incorporates the presence of measurement noise in §II-III. Along the way, we point out a direct ramification of such characterizations for robustness analysis of uncertain RSNs in §II-A. In §IV, we consider the interplay between dynamic system minimality, network observability, and estimation algorithms. We conclude the paper by pointing out a reciprocity between the geometry of the underlying RSN on one hand, and the control structure of the networked system, on the other. Simulations and examples are incorporated throughout the paper to further complement the theoretical analysis.

We start the presentation with notation and a short discussion on preliminaries.

A. Notation and preliminaries

A graph $G = (V, E)$ consists of a vertex set $V(G)$ and an edge set $E(G)$, whose elements (i.e., edges) connect pairs of vertices, making them adjacent to each other. The graphs considered in this paper are simple- multiple edges connecting a pair of vertices and those starting and ending at the same vertex (i.e., loops) are not allowed. Directed graphs consist of oriented edges with "tails" and "heads." A complete graph on n vertices contains all potential $n(n-1)/2$ edges- with or without tails and heads. We denote by $G_{j/i}$ the graph obtained by removing the edges of G_i from G_j when $V(G_i) \subseteq V(G_j)$ and $E(G_i) \subseteq E(G_j)$. In a connected graph every vertex is reachable from every other vertex by moving along its edges- for the directed case we will allow traversing an edge in any direction along the way (i.e., weak connectivity). A minimally connected graph such that removing any edge results in a disconnected graph is called a tree. A spanning tree of G is a tree on $V(G)$. Since a tree is minimally connected, it cannot contain a cycle- a subgraph where every vertex has exactly two adjacent vertices. In this paper, we use the terms "sensing network" and "information graph" synonymously to correspond to simple, directed, and connected graphs.

Our distributed system consists of dynamic units indexed as $\{1, 2, \ldots, n\}$; this set will be denoted by $[n]$. We write

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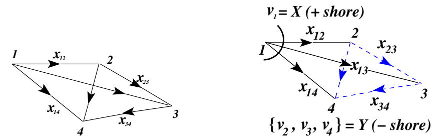


(a) (b)

Fig. 2. (a) An example of an information graph on four nodes, (b) Cut V_1 on a four noded network

$|S|$ for the cardinality of set S. Thus $|\,[n]\,| = n$. The matrix I_p denotes the $p \times p$ identity matrix; $\mathbf{1}$ is the vector of size n with all entries equal to one, **span** $\{x\}$ refers to the subspace spanned by the vector x, and **Diag** $\{x\}$ is the diagonal matrix whose diagonal entries orderly correspond to the entries of x. The Kronecker product of two matrices A and B will be denoted by $A \otimes B$ [7]; $f \circ g$ refers to the the composition of two mappings f and g. Lastly, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are, respectively, the range and the null spaces of the matrix A.

B. The linear algebra of graphs

For the information graph G, the incidence matrix $D(G)$ is defined as the $|V(G)| \times |E(G)|$ matrix in the following way: $[D(G)]_{k\,l} = 1$ if v_k is the head of e_l, $[D(G)]_{k\,l} = -1$ if v_k is the tail of e_l, and $[D(G)]_{k\,l} = 0$ if edge e_l is not incident on vertex v_k . Thus for the graph of Fig. 2(a),

$$D(G) = \begin{array}{cc} & \begin{array}{cccccc} e_{1,2} & e_{1,3} & e_{1,4} & e_{2,3} & e_{2,4} & e_{3,4} \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

Given the graph G consider a partition of $V(G)$ into two non-empty subsets X and Y. We refer to the set of edges with one end in X and the other in Y as a cut; X and Y are then called the shores of the cut. If we assign one shore as positive and the other as negative, then we have an *oriented* cut. The set of all such cuts constitute the cut space $\mathcal{T}(G)$- which for a graph on n vertices having c connected components- has dimension $n - c$. For a cut one can define a vector $z \in \mathbf{R}^{|E_G|}$, referred to as its *signed characteristic vector*, where each component is defined in the following way: $z_e = 0$ if the edge e is not in the cut, $z_e = 1$ if the head of e lies in the positive shore, and $z_e = -1$ if the head of e lies in the negative shore. Consider the cut V_i associated with a vertex v_i such that the partition (X, Y) consists of the singleton $v_i \in X$ on the positive shore, and on the negative shore Y, lie vertices v_j $(j \neq i)$. Then the characteristic signed vector associated with V_i is precisely the i-th row of the incidence matrix $D(G)$; see Fig. 2(b). In fact, the cut space can now be characterized by $\mathcal{R}(D(G)^T)$. Analogously, define the subspace orthogonal to $\mathcal{T}(G)$ as the cycle space $\mathcal{C}(G)$. Graphical construction of $\mathcal{C}(G)$ is also possible: first define an *oriented cycle* as a cycle with a particular cyclic direction; next construct a *signed characteristic vector* $w \in \mathbf{R}^{|E_G|}$ associated with this cycle such that each component of w is an element of $\{-1, 0, 1\}$, depending on the orientation of the edge with respect to the chosen cyclic direction. Linear algebraically though one can invoke the fundamental theorem of linear algebra to realize that

$$\begin{gathered} \mathcal{T}(G) := \mathcal{R}(D(G)^T), \quad \mathcal{C}(G) := \mathcal{N}(D(G)), \\ \mathcal{T}(G)^\perp = \mathcal{C}(G), \quad \text{and} \quad \mathcal{C}(G)^\perp = \mathcal{T}(G). \end{gathered}$$

