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Controlling Spatially Invariant Systems using Finite Arrays of Actuators and Sensors

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***Abstract*— Distributed parameter systems, where the underlying dynamics are spatially invariant, are common in a range of applications including the control of flows and smart structures or the cross-directional control of web processes. Bamieh *et al*. [1] have shown that controllers for these processes are also spatially invariant, leading to controller structures that are particularly simple to implement. However, these controllers use measurements that are continuous in space and generate control inputs that are are also spatially continuous. In practice, most applications use arrays of discrete, but identical actuators and sensors and this paper uses results from sampling theory to show that the controller can be implemented using these discrete arrays without loss of performance, provided that conditions on the separation between actuators and sensors are satisfied in order to avoid aliasing. It is shown that in many practical systems. Under these circumstances, it is better to incorporate the effect of aliasing by including the cross-coupling between aliased spatial frequencies in the system model.**

I. INTRODUCTION

In [1], [2] have considered a specific class of distributed parameter system [3], where the dynamics are spatially invariant. They show that spatially invariant distributed parameter systems can be controlled by using a spatially continuous measurement, $y(x,t)$, to calculate a spatially continuous input, $u(x,t)$. One of their key insights is that by transforming the control problem into the spatial frequency domain, optimal controllers can be obtained by solving a family of decoupled optimal control problems. This leads to a controller which consists of an observer that convolves (in the spatial domain) the prediction error with a kernel function, and a state feedback control law, which convolves neighboring state estimates with a second kernel function. Both kernel functions are spatially invariant, which makes the calculation of the controller computationally efficient.

Although there are a number of applications of spatially invariant systems, such as the cross-directional control of web processes [4], distributed structures [5] and stabilization of fluid flow in a channel [6], [7], it is unusual for either the control signal or the measurement signal to be continuous in the spatial domain. Instead, the measurements are obtained using an array of identical sensor distributed along the process. Similarly, the control action is applied by a discrete array of identical actuators. Although [2] does consider discrete arrays of actuators and sensors, this paper considers the effect of using discrete actuation and sensing to regulate a spatially continuous medium. By using results from sampling theory [8], this paper shows that the discrete measurements can be considered as samples of the underlying spatially continuous measurement signal, while the inputs to the individual actuators can be obtained by sampling the spatially continuous input signal, $u(x,t)$, at the location of each actuator. The separation between actuators and sensors needs to be chosen to avoid aliasing in the spatial domain, where the separation is determined by the spatial responses of the actuators and sensors. The paper points out that in practice, spatial aliasing is very common, particularly when using an array of sensors, where each sensor measures the average variation over a "zone" of the spatial domain. It is often the case that the width of the zone equals the separation between sensors so that the zones do not overlap and this leads to a high degree of spatial aliasing, which will reduce the effectiveness of the control. Under these circumstances, it is better to explicitly include the aliasing component in the controller design.

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Section II uses results from sampling theory to show that a control system for a spatially invariant system can be implemented using arrays of discrete actuators and sensors without loss of performance, provided that spatial aliasing is avoided. The conditions on spatial aliasing are related to the spatial responses of the actuators and sensors, which is illustrated by an example in Section III. Section IV points out that in practice, spatial aliasing is common and shows how the control design can be modified to explicitly include the effect of aliasing. Section V concludes the paper.

II. SYSTEM MODEL

The spatial co-ordinate of the system is taken to be the real line, $x \in \mathbb{G}$, where as shown in [1], $\mathbb{G}$ depends upon the problem being considered. In this paper, $\mathbb{G}$ will be taken to be the real line, so that $x \in (-\infty, \infty)$, although the results are applicable to the other choices for $\mathbb{G}$ given in [1] and also to higher spatial dimensions. If the state of the system, $\psi(x,t)$ is taken to be a vector valued function, then following [1], the general linear model of a spatially invariant, distributed parameter system can be written

$$\frac{\partial}{\partial t}\psi(x,t) = [A\psi](x,t) + [B_1 w](x,t) + [B_2 u](x,t) \quad (1)$$

$$z(x,t) = [C_1\psi](x,t) + [D_{12}u](x,t) \quad (2)$$

$$y(x,t) = [C_2\psi](x,t) + [D_{21}w](x,t) \quad (3)$$

where A, B_1, B_2, C_1, C_2, D_{12} and D_{21} are translation invariant operators. It is assumed that (A, B_2) and (A^*, C_2^*) are exponentially stabilizable and for simplicity, the feedthrough operators, D_{11} and D_{22} have been ignored. In this paper, B_2

and C_2 will be taken to be convolution operators, so that

$$[B_2u](x,t) = \int_{-\infty}^{\infty} B_2(x-\eta)u(\eta,t)d\eta \qquad (4)$$

$$[C_2u](x,t) = \int_{-\infty}^{\infty} C_2(x-\eta)u(\eta,t)d\eta \qquad (5)$$

One of the insights provided by [1] is that because the operators are translation invariant, taking the Fourier transform of this system with respect to the spatial co-ordinate x, diagonalizes the operators, such that

$$\frac{d}{dt}\hat{\psi}(\lambda,t) = \hat{A}(\lambda)\hat{\psi}(\lambda,t) + \hat{B}_1(\lambda)\hat{w}(\lambda,t) + \hat{B}_2(\lambda)\hat{u}(\lambda,t) \qquad (6)$$

$$\hat{z}(\lambda,t) = \hat{C}_1(\lambda)\hat{\psi}(\lambda,t) + \hat{D}_{12}(\lambda)\hat{u}(\lambda,t) \qquad (7)$$

$$\hat{y}(\lambda,t) = \hat{C}_2(\lambda)\hat{\psi}(\lambda,t) + \hat{D}_{21}(\lambda)\hat{w}(\lambda,t) \qquad (8)$$

where $\hat{\psi}(\lambda,t)$ denotes the Fourier transform of $\psi(x,t)$ and $\hat{A}(\lambda)$, $\hat{B}_1(\lambda)$, $\hat{B}_2(\lambda)$, $\hat{C}_1(\lambda)$, $\hat{C}_2(\lambda)$, $\hat{D}_{12}(\lambda)$ and $\hat{D}_{21}(\lambda)$ are multiplication operators. This has the effect of "decoupling" the responses at each spatial frequency, λ.

