

Discrete port-Hamiltonian systems: mixed interconnections

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Abstract—Either from a control theoretic viewpoint or from an analysis viewpoint it is necessary to convert smooth systems to discrete systems, which can then be implemented on computers for numerical simulations. Discrete models can be obtained either by discretizing a smooth model, or by directly modeling at the discrete level itself. The goal of this paper is to apply a previously developed discrete modeling technique to study the interconnection of continuous systems with discrete ones in such a way that passivity is preserved. Such a theory has potential applications, in the field of haptics, telemanipulation etc. It is shown that our discrete modeling theory can be used to formalize previously developed techniques for obtaining passive interconnections of continuous and discrete systems.

I. INTRODUCTION

In previous work, see e.g. [1], [2], [3], it has been shown how port-based network modeling of complex lumped-parameter physical systems naturally leads to a generalized Hamiltonian formulation of the dynamics. In fact, the Hamiltonian is given by the total energy of the energy-storing elements in the system, while the geometric structure, defining together with the Hamiltonian the dynamics of the system, is given by the power-conserving interconnection structure of the system, and is called a Dirac structure. Furthermore, energy-dissipating elements may be added by terminating some of the system ports. The resulting class of open dynamical systems has been called "port-Hamiltonian systems" ([1], [3]). The port-Hamiltonian framework offers many fundamental benefits. Firstly, it is instrumental in finding the most convenient representation of the equations of motion of the system; in the format of purely differential equations or of mixed sets of differential and algebraic equations (DAEs). From an analysis point of view it allows to use powerful methods from the theory of Hamiltonian systems. Finally, the Hamiltonian structure may be fruitfully used in control design, e.g. by the explicit use of the energy function and conserved quantities for the construction of a Lyapunov function (possibly after the connection with another port-Hamiltonian controller system), or by directly modifying by feedback the interconnection and dissipation structure and shaping the internal energy. We refer to [4], [3] for various work in this direction.

It is well known that for the study of complex physical systems, numerical simulation plays an important role. One of the most important areas of numerical analysis is in

understanding the role that the structure (conservation laws, symmetries etc.) of the physical system plays in simulations, c.f. [5]. Discrete systems themselves can be obtained in two ways. Either we can discretize continuous systems (there exist a wide variety of techniques for doing so), or we can directly model at the discrete level itself. In previous work [6], [7], [8], [9], [10] we developed a methodology to model physical systems directly at the discrete (in space and time) level and we showed that the discrete models which we obtain as a result of our modeling process *exactly* coincide with discretized models!, thus offering an *alternative approach* towards the simulation of port-Hamiltonian systems.

In [11], [12] a novel procedure for interconnecting discrete systems with continuous ones *passively* was developed, and the procedure was tested on various examples like haptic devices and telemanipulation. The work was set in the framework of port-Hamiltonian systems, i.e. the discrete and the continuous systems were assumed to be port-Hamiltonian. However since discrete port-Hamiltonian systems are not defined, the interconnection of the discrete system with the continuous one is not very clear. It is true that the interconnection preserves passivity, but the nature of the interconnection is not clear. The goal of this paper is to use our discrete port-Hamiltonian theory, c.f. [6], [10], [9] to study the nature of this interconnection.

The outline of the paper is as follows. We briefly recall discrete (port) Hamiltonian mechanics and certain geometrical concepts in Section II. In Section III we give the problem formulation. The constraints on the interconnection variables, in order to preserve passivity, are described in Section III-A. Since port-Hamiltonian systems are described as power-conserving, hence the energy conserving problem is transformed to a power conserving problem in Section III-B. Finally in Section III-C the main formal result is given concerning the interconnection of a discrete system with a continuous one.