Moreover, $\mathcal{C}(G) \oplus \mathcal{T}(G) = \mathbf{R}^{|E_G|}$. Note that for a connected graph, the rank of the incidence matrix is $|V(G)| - 1$ which is also the dimension of $\mathcal{T}(G)$. Similarly, the dimension of $\mathcal{C}(G)$ is $|E(G)| - (|V(G)| - 1)$.

C. Problem setup

We consider distributed dynamic systems of the form

$$\begin{aligned} \Sigma: \quad \dot{x}(t) &= f(x(t), u(t), w(t)) && (1) \\ y(t) &= Cx(t) + w(t) && (2) \\ z(t) &= h(x(t), u(t), w(t)), && (3) \end{aligned}$$

where $x = [\,x_1^T, x_2^T, \ldots, x_n^T\,]^T$ with $x_i \in \mathbf{R}^p$, represents the state of the system, y is the information vector available to the controller over an RSN; see Fig. 1 and the related paragraph on the feedback setup in §I. The noise-free $y(t)$ in (2) (when $w = 0$) is assumed to have components of the form

$$x_{ij}(t) := x_j(t) - x_i(t),$$

for some distinct $i, j \in [n]$. This information geometry can naturally be represented in terms of a directed graph G. For example Fig. 2(a) corresponds to the situation where

$$y(t) = [\,x_{12}(t)^T\, x_{13}(t)^T\, x_{14}(t)^T\, x_{23}(t)^T\, x_{24}(t)^T\, x_{34}(t)^T\,]^T.$$

The incident matrix provides a convenient way to represent the information geometry as

$$y_G(t) = (D(G)^T \otimes I_p)\, x(t) \qquad (4)$$

with G being the underlying information graph available to each node of the networked system Σ. Occasionally, in order to simplify the notation, we will use $D(G)$ to denote both the incidence matrix as well as its inflated version $D(G) \otimes I_p$.

Now consider the scenario where a control law, K_d, has been designed for a particular information geometry represented by oriented graph G_d. Suppose however that the actual relative states available to the controller is $y_j(t) := D(G_j)^T x(t)$. As it was shown in [13], for a system without measurement noise, there exists a linear transformation T_{dj} such that for all t, $y_d(t) = T_{dj}\, y_j(t)$. Equivalently one can view these transformations as a mechanism for control reconfiguration. Accordingly, the controller K_d constructed to operate on information geometry G_d can be updated as $K_j = K_d \circ T_{dj}$, so that it can operate on the information abstracted by the graph G_j [13]. The central idea in deriving these transformations is the realization that the RSNs contain algebraic redundancies; for example, for any $i, j, k \in [n]$ one has $x_{ik}(t) + x_{kj}(t) - x_{ij}(t) = 0$. In the graph theoretic framework, these redundancies correspond with the cycles of the graph.

In the present work, after postulating a framework for constructing the above transformations, we characterize optimal minimum variance transformations for noisy RSNs. Next we look at the interplay between spanning trees, minimality, and observability of the corresponding networked systems. Lastly, we delineate upon a case where the structure of the RSN guides the construction of local controllers for the system (1)-(3).

II. T-TRANSFORMATIONS FOR NOISE-FREE RSNs

Our discussion on transformations that facilitate mapping equivalent information geometries for RSNs, or alternatively, characterize a mechanism for controller reconfiguration, revolve around three canonical cases. These cases include: transformations from a connected graph to any of its subgraphs, from spanning trees to a connected graph, and between two arbitrary connected graphs. Such mappings will be collectively referred to as T-transformations.

