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An Encompassing Formalization of Robust Computed Torque Schemes of Robot Systems

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Abstract— **This paper deals with the tracking control of robot systems in presence of perturbations such as modelling errors and disturbance forces. More specifically, this paper aims at reviewing the well-established robust computed torque controller. The first goal consists in establishing a global and encompassing formalization for a large class of robust computed torque schemes by using a Lyapunov approach. Then the second goal is to use this formalization to improve a particular scheme. It consists in deriving lower gain thresholds by exploiting the passivity property of robot systems.**

I. INTRODUCTION

This paper deals with the tracking control of robot systems in presence of perturbations such as modelling errors and disturbance forces. More specifically, this paper aims at reviewing the well-established Robust Computed Torque (RCT) controller. Designing a RCT scheme consists in both, selecting subclasses of robot models and, establishing conditions on the control parameters leading to the robustness of the system. Generally, it amounts to the elaboration of a gain threshold beyond which robustness is achieved. One challenging problem, is to develop the minimum threshold for the less conservative conditions on the control and the model.

The Encompassing Formalization (EF) is an extension of the RCT formalization developed in [1] based on the Lyapunov direct method. Then for the specific RCT scheme [2], EF combined with passivity property will be used to elaborate lower gain thresholds. This result is presented as a theorem for which an original proof is proposed in the last part of the paper.

This paper is organized as follows. In section II, the robot nonlinear model is described and its properties are listed. Section III deals with the computed torque scheme, going from the ideal case of perfect model knowledge to the robust control problem in case of modelling errors. The encompassing formalization is presented in section IV followed by illustrative examples. Under the title of 'Living choice', section V, develops the paper theorem and demonstration and discusses the contributions with respect to the original former one.

II. ROBOT MODEL

In this paper, the system under consideration is a robot given by the nonlinear dynamical equation

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q,\dot{q},t) = \tau, \tag{1}$$

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with $t \geq 0$ indicating the time dependency[1]. $q = (q_i)_{i=1..n}$ is a n-dimensional set of coordinates describing the robot motion (for instance, the set of articular coordinates, or the vector of the end effector displacements).

- $M(q) = (m_{ij})_{i,j=1..n}$ is the mass tensor defining the kinetic energy E_K as the quadratic expression $E_K = \frac{1}{2}\dot{q}^T M(q)\dot{q}$.
- $C(q,\dot{q})\dot{q}$ is the vector of Coriolis and centripetal forces. The matrix $C = (c_{ij})_{i,j=1..n}$ is computed using the Christoffel symbols $\gamma_{ijk}(q)$:

$$c_{ij} = \sum_{k=1}^{n} \gamma_{ijk}(q)\dot{q}_k, \; i,j = 1..n, \tag{2}$$

where

$$\gamma_{ijk} = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{kj}}{\partial q_i}\right) \;, i,j,k = 1..n. \tag{3}$$

- $g(q,\dot{q},t)$ is the negative sum of the gravity forces vector obtained by derivating the system potential energy E_P in the q coordinates and all friction and disturbance forces $f(\dot{q},t)$:

$$g = -\left(-\frac{\partial}{\partial q}E_P + f(\dot{q},t)\right). \tag{4}$$

- τ is the set of forces (torques) acting on the robot (written in the q coordinates).

The formulas of C and g can be directly derived from the Lagrange Euler calculation method. In particular, one can recover formulas (2) and (3) from the quadratic expression of the kinetic energy.

Mathematical properties

1) M is symmetric positive semi-definite. In fact, singularities could appear for purely mathematical reasons[2].
2) Passivity property: $\dot{M} - 2C$ is skew symmetric. Passivity is a noteworthy feature which can be used to reduce conservatism in establishing stability for some control schemes (see section V).
3) The Coriolis and centripetal forces have a quadratic form: $C(q,\dot{q})\dot{q}$ could be rewritten as $\left(\dot{q}^T \Upsilon_i \dot{q}\right)_{i=1..n}$ where $(\Upsilon_i)_{i=1..n}$ are symmetric matrices of general term: $(\gamma_{ijk})_{j,k=1..n}$.

[1] Implicitly, time starts at $t = 0$.