II. GEOMETRY AND HAMILTONIAN MECHANICS ON DISCRETE SPACES

In this section we briefly recall certain concepts of discrete Hamiltonian mechanics, for more details c.f. [7], [8]. The first requirement is to choose an appropriate discrete analogue for the reals \mathbb{R} . We can use discrete lattices (which have a ring structure), or the space of floating point numbers \mathbb{F} which have a quasi-ring (c.f. [7], [8]) structure. Since computers use floating-point numbers, and since our main focus is numerical simulation, \mathbb{F} will be our choice. A **discrete vector** at the point $p \in \mathbb{F}^n$ is a pair (p, q) where $q \in \mathbb{F}^n$. We will denote by $T_p\mathbb{F}^n$ the set defined as

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the union of all possible vectors defined at the point p , i.e. $T_p\mathbb{F}^n = \{(p, q) \in \mathbb{F}^n \times \mathbb{F}^n\} \sim \mathbb{F}^n$. Unlike in the smooth setting, there are several representations of discrete vectors. *Each representation corresponds to a certain numerical integration technique.* We recall two representations here, the Euler discrete vector and the Runge-Kutta 2 vector. These correspond to the Euler forward difference and the second order Runge-Kutta integration techniques. In [7], [8] we have defined others like Runge-Kutta vectors of any order, Leap-Frog vectors, central difference vectors etc. Euler vectors or Runge-Kutta 2 vectors are defined as: $v(f(p)) = \frac{f(p+\varepsilon) - f(p)}{h}$. Where ε is the smallest possible distance from the point p to the next floating point number. The difference between Euler vectors and Runge-Kutta 2 vectors is of course in the actual value of $f(p + \varepsilon)$. The point we are trying to make is that discrete vectors have the same finite-difference structure, they only differ in the values! A discrete vector¹ does not satisfy the usual Leibniz (or product) rule for derivations, rather it is a linear map $v_i : A_p(\mathbb{F}^n) \rightarrow \mathbb{F}$ which satisfies the **modified Leibniz rule**: $v(f \cdot g) = v(f) \cdot g(p) + Aut_v(f(p)) \cdot v(g)$, $\forall f, g \in A_p(\mathbb{F}^n)$, where Aut_v is an automorphism which is a linear map $Aut_v : A_p(\mathbb{F}^n) \rightarrow \mathbb{F}$, corresponding to the discrete vector v , defined as: $Aut_v(f(p)) := f(p + \varepsilon)$, $p \in \mathbb{F}^n$ such that $Aut_v(f \cdot g) = Aut_v(f) \cdot Aut_v(g)$; $\forall f, g \in A_p(\mathbb{F}^n)$

Discrete covectors are defined as mapping pairs of points (i.e. discrete vectors) to a floating point number, i.e. $v^* : (p, q) \rightarrow \mathbb{F}$. The set of discrete covectors forms the discrete cotangent space.

Then, we can define **discrete vector fields** as the mapping X which assigns to each point $p \in \mathbb{F}^n$ a discrete vector, i.e. $\forall p \in \mathbb{F}^n, X(p) = (p, q), q \in \mathbb{F}^n$. The flow of the discrete vector field X is defined as the sequence of points p_0, p_1, p_2, \dots in \mathbb{F}^n such that $X(p_i) = (p_i, p_{i+1})$. Likewise we can define discrete one-forms as assigning a discrete covector to each point. A function $f : \mathbb{F}^n \rightarrow \mathbb{F}$ is said to be **discrete-differentiable** at $p \in \mathbb{F}^n$ iff there exists a mapping $G : A(\mathbb{F}^n) \rightarrow \mathbb{F}^n$ s.t. $\frac{f(p+\varepsilon) - f(p) - G(f(p) \cdot \varepsilon)}{\varepsilon} = 0$. Note that the above definition does classify discrete functions between those that are discrete differentiable and those which are not. This is easy to see, since we use floating point numbers, the computation - $\frac{f(p+\varepsilon) - f(p) - G(f(p) \cdot \varepsilon)}{\varepsilon}$ can easily result in a floating point overflow.