First, consider the scenario where the desired sensing graph, G_d, is a subgraph of the measured graph G_j. Given that G_d has m_d edges and G_j has m_j edges, the transformation T_{dj} is an $m_d \times m_j$ matrix such that $D(G_d)^T = T_{dj} D(G_j)^T$. Consider the decomposition

$$T_{dj} = [\, \widehat{T}_{dj} \mid \widetilde{T}_{dj} \,] \in \mathbf{R}^{m_d \times m_j} \tag{5}$$

with $\widehat{T}_{dj} \in \mathbf{R}^{m_d}$ and $\widetilde{T}_{dj} \in \mathbf{R}^{m_d \times m_j - m_d}$ and the corresponding partitioning

$$D(G_j) = [\, D(G_d) \,|\, D(G_{j/d}) \,];$$

hence $\widehat{T}_{dj} = I$ and $\widetilde{T}_{dj}\, D(G_{j/d})^T = 0$. The trivial solution for $\widetilde{T}_{dj}$ is the zero matrix while the general matrix solution consists of rows that belong to the cycle space $\mathcal{C}(G_{j/d})$. When $G_{j/d}$ is a tree the trivial solution is unique. The second canonical case corresponds to the transformation $T_{dj}\, D(G_j)^T = D(G_d)^T$, where G_j is a spanning tree and the target graph G_d represents an arbitrary information network. In this case, one can state the following result [13].

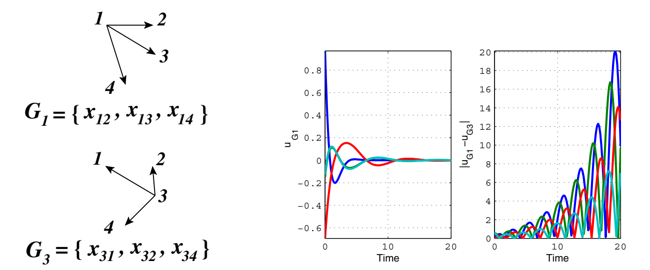
Proposition 2.1: Let G_j be a spanning tree. Then

$$T_{dj} = \left\{ \left[(D(G_j)^T D(G_j))^{-1} D(G_j)^T \right] D(G_d) \right\}^T .$$

The last scenario corresponds to the transformation T_{dj} such that $D(G_d)^T = T_{dj}\, D(G_j)^T$, where G_d and G_j are arbitrary RSNs. There are at least two approaches to finding this transformation: (a) transform the given graph G_j to a spanning tree subgraph G_k, followed by the transformation for second case above that converts this spanning tree to desired graph, i.e., $T_{dj} = T_{dk} T_{kj}$, or (b) complete the cycles of G_j to obtain a complete graph G_c then choose an appropriate subgraph of G_c (re-orient edges if necessary) that corresponds to G_d- then $T_{dj} = T_{dc} T_{cj}$. Yet another MATLAB implementable transformation between arbitrary RSNs is provided by the following procedure.

Proposition 2.2: For arbitrary RSNs G_j and G_d one has

$$T_{dj} = \left\{ \left[\widehat{D}(G_j)^T (\widehat{D}(G_j) \widehat{D}(G_j)^T)^{-1} \right] \widehat{D}(G_d) \right\}^T , \tag{6}$$



(a) (b)

Fig. 3. (a) Robustness analysis for the feedback control law designed for RSN G_1 when the actual RSN is G_3, (b) Nominal control operating on RSN G_1 (left plot) and the difference between the resulting controls as applied to RSN G_1 and RSN G_3 (right plot)

where $\widehat{D}(G_j)$ and $\widehat{D}(G_d)$ consist of any $(n-1)$ rows (or cuts) of graphs G_j and G_d. As in the second case above, if the initial RSN is a spanning tree, then the unique T_{dj} is $\widehat{D}(G_d)^T (\widehat{D}(G_j)^T)^{-1}$. However for an initial RSN with cycles, such a transformation is not unique.

A. Robustness analysis for uncertain RSNs via T-transformations

Consider the scenario where a stabilizing controller has been designed for Σ (1)-(3), assuming the availability of the RSN G_1 in Fig. 3(a), while each node has in fact the measurement graph G_3 available to it. The control objective is to drive the error between the desired and actual states of the RSN G_1 to the origin. Assume that each node has double integrator dynamics and the relative states associated with G_1 are: $[\, z_1(t)^T \, \dot{z}_1(t)^T \,]^T := [\, (D(G_1)^T x(t))^T \, (D(G_1)^T \dot{x}(t))^T \,]^T$. The transformation of interest $T_{13} : D(G_3)^T \rightarrow D(G_1)^T$, accounts for the discrepancy between expected and available RSNs. In the case when this transformation is not applied, however, the feedback configuration can become unstable; see Fig. 3(b). The robustness of the designed control mechanism with respect to variations in the underlying RSN can be addressed in the robust control framework [3], [15]. For this purpose, assume that the allowable variation of RSNs is among the class of graphs consisting of spanning trees denoted by $\mathcal{G}$. Denote by $K_d(s)$ the stabilizing linear control law designed for $G_d \in \mathcal{G}$ for the system of relative states, $\dot{z}_d(t) = A z_d(t) + B u(t)$ and $y(t) = z_d(t)$; let $P_d(s)$ be the corresponding input-output transfer matrix from u to y. Furthermore assume that $G_j \in \mathcal{G}$ is an unknown RSN available to each node. Setting $\Delta_{jd} = T_{jd} - I$ with $D(G_j)^T = T_{jd}\, D(G_d)^T$, we realize that the transfer matrix from u to y for this "perturbed" system is now $T_{jd}\, P_d(s)$. The following result is a consequence of the small gain theorem applied in the context of feedback configurations that operate on uncertain RSNs.