[2] For example, when the system of coordinates q is not minimal, or for some three dimensional rotation parameterization.

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III. THE COMPUTED TORQUE

As presented in [8], "The computed torque controller was developed in the early seventies and has been analyzed and modified extensively. Essentially, this method calculates a torque to negate the effects of load disturbances due to cross-coupling from adjacent links, so that the closed-loop robotic manipulator resembles a linear decoupled system".

A. Perfect model knowledge

The computed torque is a feedback linearization tracking algorithm. Let $q_d(t)$ be the desired regular trajectory. This algorithm consists in using the control:

$$\tau = M(q)(\ddot{q}_d + K_v\dot{e} + K_p e) + C(q,\dot{q})\dot{q} + g(q,\dot{q},t). \quad (5)$$

The position tracking error is defined by

$$e = q_d - q. \quad (6)$$

K_v and K_p are symmetric positive definite (s.p.d.) matrices. Consequently the closed loop system (1) and (5) is an exponentially stable second order system in e:

$$\ddot{e} + K_v\dot{e} + K_p e = 0. \quad (7)$$

Remarks:

- It is implicitly assumed that M is s.p.d. This hypothesis will be assumed throughout the paper.
- There are variations of this formalization. One could, for instance, add an integral term to transform the PD into a PID controller.

B. Modelling errors

Unfortunately the former technique requires a perfect knowledge of the robot (M, C and g). Generally, models used in control are different from the real ones. There are two major reasons for it: (a) *modelling difficulties* (the robot might be quite complex, some dynamical parameters could be difficult to estimate,...) (b) *implementation requirements* such as high computation speed for real time applications. The *standard* computed torque controller accounts for the last two points and takes advantage of the structure of (5). It is defined by

$$\tau_c = M_c(q)(\ddot{q}_d + K_v\dot{e} + K_p e) + N_c(q,\dot{q},t), \quad (8)$$

where M_c and N_c are the new design parameters besides K_v and K_p. The parameter N_c can always be written as

$$N_c(q,\dot{q},t) = C_c(q,\dot{q})\dot{q} + g_c(q,\dot{q},t) + u_c. \quad (9)$$

The designer could actually express his partial knowledge of the model thanks to the triplet (M_c, C_c, g_c) by using the Lagrange Euler method. The computed parameters (indexed by 'c') could actually be interpreted as the estimations of the real entities. Furthermore u_c is a latitude parameter which could help the designer to act freely (in an unstructured way) and directly on the robot. The general closed loop system (1) and (8) is

$$\ddot{e} + D\dot{e} + Fe + w = 0, \quad (10)$$

involving the following quantities

$$\begin{cases} F = M^{-1}M_cK_p \\ D = M^{-1}\left[M_cK_v + \Delta C\right] \\ w = M^{-1}\left[\Delta M\ddot{q}_d + \Delta C\dot{q}_d + \Delta g - u_c\right]. \end{cases} \quad (11)$$

$\Delta a = a - a_c$ with the alphabetic letter $a \in \{M, C, g\}$. The question that will be studied and formalized is:

Under which conditions on the control parameters (M_c, C_c, g_c, u_c, K_p and K_v) the system (10) is 'robustly stable' i.e. uniformly ultimately bounded (the error vector $(e,\dot{e})$ converges to a ball around the origin within a finite time)?

Unfortunately, due to the modelling errors, this system presents nonlinear terms F, D and w and no direct conclusion can be drawn on its stability. Subclasses have been derived for which several results have been carried out. For these subclasses the standard computed torque scheme is said to be robust. The first goal of this paper, is precisely to develop an encompassing formalization that gathers the subclasses found in the literature.

IV. THE ENCOMPASSING FORMALIZATION

A. Preliminary classification

RCT schemes, aim at proving the intuitive idea that is: by choosing the gains K_p and K_v sufficiently high[3], the non-linearities induced by modelling errors are compensated and the system is uniformly bounded. In fact, each scheme states hypothesis and conditions on the model and the control parameters to extract lower bounds on the gains. One challenging problem, is to develop the minimum lower bound for the less conservative conditions on the model and the control.