The *discrete exterior differential* is a mapping: $\Delta : \bigwedge^k(\mathbb{F}^n) \rightarrow \bigwedge^{k+1}(\mathbb{F}^n)$, defined in the following way. Consider, for instance, a function $f \in A(\mathbb{F}^n)$. The function corresponds to the assignment of an element of \mathbb{F} at each point of the discrete space. The definition of a discrete one-form implies that we must construct a covector at each point. We can do that in many different ways, but if we want to preserve at the discrete level the smooth property $X(f) = \langle X, \Delta f \rangle$, the definition of the exterior differential must take into account the type of action that vector fields have on functions. For the forward difference method, this leads us to a definition

¹In [7], [8] we have shown that a collection of discrete vectors (Euler vectors, Runge-Kutta vectors etc.) form a 'discrete' tangent space.

of the exterior differential such as to define the one-form $\Delta f \in \bigwedge^1(\mathbb{F}^n)$ which for every point $p \in \mathbb{F}^n$ assigns to the one-dimensional hypersurface (i.e. a link) connecting each pair of points (p, q) , where the pair of points are defining a discrete vector, the value $f(q) - f(p)$ (note that this definition can be easily extended for higher-order forms, c.f. [6], [7]). Hence, in the natural basis, we would obtain as a representation: $\Delta f(p) = \sum_i (f(p + h\varepsilon_i) - f(p)) dx^i$, where h is the smallest possible distance from the point p to the next floating point number in the i -th direction of the point p , and $\varepsilon_i = [0, \dots, 1, 0, \dots]^T$. The concept of **discrete manifolds** has been introduced in [7], [8]. Discrete manifolds are those that locally look like \mathbb{F}^n , on these we can define the discrete analogues of charts, atlases etc. Since \mathbb{F}^n has a discrete-differentiable structure, this structure can be transferred onto discrete manifolds via chart mappings.

Let us conclude this section with discrete Hamiltonian mechanics. One way to do that would be by defining a discrete Poisson bracket as follows. Let \mathcal{Z} be a discrete manifold and consider the algebra of discrete differentiable functions $A(\mathcal{Z})$ on \mathcal{Z} . This is endowed with a discrete Poisson structure if there exists a mapping from $A(\mathcal{Z})$ to the set of discrete vector fields $\mathfrak{X}(\mathcal{Z})$ which defines an intrinsic operation as: $\{f, g\} := X_f(g)$. This definition easily satisfies the required properties of skew-symmetry, bilinearity and the modified Leibniz rule. And then we can define discrete Hamiltonian dynamics as follows. We have a canonical mapping from the algebra $A(\mathcal{Z})$ onto the space of discrete vector fields $\mathfrak{X}(A)$ of the algebra: $f \mapsto X_f = \{f, \cdot\}$, $\forall f \in A(\mathcal{Z})$. The discrete Poisson dynamics are defined as follows. for any $f \in A(\mathcal{Z})$: $\frac{\Delta f(t)}{\Delta t} = \{f, H\} \Rightarrow f_{n+\delta} = f_n + \delta X_H(f_n)$. So in the limit as $\delta \rightarrow 0$ we recover the definition of dynamics in the smooth case using the smooth Poisson bracket: $\dot{f} = \{f, H\} = X_H(f)$.

Now let us briefly recall discrete port-Hamiltonian systems from [6], [9]. A constant discrete Dirac structure on a finite-dimensional q-module \mathbb{F}^n is a n -dimensional subspace $\mathcal{D} \subset \mathbb{F}^n \times \mathbb{F}^{n*}$ with the property that

$$\langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle = 0, \quad \forall (f_1, e_1), (f_2, e_2) \in \mathcal{D} \quad (1)$$