Theorem 2.3: The linear control law $K_d(s)$ robustly stabilizes the system of relative states for an uncertain RSN G_j if $\|\Delta_{jd}\| < 1/\|\underline{S}(M_d, K_d)\|$, where

$$M_d(s) := \begin{bmatrix} 0 & P_d(s) \\ I & P_d(s) \end{bmatrix};$$

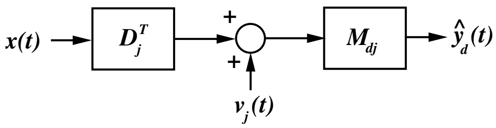


Fig. 4. Optimal (static) state estimation over graphs

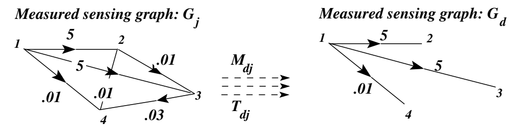


Fig. 5. Transformation between RSNs G_j and G_d in presence of noise

$\underline{S}(M_d, K_d)$ denotes the lower linear fractional transformation of $M_d(s)$ and $K_d(s)$, and the norm for a transfer matrix is its maximum singular value across all frequencies (i.e., its H_∞-norm).

III. RSN TRANSFORMATIONS FOR NOISY NETWORKS

Consider now the scenario where a controller has been designed for the minimal system specified by the sensing spanning tree $y_d(t) = D(G_d)^T x(t)$. Suppose that the measured sensing topology for one of the nodes is $\widetilde{y}_j(t) = D(G_j)^T x(t) + v_j(t)$, where v_j is the noise on the corresponding edges. In this section we consider the problem of finding the transformation T_{dj} that results in the minimum variance estimate for $y_d(t)$. We will denote this estimate by $\widehat{y}_d(t)$; see Fig. 4. For this purpose, we assume a zero mean measurement noise with measurement covariance σ_{ik}^2 for each relative state measurement x_{ik}. The covariance matrix for the complete graph is given by the diagonal matrix R:

$$R = \textbf{Diag}\,([\,\sigma_{12}^2\;\sigma_{13}^2\;\ldots\;\sigma_{(n-1)n}^2\,]^T).$$

The noise covariance on the graph G_j is denoted by R_j with diagonal entries that coincide, in an orderly way, with the diagonal entries of R for each measured edge of G_j.

We first show that the transformation M_{dj} satisfies the cycle constraints. Observe that

$$\widetilde{y}_j(t) = y_j(t) + v_j(t) = T_{jd}\, y_d(t) + v_j(t).$$

The objective is to find the transformation $M_{dj} : \widetilde{y}_j(t) \to \widehat{y}_d(t)$ such that at time t the mean value of the error,

$$\frac{1}{2}\sum_i \mathbf{E}\,[\,e_i^2(t)\,] := \frac{1}{2}\sum_i \mathbf{E}\,[\,(\widehat{y}_d(t) - y_d)_i^2\,],$$

is minimized. In the absence of noise one has $\widetilde{y}_j(t) = T_{jd}\, y_d(t) = y_j(t)$, $\widehat{y}_d(t) = y_d(t)$, and $y_d(t) = M_{dj}\, y_j(t) = M_{dj}\, T_{jd}\, y_d(t)$. Thus $M_{dj}\, T_{jd} = I$. Since each element of the error vector is,

$$e_i(t) = (\widehat{y}_d(t) - y_d(t)\,)_i = (\,M_{dj}\,(y_j(t) + v_j(t)) - y_d(t))_i$$

the problem of minimizing the error covariance for estimated relative states associated with G_d can be written as

$$\min_{M_{dj}} \quad \frac{1}{2}\,\textbf{Trace}\; M_{dj}\, R_j\, M_{dj}^T \tag{7}$$

subject to the equality constraint $M_{dj} T_{jd} = I$. The optimal transformation solving (7) is then given as

$$M_{dj} = (T_{jd}^T\, R_j^{-1}\, T_{jd})^{-1}\, T_{jd}^T\, R_j^{-1}.$$

Viewing the minimum variance transformation from yet another perspective, we note that for the perfect measurement case, $\widetilde{y}_j(t) = y_j(t)$, and hence $\widehat{y}_d(t) = y_d(t) = M_{dj}\, y_j(t)$. Thereby, $y_d(t) = M_{dj}\, y_j(t)$ and $\widehat{D}(G_d)^T = M_{dj}\, \widehat{D}(G_j)^T$, where $\widehat{D}(G_d)$ and $\widehat{D}(G_j)$ are defined in the paragraph following (6). Consequently, the optimization problem (7) can also be represented by

$$\min_{M_{dj}} \quad \frac{1}{2}\,\textbf{Trace}\; M_{dj}\, R_j\, M_{dj}^T \tag{8}$$

subject to $\widehat{D}(G_d)^T = M_{dj}\, \widehat{D}(G_j)^T$ that captures the cycle constraints. The optimal transformation is now given by

$$M_{dj} = \widehat{D}(G_d)^T \left[\widehat{D}(G_j) R_j^{-1} \widehat{D}(G_j)^T\right]^{-1} \widehat{D}(G_j) R_j^{-1}.$$