To the authors knowledge, RCT schemes can be classified following the following diagram. The general idea consists in making a linear change of coordinates. The error vector is transformed into

$$(e,\dot{e}) \rightarrow (e,s), \quad (12)$$

where depending on the new variable s, the tree of possibilities can be drawn

- The transformation is

  $$s = \dot{e} + Le, \quad (13)$$

  where L is a strictly positive[4] matrix (not necessarily symmetric). There are two sub-cases
  - $L(t)$ depends on time. This is called : the *living choice*. Indeed L is determined *naturally* by the system dynamics.
  - L constant and arbitrarily set by the control designer.
- The transformation is the identity $s = \dot{e}$. It could be seen as the complementary case of the previous linear transformation (examples in [3]).

The first case $L > 0$ (and particularly, the *living* choice), will be studied further in this paper. Indeed, it presents

[3]In a more rigorous way: the lower eigenvalue is sufficiently high

[4]The definition of $L > 0$ is $\forall x \neq 0$, $x^T Lx > 0$

an interesting interpretation. The new dynamical equation written with this change of coordinates are

$$\left\{ \begin{array}{l} \dot{s} = (L - D)s + \left[\dot{L} + \Pi\right] e - w \\ \dot{e} + Le = s, \end{array} \right. \tag{14}$$

with

$$\Pi = -F - (L - D)L. \tag{15}$$

If $\dot{L} + \Pi = 0$, this transformation could be seen as a back-stepping strategy where the control would act in two stages. The first action is a minimization of $\|s\|$ (first equation of the system). Then s acts subsequently (as a control) on e thanks to the second equation. As $L > 0$ then e will converge to 0 if s does. More generally, if s converges to a hyper-ball around the origin then e will do as well.

B. Mathematical formalization

To illustrate this classification, one needs a unifying mathematical formalization. Here the direct Lyapunov method developed by Qu and Dawson [1] is adopted and extended. The stability of the closed loop system (10) is studied through the use of two general Lyapunov candidate functions V_1 and V_2 (subject to an adequate choice of P and K)

$$V_1 = \frac{1}{2} s^T P s, \; V_2 = \frac{1}{2} e^T K e. \tag{16}$$

These Lyapunov candidate functions are parameterized by P and K which are s.p.d. matrices possibly dependent on time. The goal of this formalization is to build Lyapunov functions that are strictly negative except around the origin. This result implies the uniform ultimate boundedness of the system i.e. the error vector $(e, \dot{e})$ starts to be bounded after a finite time (Khalil [6] theorem 4.18).

Illustration by some examples:

- The scheme given in [2] is used as an example for the *living* choice of L (that will be improved). By choosing $V = V_1$ with $P = M_c$ as the system Lyapunov function, one can recognize the terms involved in it. The general idea can now be seen as the cancellation of the cross term $s^T M_c \left[\dot{L} + \Pi\right] e$ in the time derivative of V:

$$\begin{aligned} \dot{V} = \; & s^T \left[M_c(L - D) + \tfrac{1}{2}\dot{M}_c\right] s + \\ & s^T M_c \left[\dot{L} + \Pi\right] e - s^T M_c w, \end{aligned} \tag{17}$$

  by setting L as a positive definite solution[5] of the non-autonomous non symmetric Riccati equation [5]

$$\dot{L} = -\Pi = F + (L - D)L. \tag{18}$$

  Thus the time derivative of V is

$$\dot{V} = s^T \left[M_c(L - D) + \frac{1}{2}\dot{M}_c\right] s - s^T M_c w, \tag{19}$$

  and, given that $\|M_c w\|$ is bounded, a sufficient condition on L for the uniform ultimate boundedness of the system is that:

$$M_c(L - D) + \frac{1}{2}\dot{M}_c < 0 \tag{20}$$

- Qu and Dawson in [1], choose to set L constant.

[5]It could also be seen as the triangulation of the system (14).

V. THE LIVING CHOICE THEOREM

This section will first present the living choice theorem. It is an improvement of the theorem given in [2] thanks to the passivity property and the use of the spectral norm which is defined for a matrix A, by

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}. \tag{21}$$

Afterwards, the passivity property is given up. A second theorem is derived. The differences between these results are discussed.