where $\langle | \rangle$ denotes the natural pairing between \mathbb{F}^n and \mathbb{F}^{n*} . Consider the finite-dimensional space $\mathcal{F} := \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_P$, with \mathcal{F}_S denoting the space of flows f_S connected to the energy-storing elements, \mathcal{F}_R denoting the space of flows f_R connected to the energy dissipating elements, and \mathcal{F}_P denoting the space of external flows f_P which can be connected to the environment. Dually we write $\mathcal{E} := \mathcal{E}_S \times \mathcal{E}_R \times \mathcal{E}_P$ with the efforts $e_S \in \mathcal{E}_S, e_R \in \mathcal{E}_R, e_P \in \mathcal{E}_P$ being the corresponding dual variables of $f_S \in \mathcal{F}_S, f_R \in \mathcal{F}_R, f_P \in \mathcal{F}_P$, i.e. with $\mathcal{E}_S = \mathcal{F}_S^*, \mathcal{E}_R = \mathcal{F}_R^*, \mathcal{E}_P = \mathcal{F}_P^*$.

Let \mathcal{Z} be a discrete n -dimensional manifold of energy variables, and let $H : \mathcal{Z} \rightarrow \mathbb{F}$ be a discrete Hamiltonian. Furthermore, let \mathcal{F}_P be the space of external flows f , with \mathcal{E}_P the dual space of external effort e . Consider a Dirac structure on the product space $\mathcal{Z} \times \mathcal{F}_P$, only depending on z . The implicit discrete port-Hamiltonian system corresponding to

\mathcal{L} , \mathcal{D} , H and \mathcal{F}_P is defined by the specification:

$$\left(-\frac{\Delta z}{\Delta t}, f, \partial_z H(z), e \right) \in \mathcal{D}(z)$$

III. INTERCONNECTION OF SMOOTH AND DISCRETE PORT-HAMILTONIAN SYSTEMS

There are certain applications, [11], [12] wherein we are required to interconnect smooth systems with discrete ones. One such application is the field of haptics where the discrete virtual environment should ‘feel like’ the equivalent physical system. Figure 1 shows the energetic exchange among the system components of a haptics setup by means of the bond-graph formalism, with the double vertical bar indicating an exchange of energy which occurs in discrete time. The element denoted by ‘S/H’ represents the Sample and Hold component.

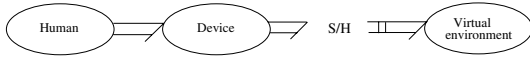


Fig. 1. Energetic representation of a haptic display.

In the haptics setting there are two main problems regarding the energy. These problems can lead to a non passive behavior and therefore to haptic device instability, c.f. [11], [12]. The two sources of energy problems are:

- the discrete virtual environment is usually not energy conserving (due to numerical errors).
- the Sample and Hold system, because it keeps a power variable to a constant value during the sample period, regardless of the actual behavior of its conjugate variable.

There are many ways to preserve energy for the discrete system which is implemented on computers, c.f. [5], [6], [7]. Hence we will not focus on that issue - we focus only on the second problem. The basic goal in the haptic setting is to ensure the stability of the haptics device, and in this work we will show a way of interconnecting a smooth port-Hamiltonian system (the haptics device) with a discrete port-Hamiltonian system (the virtual environment) and still preserve passivity². Also this study enables us to understand the physical and geometric properties of the interconnection of discrete port-Hamiltonian systems with smooth port-Hamiltonian systems. Consider Figure 2.

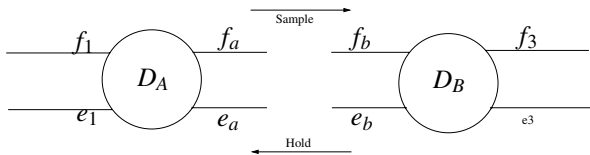


Fig. 2. Composition of Dirac structures.

We consider \mathcal{D}_a to be the Dirac structure corresponding to a smooth port-Hamiltonian system, and \mathcal{D}_b the Dirac

²The theory here can be used for other applications, like telemanipulation i.e. for the implementation of a passive master-slave system on a digital transmission line with varying time delays and possible loss of packets (e.g. the internet).

structure corresponding to a discrete one. The two systems are interconnected by a *sample and hold* system. The flow f_a is considered as the output corresponding to \mathcal{D}_a and f_b is the input corresponding to \mathcal{D}_b . Similarly, e_b is the output effort corresponding to \mathcal{D}_b and e_a is the input corresponding to \mathcal{D}_a . The discrete system corresponding to \mathcal{D}_b is implemented on a computer and hence is defined on a finite space, i.e. the space of floating point numbers \mathbb{F}^n . Let us assume that the smooth system is defined³ on \mathbb{R}^n .