Note that when variances on all edges are equal, i.e., $R = \rho I$, this transformation is exactly the transformation T_{dj}, given in (6), for the perfect measurement case. Moreover, for the case when the sensing geometry is a spanning tree the resulting transformation is again $M_{dj} = T_{dj}$. The uniqueness of this solution follows from the fact that the RSN transformation from a spanning tree to any other graph is unique. In other words, when the measured sensing graph is minimal, the variance on the edges is irrelevant as the only feasible solution is specified by (6).

Example 3.1: Consider a 4-noded network of damped harmonic oscillators with identical dynamics $\ddot{\theta}_i(t) = -\omega_n^2 \theta_i(t) - 2\zeta\omega_n \dot{\theta}_i(t)$ with $\zeta = \sqrt{2}/2$ and $\omega_n = 1$. The desired and measured RSNs are shown in Fig. 5 with covariances labeled on the edges. Simulation results that compare the estimate $\widehat{y}_d(t)$ corresponding to the optimal transformation M_{dj} versus the estimate obtained by applying the T-transformation T_{dj} (6), is shown in Fig. (6). The estimated edges correspond to the relative angle measurements θ_{ij}.

IV. MINIMALITY, OBSERVABILITY, AND SPANNING TREES

In this section, we consider the interplay between system theoretic concepts of minimality and observability on one hand, and the graph theoretic constructions such as spanning trees, on the other. In this venue, let us first consider the networked system consisting of n double integrators,

$$\ddot{x}_i(t) = u_i(t),\; y_i(t) = D(G_d)^T x(t),\; i = 1, \ldots, n, \tag{9}$$

where $x_i \in \mathbf{R}$, $x := [\,x_1, \ldots, x_n\,]^T$ and G_d corresponds with an RSN. We note that system (9) describing the "node dynamics" is controllable but not observable as the inertial states can not be reconstructed via the relative state measurements. To make this observation more precise construct the observability matrix for (9) as

$$\mathcal{O}^{\mathcal{T}} = \mathbf{1} \otimes \begin{bmatrix} D(G_d) & 0 & 0 & \ldots & 0 \\ 0 & D(G_d) & 0 & \ldots & 0 \end{bmatrix}.$$

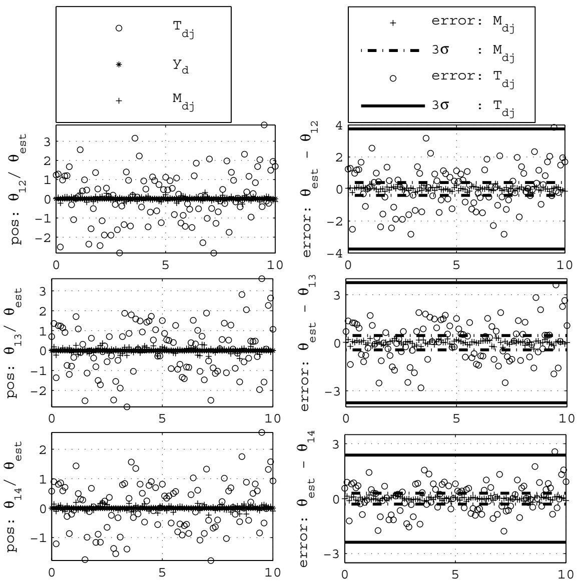


Fig. 6. The estimate $\widehat{y}_d(t)$ under M_{dj} and T_{dj} transformations

Thereby, $\mathbf{rank}\,\mathcal{O} = 2\,\mathbf{rank}\,D(G_d) = 2\,(n-1)$. The observable subspace is thus completely characterized by $\mathcal{R}(D(G_d))$. Since each column of $D(G_d)$ corresponds to an edge of G_d, any $(n-1)$ linearly independent edges define the observable subspace. These independent edges are however exactly the edges of a spanning tree of G_d. Similarly, the unobservable subspace of (9) is characterized by $\mathcal{N}(D_d^T)$. For a connected graph this subspace is spanned by $\mathbf{1}$, often referred to as the agreement subspace (see (15)). Since (9) is not observable, there exists a similarity transformation that brings into focus its observable component. Let $\bar{x}(t) = Px(t)$ and P is such that $y(t) = CP^{-1}\bar{x}(t) = \mathbf{1} \otimes \left[\begin{array}{cc} \tilde{C}_1 & 0 \end{array}\right] \bar{x}(t)$. The transformed state can now be partitioned into its observable and unobservable components as $\bar{x}(t) = [\,x_o(t)^T\, x_{\bar{o}}(t)^T\,]^T$. This transformation is given by