A. Living choice theorem

The theorem *hypothesis* are:

- **H1** M, C, g and their corresponding computed terms are continuous with respect to q and $\dot{q}$. M and M_c are s.p.d.
- **H2** $K_p = k_p I$ and $K_v = k_v I$ are homothetic transformations (possibly dependent on time). The strictly positive gains k_p and k_v are related by a proportionality constant coefficient $\mu > 0$

$$k_p(t) = \mu k_v(t). \tag{22}$$

- **H3** There exist two constants σ and σ_c such that the coercivity conditions are satisfied

$$0 < \sigma < \|M\| \text{ and } 0 < \sigma_c < \|M_c\| \tag{23}$$

- **H4** M is continuously differentiable. C_c, ΔC and w are bounded.
- **H4c** M_c is continuously differentiable. $\dot{M}_c$, ΔC and w are bounded.

Theorem 1: (Living choice theorem) Under hypothesis H1, H2, H3 and H4, if $k_v > k_{life}$ with

$$\begin{aligned} k_{life} \; & = \kappa_{life} \left(\|C_c\| \left\|M^{-1}\right\| + \|\Delta C\| + 3\mu \right. \\ & \left. + \mu \sqrt{\tfrac{\|M\|}{\sigma}} + \alpha \sqrt{\|M\|} \, \|w\| \right), \end{aligned} \tag{24}$$

and

$$\kappa_{life} = \left\|M_c^{-1}\right\| \|M\|, \tag{25}$$

$\alpha > 0$ being a parameter, then the system (10) is uniformly ultimately bounded.

In this theorem, passivity was used to eliminate $\dot{M}$ in the expression of the minimum gain k_{life} (it was replaced by C_c). As $\dot{M}$ is unknown (and not necessarily bounded), this substitution with C_c is very useful since the latter quantity is a control parameter (which bound is set by the designer).

B. Other theorems

The authors used the same technique as for the living choice theorem and relaxed the passivity property. They obtained the “relaxed theorem”. This theorem answers the following question: in the absence of passivity, how could we derive a minimum gain threshold without $\dot{M}$?

Theorem 2: (Relaxed theorem) Under hypothesis H1, H2, H3, H4c, if $k_v > k_{relax}$ with

$$k_{relax} = \kappa_{relax}\left(\sqrt{\tfrac{\|M_c\|}{\sigma_c}}\left(2\|\Delta C\|\left\|M^{-1}\right\| + \mu\right)\right.$$
$$\left. +\alpha\|w\|\sqrt{\|M_c\|} + \tfrac{1}{2}\left\|\dot{M}_c\right\|\left\|M_c^{-1}\right\| + 3\mu\right), \quad (26)$$

and

$$\kappa_{relax} = \left\|M_c^{-\frac{1}{2}} M M_c^{-\frac{1}{2}}\right\| \quad (27)$$

then the system is uniformly ultimately bounded.

Actually, the original theorem [2], answers the same question as before:

Theorem 3: (Original theorem) Under hypothesis H1, H2, H3, H4c, if $k_v > k$ with

$$k = \kappa\left(\sqrt{\tfrac{\|M_c\|}{\sigma_c}}\left((n+1)\|\Delta C\|\left\|M^{-1}\right\| + (n+1)\mu\right)\right.$$
$$\left. +\alpha\|w\|\sqrt{\|M_c\|} + \tfrac{1}{2}\left\|\dot{M}_c\right\|\left\|M_c^{-1}\right\| + 2\mu\right), \quad (28)$$

and

$$\kappa = \kappa_{relax} \quad (29)$$

then the system (10) is uniformly ultimately bounded.

Then with respect to our theorems one major drawback emerges: the system size n appears in the threshold. This means that k *increases with the size of the system.*

C. Proof of the living choice theorem

This theorem is based on the study of the Lyapunov function $V = V_1$ with $P = M$ (choosing M instead of M_c in IV-B, will allow to exploit passivity). Its time derivative is

$$\begin{aligned}\dot{V} &= s^T\left[M(L-D) + \tfrac{1}{2}\dot{M}\right]s \\ &+ s^T M\left[\dot{L} + \Pi\right]e - s^T M w.\end{aligned} \quad (30)$$

Then, the general idea is to cancel the cross term $s^T M\left[\dot{L} + \Pi\right]e$ and to ensure the negativity of matrix $M(L-D) + \frac{1}{2}\dot{M}$. The following lemma is at the center of the proof.