A. Constraints on the boundary flows and efforts.

In applications like haptics or telemanipulation, power conserving interconnections of the continuous physical system with the discrete virtual environment is not the key issue - the interconnections should be *energy conserving* at each sampling time T . Let us make this more clear. Consider again Figure 2. For the haptics system above we define that for each sampling instance

$$f_a(kT) = f_b(kT), \quad e_a(t) = e_b(kT), \quad t \in [kT, (k+1)T]$$

In other words the power at each port is equal on the sampling time instances. However for the energies to be equal now we require the flows to be related as:

$$f_b(k+1)T = \int_{kT}^{(k+1)T} f_a(s) ds$$

The efforts are related just as before, and since the product of the flows and efforts must be equal, so:

$$\begin{aligned} e_b(k) \cdot f_b(k+1) &= \int_k^{(k+1)} e_a \cdot f_a(s) ds \\ &= e_a(t) \cdot \int_k^{(k+1)} f_a(s) ds \end{aligned} \quad (2)$$

The product of the port variables now is no longer power at the sampling time $(k+1)T$, rather it is the total energy in the sampling interval $[kT, (k+1)T]$. Now it is obvious to see that if the discrete system is designed so as to respect the above energy relation, then we have energy-conserving interconnections. This problem has been studied in [11], [12] and the setting has been tested on various examples. In the following we attempt to formalize such energy conserving interconnections. Since the port-Hamiltonian framework is meant for power-conserving interconnections we will need to extend the framework for energy-conserving interconnections.

B. Transforming energy conserving interconnection structures to power conserving interconnection structures

In this subsection we attempt to formalize Dirac structures *whose port variables are required to be energy-duals, instead of the usual power-duals*. Since one of the port variables is constant over a sampling interval, this will make it possible to rewrite the problem as a power-conservation problem, as

³The analysis also holds for smooth and discrete manifolds.

shown below. Consider a smooth port-Hamiltonian system with Dirac structure defined as:

$$\mathcal{D}_a = \{(f_1, e_1, f_a, e_a) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^* | F_1 f_1 + E_1 e_1 + F_2 f_a + E_2 e_a = 0\}$$

Replace \mathcal{D}_a with a new Dirac structure $\tilde{\mathcal{D}}_a$ defined as

$$\tilde{\mathcal{D}}_a = \{(f_1, e_1, \tilde{f}_a, e_a) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \tilde{\mathcal{F}}_2 \times \mathcal{F}_2^* | F_1 f_1 + E_1 e_1 + \tilde{F}_2 \tilde{f}_a + E_2 e_a = 0\}$$

where

$$\tilde{f}_a(k+1) = \int_k^{(k+1)} f_a(s) ds$$

We have assumed that the (new) smooth port-Hamiltonian system has admittance causality (i.e. effort in/flow out) and the discrete one as having impedance causality (i.e. flow in/ effort out). So the flow \tilde{f}_a is the input to the discrete system. Of course this does *not* mean that \tilde{f}_a is the output of the original smooth system at any sampling time instance - however on the discrete side we have *chosen* $f_b = \tilde{f}_a$ so as to enable energy conservation. So in some sense (to be made clear later) the output of the original smooth system, f_a , before reaching the input port of the discrete system gets modified to \tilde{f}_a . Now let us see how to formalize this.