$$P^{-1} \;=\; [\,\mathcal{V}_o\,|\,\mathcal{V}_{\bar{o}}\,] = \left[\begin{array}{cccc} D(G_t) & 0 & \mathbf{1} & 0 \\ 0 & D(G_t) & 0 & \mathbf{1} \end{array}\right]$$

and hence

$$P = \left[\begin{array}{cc} D(G_t)^{\dagger} & 0 \\ 0 & D(G_t)^{\dagger} \\ (1/n)\,\mathbf{1}^T & 0 \\ 0 & (1/n)\,\mathbf{1}^T \end{array}\right] \tag{10}$$

where G_t is a spanning tree subgraph of G_d and

$$D(G_t)^{\dagger} := (D(G_t)^T D(G_t))^{-1} D(G_t)^T. \tag{11}$$

Let $[\,z(t)^T\,\dot{z}(t)^T\,] := [\,(D(G_t)^T x(t))^T\,(D(G_t)^T \dot{x}(t))^T\,]$ denote the vector of relative states associated with G_t; we refer to the corresponding dynamics as the "edge dynamics." The observable component of node states after applying the transformation (10) is thereby $x_o(t) = (D(G_t)^T D(G_t))^{-1}[\,z(t)^T \;\; \dot{z}(t)^T\,]^T$. Consequently, the observable component of the state for a networked system of double integrators is in direct correspondence with the edge dynamics of an underlying spanning tree.

A. Node to edge dynamics: the general case

Consider now the more general scenario where each node dynamics is governed by $\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t)$, with $x_i \in \mathbf{R}^p$ and $u_i(t) \in \mathbf{R}^m$. The dynamic model for the distributed system is then given by

$$\dot{x}(t) \;=\; Ax(t) + Bu(t), \tag{12}$$

$$y(t) \;=\; \mathbf{1} \otimes (D(G_d)^T \otimes I_p)\, x(t), \tag{13}$$

where $x(t) \in \mathbf{R}^{pn}$, $u(t) \in \mathbf{R}^{mn}$, and $A \in \mathbf{R}^{pn \times pn}$ and $B \in \mathbf{R}^{pn \times mn}$ are block diagonal matrices with the (i,i)-block entry of A_i and B_i, respectively; as always, G_d is a connected RSN. The observability matrix for (12)-(13) is given by

$$\mathcal{O} \;=\; \mathbf{1} \otimes \left[\begin{array}{c} \big(D(G_d)^T \otimes I_p\big) \\ \big(D(G_d)^T \otimes I_p\big)\, A \\ \vdots \\ \big(D(G_d)^T \otimes I_p\big)\, A^{n-1} \end{array}\right]. \tag{14}$$

Recall that for a quadruple A_1, A_2, A_3, A_4 of appropriate dimensions $(A_1 \otimes A_2)(A_3 \otimes A_4) = A_1 A_3 \otimes A_2 A_4$. Hence for any $x \in \mathbf{R}^p$, $(D(G_d)^T \otimes I_p)(\mathbf{1} \otimes x) = (D(G_d)^T\,\mathbf{1}) \otimes x = 0$. We observe that if for some $x \in \mathbf{R}^p$ and for all $j = 1, ..., n-1$, $((D_d^T \otimes I_p)\, A^j)\,(\mathbf{1} \otimes x) = 0$, then $(\mathbf{1} \otimes x) \in \mathcal{N}(\mathcal{O})$ and (12)-(13) is unobservable. In this case, the linear system that describes the edge dynamics can be derived from the node dynamics.

Let us refer to the set

$$\mathcal{A} := \{\mathbf{1} \otimes x \,|\, x \in \mathbf{R}^p\} \tag{15}$$

as the agreement subspace. Recall that a subspace $V \in \mathbf{R}^n$ is called A-invariant if for all $x \in V$, $Ax \in V$.

Proposition 4.1: The node dynamical system (12)-(13) is unobservable if the agreement subspace $\mathcal{A}$ (15) is A-invariant. In this case, there exists a similarity transformation such that the observable component of the node dynamics coincides with the edge dynamics.