Lemma 1: Under hypothesis H1 to H4, if $k_v > k_1$ with

$$k_1 = \kappa_{life}\left(\|C_c\|\left\|M^{-1}\right\| + \|\Delta C\| + 3\mu + \mu\sqrt{\frac{\|M\|}{\sigma}}\right), \quad (31)$$

then, there exists $L(t)$ on $[0, \infty[$, solution of the system $\dot{L} + \Pi = 0$ with the initial condition $L(0) = \mu I$ (I is the identity matrix) that fulfills the following inequality

$$\|L(t) - \mu I\| < \mu, \forall t \geq 0. \quad (32)$$

This result is crucial because it allows to ensure the positiveness and the boundedness of L:

Corollary 1: $\forall t \geq 0$, the following inequalities are satisfied

$$0 < L(t) < 2\mu I \quad (33)$$

The boundedness of L is then used to show the second corollary:

Corollary 2: The first term of the Lyapunov function $\dot{V}$ is negative because

$$M(L-D) + \frac{1}{2}\dot{M} < 0 \quad (34)$$

This corollary implies that in (30) $\dot{V}$ is negative when s is outside a hyper-ball *which radius depends on Mw*. Consequently, the final step is to control this radius and to bound it:

Lemma 2: $\forall\alpha > 0$, if

$$k_v > k_{life} = k_1 + \kappa\alpha\sqrt{\|M\|}\,\|w\|, \quad (35)$$

then the system is uniformly ultimately bounded

$$\forall t \geq T_b,\ \|s(t)\| \leq 1/\alpha\sqrt{\sigma}, \quad (36)$$

with $T_b = \left\|(M^{\frac{1}{2}}s)(t=0)\right\|/\mu\alpha^2$.

Remarks:

- The ultimate bound $1/\alpha\sqrt{\sigma}$ and T_b can be reduced by increasing α.
- Here one can use the *sliding control sign term* [7] instead of adding $\kappa_{life}\alpha\sqrt{\|M\|}\,\|w\|$. If $k > k_1$ and $u_c = -\beta\,\mathrm{sgn}(s)$ with

  $$\beta > \|\Delta M\ddot{q}_d + \Delta C\dot{q}_d + \Delta g\| \quad (37)$$

  then $\dot{V}$ is strictly negative and the system is asymptotically stable.

VI. PROOFS

A. Lemma 1

There exists $T > 0$ such that the differential Riccati equation (18) with initial condition $L(0) = \mu I$ admits a (local) solution $L(t)$ for $t \in [0, T]$. On this time interval, we define Q as $L - \mu I$. Then Q is given by a Riccati equation derived from (18)

$$\left\{\begin{array}{l}\dot{Q} = -DQ + (F - \mu D) + (Q + \mu I)^2 \\ Q(0) = 0\end{array}\right. \quad (38)$$

From (22) and the formulas of D and F in (10), we obtain

$$\dot{Q} = -k_v M^{-1} M_c Q - M^{-1}\Delta C(Q + \mu I) + (Q + \mu I)^2, \quad (39)$$

in which, the gain k_p disappears from the equation of evolution of Q.

Assume momentarily the matrix $M^{-1}M_c$ positive definite, and as k_v can be as high as needed then in equation (39) $\dot{Q}$ can always be 'set negative definite'[6]. This idea will be used in the following. However, despite the fact that both M^{-1} and M_c are positive, their product is not. This justifies the transformation

$$Y = M^{\frac{1}{2}}Q, \quad (40)$$

where $M^{\frac{1}{2}}$ is the s.p.d. square root of M. Then, to fulfill condition (32), it is sufficient to have (from (23))