We rewrite the modified flow variable at each sampling instance as:

$$\tilde{f}_a(k+1) = f_a(k+1) + \frac{1}{T} \int_{kT}^{(k+1)T} [f_a(s) - f_a(k+1)] ds \quad (3)$$

Doing so would result in a modification of Figure 2 to Figure 3. Since this is always the case, we can think of the above as the original port-Hamiltonian system (with output flow f_a) *appended with an external flow source*. For example if our system is an LC-circuit, then the output current i_{out} is:

$$i_{\text{out}}((k+1)) = i(k+1) + \frac{1}{T} \int_{kT}^{(k+1)T} [i(s) - i(k+1)] ds$$

where $\int_k^{(k+1)^-} i(s) ds$ can be thought of as the amount of current delivered at the output port at time $(k+1)T$ via a *current source* appended to the system. In this manner we do have a well defined new power-balance:

$$P^{\text{net}} = P_{\text{internal}} - P_{\text{boundary}} = 0$$

Hence we can transform the problem of energy-conserving interconnection on sampling instances, to a power-conserving interconnection with an external flow source.

Remark 1: Figure 3 may seem a bit confusing. The original port-Hamiltonian system is the one with f_a as its output flow variable after the sampling operation. And if our concern was power-conservation, and not energy conservation, then $f_b = f_a$ would be the obvious input to \mathcal{D}_b . However since we desire energy matching on sampling intervals - we *choose* the input to \mathcal{D}_b to be \tilde{f}_a , even though f_a is the output of the smooth system. For this kind of an interconnection to make sense - what we can do is to append to the smooth system an extra flow source as we have done in Figure 3. And once we do this we have an equality among the

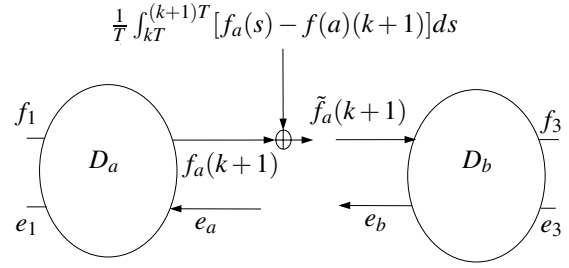


Fig. 3. A modified composition of Dirac structures.

interconnecting port variables, and then we are once again in the usual port-Hamiltonian setting.

Let us summarize the above discussion. We require the interconnection of the discrete and the continuous system to be energy-conserving. However the theory of port-Hamiltonian systems is applicable only for power-conserving interconnections. Our claim is that we can *rewrite* energy-conserving interconnections into a power-conserving setting. The basic constraint on the boundary flows and efforts is as shown in Eq. (2). We can rewrite the flow variable as in Eq. (3). And then from Figure 3, we see that - if we assume $\frac{1}{T} \int_{kT}^{(k+1)T} [f_a(s) - f_a(k+1)] ds = 0$ - then at the sampling instances $f_b(k+1)T = f_a(k+1)T$, and $\int_k^{(k+1)} f_a(s) ds$ can then be treated as an *external flow source*. With this, if we can prove that the resulting interconnected system is again port-Hamiltonian (it will be some type of a mixed interconnection) then *passivity is guaranteed*. This is what do in the next subsection, where we shall prove that the interconnection of a smooth port-Hamiltonian system with a discrete port-Hamiltonian system *at the sampling instances* results in a *mixed port-Hamiltonian system*.

C. Energy-conserving Interconnection

The title of this subsection may be confusing - we shall prove that the interconnection of a discrete Dirac structure with a continuous Dirac structure results in a special mixed Dirac structure. So in that sense we are talking about power-conserving interconnections. However in the *specific case* of the applications that we have discussed in the previous subsections, such an interconnection would lead to the energy balance - but not power balance. The goal of this section is to provide a formal way to interconnect a smooth physical system with a discrete one via a S/H. The Sample part of the S/H has been taken 'inside' the physical system and the Hold part 'inside' the discrete system. Our smooth system is defined as in Figure 3, i.e. including the flow source, meaning that its output flow is \tilde{f}_a . The discrete system has \tilde{f}_a as its flow input. Our discrete port-Hamiltonian system is defined in the usual way, i.e.

$$\mathcal{D}_b = \{(f_b, e_b, f_3, e_3) \in \mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^* | F_2 f_b + E_2 e_b + F_3 f_3 + E_3 e_3 = 0\}$$

This is the discrete port-Hamiltonian system that is inter-connected to the modified smooth port-Hamiltonian system. We are concerned with the power/energy exchange that

takes place at the sampling instances, so we will define the interconnection structure at these time instances - in between the sampling time instances the smooth system gets a constant effort input from the discrete system.