Proof: The proof follows from the fact that for a connected graph the only elements of the null space of $D(G_d)^T \otimes I_p$ are of the form $\mathbf{1} \otimes x$, for some $x \in \mathbf{R}^p$. The similarity transformation of interest is now similar to those derived previously (see (10)). In this case,

$$P^{-1} = [\,D(G_t) \;\; \mathbf{1}\,] \otimes I_p \quad \text{and} \quad P = \left[\begin{array}{c} D(G_t)^{\dagger} \\ (1/n)\,\mathbf{1}^T \end{array}\right] \otimes I_p,$$

where $D(G_t)^{\dagger}$ is defined as in (11). Applying this transformation to system matrices of (13) results in

$$PAP^{-1} \;=\; \left[\begin{array}{cc} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{array}\right];\; PB = \left[\begin{array}{c} \widetilde{B}_1 \\ \widetilde{B}_2 \end{array}\right]$$

$$CP^{-1} \;=\; \mathbf{1} \otimes \left[\begin{array}{cc} (D(G_d)^T\, D(G_t) & 0 \end{array}\right] \otimes I_p$$

with $\widetilde{A}_{12} = (D(G_d)^{\dagger} \otimes I)\, A\, (\mathbf{1} \otimes I_p)$. Since each column of $\mathbf{1} \otimes I_p$ is of the form $\mathbf{1} \otimes x$ and $\mathcal{A}$ (15) is A-invariant, $\widetilde{A}_{12} = 0$. ■

Corollary 4.2: Given a distributed system with identical state matrices $A_i = \bar{A}$ $(i = 1, ..., n)$, the system (12)-(13) is unobservable. In this case, linear edge dynamics can be obtained from the node dynamics.

Proof: In view of Proposition 4.1 it suffices to show that the agreement subspace (15) is A-invariant. This follows by observing that $A(\mathbf{1}\otimes x) = (I_n \otimes \bar{A})(\mathbf{1}\otimes x) = \mathbf{1}\otimes \bar{A}x \in \mathcal{A}$. Let G_t be the spanning tree subgraph of G_d. Note that

$$\tilde{A}_{12} = (D(G_t)^\dagger \otimes I_p)(\mathbf{1} \otimes \bar{A}) = (D(G_t)^\dagger \mathbf{1} \otimes \bar{A}) = 0$$

and the corresponding minimal system assumes the form $(D(G_t)^\dagger \otimes I_p)\,\dot{x}(t) = (I_{n-1} \otimes \bar{A})\,(D(G_t)^\dagger \otimes I_p)\,x(t) + (D(G_t)^\dagger \otimes I_p)\,\tilde{B}_1 u(t)$. Multiplying both sides of this dynamic equation by $D(G_t)^T D(G_t)$ and defining the relative state vector $z(t) = (D(G_t)^T \otimes I_p)x(t)$ results in the linear system $\dot{z}(t) = (I_{n-1} \otimes \bar{A})\,z(t) + (D(G_t)^T \otimes I_p)\,\tilde{B}_1 u(t)$, $y(t) = \mathbf{1} \otimes \left[D(G_d)^T D(G_t)\left(D(G_t)^T D(G_t)\right)^{-1}\right] z(t)$, describing the edge dynamics. Note that if the original RSN G_d is a spanning tree, then $y(t) = \mathbf{1} \otimes z(t)$. ■

The node dynamics (12)-(13) however can be observable when the block diagonal system matrix A consists of heterogeneous blocks. In this case a network observer that outputs the inertial node states given the RSN measurements (and the control input) can be designed.

Example 4.3: Consider two first order systems, $\dot{x}_i(t) = a_i x_i(t) + b_i u_i(t)$, $i = 1, 2$, with relative state measurement $y(t) = x_2(t) - x_1(t) = [\,-1\ \ 1\,]\,x(t)$. With $a_1 = a_2$ the agreement subspace $\mathcal{A}$ is not observable. In the meantime, when $a_1 = -2, a_2 = 1$, the observer gain $L = [\,0\ \ -2\,]^T$ leads to a Hurwitz observer error state matrix $A + LC$ with eigenvalues -1 and -2.

V. Local control laws for the RSN

In this section, consider again the relative state dynamics of a system of double integrators in 1-D associated with a spanning tree G; denote this relative state by $z(t)$. Assume that the relative state measurement associated with an edge is available to both incident nodes. Next, construct the error dynamics that corresponds to a given reference signal, $\zeta_r(t) := [\,z_r(t)^T\ \dot{z}_r(t)^T\,]^T$, taking the form $\ddot{e}(t) = -D(G)^T\,\ddot{x}(t) = -D(G)^T\,u(t)$. Consider the state feedback controller $u(t) = K\,[\,e(t)^T\ \dot{e}(t)^T\,]^T$, where the i-th row of K is denoted by $K^{(i)}$. Set $K = [\,D(G)\ D(G)\,]$; note that this controller takes into account the incidence relation for each node since the j-entry of $K^{(i)}$ is zero if the j-th edge is not incident to node i. The resulting closed loop system is

$$\begin{bmatrix} \dot{e}(t) \\ \ddot{e}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -D(G)^T D(G) & -D(G)^T D(G) \end{bmatrix} \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix}.$$