$$\|Y\| < \sqrt{\sigma}\mu, \quad (41)$$

[6] i.e. $\forall x$, $x^T\dot{Q}x < 0$

Eigenvalues variations

In order to prove the boundedness of the spectral norm of Y, we will study the time variations of its eigenvalues. Let $(\lambda_i)_{i=1..n}$ be the set of Y^TY eigenvalues. As Y^TY is continuously differentiable, then (Kato [4], theorem 5.4) there exists a unitary x_i belonging to the proper subspace associated to λ_i such that

$$\dot{\lambda}_i = x_i^T \left(\frac{d}{dt} Y^T Y\right) x_i. \tag{42}$$

Moreover λ_i is continuously differentiable (Kato [4], theorem 6.8). From (40)

$$\frac{d}{dt} Y^T Y = \frac{d}{dt} Q^T M Q = 2Q^T M \dot{Q} + Q^T \dot{M} Q. \tag{43}$$

Therefore the equation of evolution of λ_i can be summarized as:

$$\dot{\lambda}_i = -2k_v \upsilon_i + \omega_i, \tag{44}$$

where

$$\upsilon_i = x_i^T Q^T M_c Q x_i \geq 0, \tag{45}$$

and

$$\begin{aligned} \omega_i = \; & 2x_i^T Q^T \left(-\Delta C(Q+\mu I) + M(Q+\mu I)^2\right) x_i \\ & + x_i^T \left(Q^T \dot{M} Q\right) x_i. \end{aligned} \tag{46}$$

The passivity property simplifies the equation of ω_i and gives

$$\omega_i = 2x_i^T Q^T \left(C_c Q - \mu \Delta C + M(Q+\mu I)^2\right) x_i, \tag{47}$$

which satisfies (after replacing Q by $M^{-\frac{1}{2}}Y$ and bounding)

$$|\omega_i| \leq 2W, \tag{48}$$

with

$$\begin{aligned} W = \; & \|Y\|^2 \left(\|C_c\| \left\|M^{-1}\right\| + 2\mu + \left\|M^{-\frac{1}{2}}\right\| \|Y\|\right) \\ & + \|Y\| \left(\mu \|\Delta C\| \left\|M^{-\frac{1}{2}}\right\| + \mu^2 \left\|M^{\frac{1}{2}}\right\|\right). \end{aligned} \tag{49}$$

Besides, υ_i fulfills the inequality

$$\upsilon_i \geq \left\|M_c^{-1}\right\|^{-1} \|M\|^{-1} x_i^T Q^T M Q x_i = \lambda_i / \kappa_{life}, \tag{50}$$

with

$$\kappa_{life} = \left\|M_c^{-1}\right\| \|M\|. \tag{51}$$

Final step: proof by contradiction: Now assume that there exists an instant $t_c \in [0, T]$ at which $\|Y\|$ reaches for the first time the bound $\sqrt{\sigma}\mu$ i.e. for $0 \leq t < t_c$

$$\|Y(t)\| < \sqrt{\sigma}\mu \text{ and } \|Y(t_c)\| = \sqrt{\sigma}\mu. \tag{52}$$

We will show that by choosing $k_v > k_1$, the derivative $\frac{d}{dt}\|Y\|$ at $t = t_c$ is strictly negative. Consequently, such instant t_c doesn't exist and condition (41) is fulfilled for $t \in [0, T]$. Moreover T can be extended to infinity, because otherwise, there would exist an escape time which contradicts the non existence of a finite t_c.

The contradiction: Let $t = t_c$ and λ_j one of the maximum eigenvalues of Y^TY at this instant, then

$$\|Y(t_c)\|^2 = \lambda_j(t_c) = \sigma\mu^2, \tag{53}$$

and from (44), (48), (50):

$$\dot{\lambda}_j \leq -2(k_v \frac{\lambda_i}{\kappa_{life}} + 2W). \tag{54}$$

So if $k_v > k_1$ with

$$k_1 = 2W \frac{\kappa_{life}}{2\lambda_j(t_c)}, \tag{55}$$

(k_1 expression is expanded in (31)) then

$$\dot{\lambda}_j(t_c) < 0, \tag{56}$$

and

$$\frac{d}{dt}\|Y\|\bigg|_{t=t_c} = \frac{1}{2}\dot{\lambda}_j(t_c)\lambda_j^{-1}(t_c) < 0. \tag{57}$$