At the sampling time instance we have the interconnecting variables as:

$$\begin{aligned} f_b((k+1)T) &= \tilde{f}_a((k+1)T), \\ e_a(t) &= e_b(kT), \quad t \in [kT, (k+1)T] \end{aligned}$$

Before we continue we shall need the following result from [6], [7]:

Proposition 1:

- Every Dirac structure $\mathcal{D} \subset \mathbb{F}^n \times \mathbb{F}^{n^*}$ can be written as $\mathcal{D} = \ker(F + E)$ for linear maps as defined above. Furthermore any such E and F satisfy: $EF^* + FE^* = 0$.
- Every n -dimensional subspace $\mathcal{D} = \ker(F + E)$ defined by the above linear maps and satisfying $EF^T + FE^T = 0$, defines a Dirac structure.
- \mathcal{D} can be written in an image representation as: $\mathcal{D} = \{(f, e) \in \mathbb{F}^n \times \mathbb{F}^{n^*} \mid f = E^T \lambda, e = F^T \lambda, \lambda \in \mathbb{F}^n\}$

The following theorem says that the interconnection of a smooth Dirac structure with a discrete one results in a special kind of mixed Dirac structure. The important thing to understand here is that the interconnection structure is defined only for certain discrete time instances.

Theorem 1: Let $\mathcal{D}_a, \mathcal{D}_b$ be Dirac structures w.r.t. $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ and $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$, with $\tilde{\mathcal{F}}_2 \subset \mathcal{F}_2$ and $\tilde{\mathcal{F}}_2^* \subset \mathcal{F}_2^*$. Then $\mathcal{D}_a \parallel \mathcal{D}_b$ is a Dirac structure with respect to the bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$.

Proof: Consider the Dirac structures \mathcal{D}_a (the Dirac structure for the smooth system) and \mathcal{D}_b (the Dirac structure for the discrete system) defined in matrix (or ‘module’) kernel representation by:

$$\mathcal{D}_a = \{(f_1, e_1, f_a, e_a) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^* \mid F_1 f_1 + E_1 e_1 + F_{2a} f_a + E_{2a} e_a = 0\}$$

$$\mathcal{D}_b = \{(f_b, e_b, f_3, e_3) \in \tilde{\mathcal{F}}_2 \times \tilde{\mathcal{F}}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^* \mid F_{2b} f_b + E_{2b} e_b + F_3 f_3 + E_3 e_3 = 0\}$$

We require $\tilde{\mathcal{F}}_2 \subset \mathcal{F}_2, \tilde{\mathcal{F}}_2^* \subset \mathcal{F}_2^*$ because on the sampling instances the interconnecting flow and effort variables must be the same, in other words the discrete interconnecting flow and effort spaces must be a subset of the continuous interconnecting flow and effort spaces. Using Proposition 1. we can easily see that \mathcal{D}_a and \mathcal{D}_b are alternatively given in the ‘matrix’ image representation as:

$$\mathcal{D}_a = [E_1^T \ F_1^T \ E_{2a}^T \ F_{2a}^T \ 0 \ 0]^T, \quad \mathcal{D}_b = [0 \ 0 \ E_{2b}^* \ F_{2b}^* \ E_3^* \ F_3^*]^*$$

Matrices $(E_1^T, F_1^T, E_{2a}^T, F_{2a}^T)$ are matrices in \mathbb{R}^n , and matrices (more correctly ‘module’ elements) $(E_{2b}^*, F_{2b}^*, E_3^*, F_3^*)$ are elements of \mathbb{F}^n .