For the above state matrix A_{cl} the characteristic equation is $\det A_{cl} = \det(\lambda^2 I + (\lambda+1) D(G)^T D(G)) = 0$. Since $\lambda = -1$ does not satisfy this equation it is not an eigenvalue of A_{cl}. The eigenvalues of A_{cl} thus satisfy $\det(\lambda^2/(\lambda + 1)\,I + D(G)^T D(G)) = 0$. Denoting the eigenvalues of $-D(G)^T D(G)$ by μ, one has for each i, $\mu_i = \lambda_i^2/(\lambda_i + 1)$, and hence $\lambda_i = \left(\mu_i \pm \sqrt{\mu_i^2 + 4\mu_i}\right)/2$; see also [12], [18]. As $-D^T D < 0$, $\mu_i < 0$ for all i, and A_{cl} is Hurwitz guaranteeing that $\{e(t), \dot{e}(t)\} \to 0$ as $t \to \infty$. If we convert the edge dynamics back to the node dynamics, the closed loop system assumes the form

$$\dot{\eta}(t) = \mathcal{L}(G)\ \eta(t) + \mathcal{D}(G)\ \zeta_r(t),$$

where $\eta(t) := [\,x(t)^T\ \dot{x}(t)^T\,]^T$,

$$\mathcal{L}(G) = \begin{bmatrix} 0 & I \\ -L(G) & -L(G) \end{bmatrix},\ \mathcal{D}(G) = \begin{bmatrix} 0 & 0 \\ D(G) & D(G) \end{bmatrix},$$

and $L(G) = D(G)\,D(G)^T$ is the graph Laplacian [5]. The closed loop system that corresponds to the relative state dynamics with $\zeta(t) := [\,z(t)^T\ \dot{z}(t)^T\,]^T$ is now

$$\dot{\zeta}(t) = -\mathcal{E}(G)\ \zeta(t) + \mathcal{E}(G)\ \zeta_r(t)$$

where

$$\mathcal{E}(G) := \begin{bmatrix} 0 & I \\ E(G) & E(G) \end{bmatrix}$$

and $E(G) = D(G)^T D(G)$; this last matrix can be viewed as the edge version of the graph Laplacian. Choosing $\zeta_r(t) = 0$ now implies that the relative states asymptotically converge to the origin. This observation parallels the general results on the agreement protocol as considered for example in [6], [8], [11], [12], [17].

References

[1] R. D'Andrea and G. E. Dullerud, Distributed control design for spatially interconnected systems. *IEEE Transactions on Automatic Control*, (48) 9: 1478-1495, 2003.

[2] R. Diestel. *Graph Theory*, Springer, 2000.

[3] G. Dullerud and F. Paganini. *A Course in Robust Control Theory*, Springer, 2000.

[4] A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations, *IEEE Transactions on Automatic Control*, (49) 9: 1465- 1476, 2004.

[5] C. Godsil and G. Royle. *Algebraic Graph Theory*, Springer, 2001.

[6] Y. Hatano and M. Mesbahi. Agreement over random networks. *IEEE Transactions on Automatic Control* (to appear).

[7] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*, Cambridge, 1991.

[8] A. Jadbabaie, J. Lin and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules, *IEEE Transactions on Automatic Control*, (48) 6: 988–1001, 2003.

[9] M. Mesbahi and F. Y. Hadaegh. Formation flying control of multiple spacecraft via graphs, matrix inequalities, and switching, *AIAA Journal of Guidance, Control, and Dynamics*, (24) 2: 369–377, 2001.

[10] M. Mesbahi. State-dependent graphs and their controllability properties, *IEEE Transactions on Automatic Control*, (50) 3: 387 - 392, 2005.

[11] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays, *IEEE Transactions on Automatic Control*, (49) 9: 1520- 1533, 2004.

[12] W. Ren and E. Atkins. Second-order Consensus Protocols in Multiple Vehicle Systems with Local Interactions, *AIAA Conference on Guidance, Navigation, and Control*, 2005.

[13] J. Sandhu, M. Mesbahi, and T. Tsukamaki. Cuts and Flows of Relative Sensing and Control of Spatially Distributed Systems, *American Control Conference*, 2005.

[14] R. S. Smith and F. Y. Hadaegh. Control of deep-space formation-flying spacecraft: relative sensing and switched information, *AIAA Journal of Guidance, Control, and Dynamics*, (28) 1: 106-114, 2005.

[15] R. S. Sánchez-Peña and M. Sznaier. *Robust Systems Theory and Applications*, Wiley, 1998.

[16] P. K. C. Wang and F. Y. Hadaegh. Coordination and control of multiple microspacecraft moving in formation, *Journal of the Astronautical Sciences*, 44: 315–355, 1996.

[17] L. Xiao and S. Boyd. Fast linear iterations for distributed averaging. *Systems and Control Letters*, (53): 65-78, 2004.

[18] F. Zhang. *Matrix Theory*, Springer, 1999.