B. Corollary 1

Using the singular value decomposition of $Q = U\Sigma V^T$ we have $Q^TQ = V\Sigma^2 V^T$. As Σ^2 is the diagonal matrix of Q^TQ eigenvalues, what we have been showing so far is that they are bounded by μ^2. Consequently (as U and V are unitary) $\forall x$,

$$\left|x^T Q x\right| = \left|x^T U \Sigma V^T x\right| < \mu \|x\|_2^2. \tag{58}$$

Consequently

$$-\mu x^T x < x^T Q x < \mu x^T x, \tag{59}$$

so

$$0 < x^T L x < 2\mu x^T x, \tag{60}$$

and finally

$$0 < L < 2\mu I. \tag{61}$$

C. Corollary 2

In order to show $M(L-D) + \frac{1}{2}\dot{M} < 0$, examine further

$$M(L-D) + \frac{1}{2}\dot{M} = -k_v M_c - \Delta C + ML + \frac{1}{2}\dot{M}. \tag{62}$$

Then (using passivity) it is equivalent to show the negativity of

$$M(L-D) + \frac{1}{2}\dot{M} = -k_v M_c + M(M^{-1}C_c + L). \tag{63}$$

The decomposition in two blocks gives this sufficient condition on k_v

$$k_v > \kappa_{life}(\left\|M^{-1}\right\| \|C_c\| + 2\mu). \tag{64}$$

As k_1 includes $\kappa_{life}(\left\|M^{-1}\right\| \|C_c\| + 2\mu)$ then it is sufficient to have $k > k_1$. Moreover we have

$$k_1 > \kappa_{life}\mu + \kappa_{life}(\left\|M^{-1}\right\| \|C_c\| + 2\mu). \tag{65}$$

D. Lemma 2

Actually, so far, no attention has been paid to the residual error w. If it is bounded then the last conditions are sufficient to show the uniform ultimate boundedness of the system (Khalil [6], theorem 4.18). Otherwise, a correction term has to be added to the minimum gain to ensure the boundedness of the convergence radius. It is sufficient to add $\kappa_{life}\alpha\sqrt{\|M\|}\,\|w\|$ to the gain k_1 so that the new gain is

$$\begin{aligned} k_{life} &= \kappa_{life}\left(\|C_c\|\left\|M^{-1}\right\| + \|\Delta C\| + 3\mu \right. \\ &\left. +\mu\sqrt{\tfrac{\|M\|}{\sigma}} + \alpha\sqrt{\|M\|}\,\|w\|\right), \end{aligned} \tag{66}$$

where α is a strictly positive parameter. Indeed if $k_v > k_{life}$ the Lyapunov derivative

$$\dot{V} = s^T\left[M(L-D) + \frac{1}{2}\dot{M}\right]s - s^T M w, \tag{67}$$

satisfies, thanks to (65) and (66)

$$\dot{V} < \|M\|\,\|w\|\left\|M^{\frac{1}{2}}s\right\|\left(-\alpha\left\|M^{\frac{1}{2}}s\right\| + 1\right) - \mu\left\|M^{\frac{1}{2}}s\right\|^2. \tag{68}$$

Consequently, if $\left\|M^{\frac{1}{2}}s\right\| \geq \alpha^{-1}$ then $\dot{V} < 0$ and

$$\dot{V} < -\mu\alpha^2, \tag{69}$$

which means that $\left\|M^{\frac{1}{2}}s\right\|$ goes to the hyper-ball of radius α^{-1} in a time less than T_b:

$$T_b = \left\|(M^{\frac{1}{2}}s)(t=0)\right\|/\mu\alpha^2. \tag{70}$$

VII. CONCLUSION

The RCT controller has been studied. An encompassing formalization has been carried out and a particular scheme was improved. For this scheme, a theorem deriving a lower gain threshold has been established and proved. Eventually, the differences between this paper result and the original one have been discussed. Besides the passivity property contribution, the use of the spectral norm, has made the minimum lower threshold independent of the size system n.

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