Hence,

$$(f_1, e_1, f_3, e_3) \in \mathcal{D}_a \parallel \mathcal{D}_b \Leftrightarrow$$

$$\Leftrightarrow \exists \lambda_a \in \mathbb{R}^n, \lambda_b \in \mathbb{F}^n \text{ s.t. } \begin{bmatrix} f_1 \\ e_1 \\ 0 \\ f_3 \\ e_3 \end{bmatrix} = \begin{bmatrix} E_1^T & 0 \\ F_1 & 0 \\ E_{2a}^T & E_{2b}^* \\ F_{2a}^T & -F_{2b}^* \\ 0 & E_3^* \\ 0 & F_3^* \end{bmatrix} \begin{bmatrix} \lambda_a \\ \lambda_b \end{bmatrix} \Leftrightarrow$$

In the above we have:

$$\begin{aligned} E_{2a}^T \lambda_a + E_{2b}^* \lambda_b &= f_a - f_b = 0 \\ F_{2a}^T \lambda_a - F_{2b}^* \lambda_b &= e_a - e_b = 0 \end{aligned}$$

which is true only on the sampling time instances, it is not true for all time. This is because f_b is the sampled signal of f_a and hence matches f_a only on those sampled time instances.

$$\Leftrightarrow \forall (\beta_1, \alpha_1, \beta_2, \alpha_2, \beta_3, \alpha_3) \text{ s.t.}$$

$$(\beta_1^T \ \alpha_1^T \ \beta_2^T \ \alpha_2^T \ \beta_3^T \ \alpha_3^T) \begin{bmatrix} E_1^T & 0 \\ F_1 & 0 \\ E_{2a}^T & E_{2b}^* \\ F_{2a}^T & -F_{2b}^* \\ 0 & E_3^* \\ 0 & F_3^* \end{bmatrix} = 0$$

In the above β_2^T multiplies E_{2a}^T and also E_{2b}^* - so the question is to what space (\mathbb{R}^n or \mathbb{F}^n) does β_2^T belong to? Well, it does not matter since we are interested in only those specific time instances where $f_a = f_b$, so it makes no difference to consider β_2^T to lie in either \mathbb{R}^n or in \mathbb{F}^n . The same situation holds for α_2^T .

$$\Rightarrow \beta_1^T f_1 + \alpha_1^T e_1 + \beta_3^T f_3 + \alpha_3^T e_3 = 0 \Leftrightarrow$$

$$\Leftrightarrow \forall (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) \text{ s.t.}$$

$$\begin{bmatrix} F_1 & E_1 & F_{2a} & E_{2a} & 0 & 0 \\ 0 & 0 & -F_{2b} & E_{2b} & F_3 & E_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \alpha_3 \\ \beta_3 \end{bmatrix} = 0$$

$$\beta_1^T f_1 + \alpha_1^T e_1 + \beta_3^T f_3 + \alpha_3^T e_3 = 0 \Leftrightarrow$$

$$\Leftrightarrow \forall (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) \in \mathcal{D}_a \parallel$$

$$\mathcal{D}_b, \beta_1^T f_1 + \alpha_1^T e_1 + \beta_3^T f_3 + \alpha_3^T e_3 = 0$$

$$\Leftrightarrow (f_1, e_1, f_3, e_3) \in (\mathcal{D}_a \parallel \mathcal{D}_b)^\perp$$

Thus $\mathcal{D}_a \parallel \mathcal{D}_b = (\mathcal{D}_a \parallel \mathcal{D}_b)^\perp$, and hence it is a Dirac structure. \blacksquare

IV. CONCLUSIONS

In applications like haptics, telemanipulation etc. a primary goal is to have a passive interconnection of a continuous system with the discrete virtual environment. In this paper we studied the geometric structure that results from such an interconnection. More specifically we work in the framework of port-Hamiltonian systems, and on the discrete side we used a previously developed discrete port-Hamiltonian modeling technique. The interconnection was shown to be passive, and a new type of mixed Dirac structure was defined - this Dirac structure is defined only on the sampling time instances.

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