Given the decoupled model of the system in the spatial frequency domain, [1] shows that optimal controllers based upon quadratic signal norms (e.g. LQR, $\mathcal{H}_2$ or $\mathcal{H}_\infty$) can be designed. For example, if T_{zw} denotes the closed loop map from $\hat{w}(\lambda,t)$ to $\hat{z}(\lambda,t)$ for the standard time invariant disturbance rejection problem, then a controller that minimizes $\|T_{zw}\|_2$, takes the form [2]

$$\frac{d}{dt}\hat{\psi}_K(\lambda,t) = \hat{A}_K(\lambda)\hat{\psi}_K(\lambda,t) + \hat{P}_2(\lambda)\hat{C}_2^*(\lambda)\hat{y}(\lambda,t) \qquad (9)$$

$$\hat{u}(\lambda,t) = -\hat{B}_2^*(\lambda)\hat{P}_1(\lambda)\hat{\psi}_K(\lambda,t) \qquad (10)$$

with

$$\hat{A}_K(\lambda) = \hat{A}(\lambda) - \hat{B}_2(\lambda)\hat{B}_2^*(\lambda)\hat{P}_1(\lambda) - \hat{P}_2(\lambda)\hat{C}_2^*(\lambda)\hat{C}_2(\lambda) \qquad (11)$$

where $\hat{P}_1(\lambda)$ and $\hat{P}_2(\lambda)$ are multiplicative operators that satisfy the standard algebraic Riccati equations [9] at each λ

$$\hat{A}^*(\lambda)\hat{P}_1(\lambda) + \hat{A}(\lambda)\hat{P}_1(\lambda) - \hat{P}_1(\lambda)\hat{B}_2(\lambda)\hat{B}_2^*(\lambda)\hat{P}_1(\lambda) + \hat{C}_1^*(\lambda)\hat{C}_1(\lambda) = 0 \qquad (12)$$

$$\hat{A}(\lambda)\hat{P}_2(\lambda) + \hat{A}^*(\lambda)\hat{P}_2(\lambda) - \hat{P}_2(\lambda)\hat{C}_2^*(\lambda)\hat{C}_2^(\lambda)\hat{P}_2(\lambda) + \hat{B}_1(\lambda)\hat{B}_1^*(\lambda) = 0 \qquad (13)$$

For simplicity, it is assumed that the matrices are normalized, so that $\forall\lambda \in \mathbb{G}$, $\hat{D}_{12}^*(\lambda)[\hat{C}_1(\lambda)\ \hat{D}_{12}(\lambda)] = [0\ I]$ and $\hat{D}_{21}(\lambda)[\hat{B}_1^*(\lambda)\ \hat{D}_{21}^*(\lambda)] = [0\ I]$.

Applying the inverse Fourier transform to (9) and (10), the controller becomes

$$\frac{\partial}{\partial t}\psi_K(x,t) = [A_K\psi_K](x,t) + L(x)\star y(x,t) \qquad (14)$$

$$u(x,t) = F(x)\star\psi_K(x,t) \qquad (15)$$

where $\star$ denotes spatial convolution and $L(x)$ and $F(x)$ are spatial convolutions operators, whose Fourier transforms are given by $\hat{L}(\lambda) = \hat{P}_2(\lambda)\hat{C}_2^*(\lambda)$ and $\hat{F}(\lambda) = -\hat{B}_2^*(\lambda)\hat{P}_1(\lambda)$.

The $\mathcal{H}_\infty$ controller, which ensures that $\|T_{zw}\|_\infty < \gamma$, has the same form with $\hat{L}(\lambda) = \hat{Z}_2(\lambda)\hat{P}_2(\lambda)\hat{C}_2^*(\lambda)$ and $\hat{F}(\lambda) = -\hat{B}_2^*(\lambda)\hat{P}_1(\lambda)$, where $\hat{Z}_2(\lambda) = (I - \gamma^{-2}\hat{P}_2(\lambda)\hat{P}_1(\lambda))$, with $\hat{P}_2(\lambda)$ and $\hat{P}_2(\lambda)$ being the solution to the corresponding $\mathcal{H}_\infty$ algebraic Riccati equations [1], [2].

When the control input, $\hat{u}(\lambda,t)$ generated by the controller in (9) and (10) is converted back to the spatial domain by applying an inverse Fourier transform, the resulting signal, $u(x,t)$ is continuously distributed in x, so the input is applied throughout $x \in \mathbb{G}$. However, in most practical systems, control is applied using a discrete array of identical actuators. If the actuators are equally spaced at positions $\{x_n^{(a)} = nd_a : n \in \mathbb{Z}\}$, where d_a is the distance between adjacent actuators, then when B_2 is a convolution operator as in (4), the response of a discrete set of actuators is given by

$$\sum_{n=-\infty}^{\infty} B_2\left(x - x_n^{(a)}\right)u_n^{(a)}(t) \qquad (16)$$

where $\{u_n^{(a)}(t) : n \in \mathbb{Z}\}$ represents the input signals applied to each of the actuators. The following two lemmas show that provided the separation between the actuators is chosen so that spatial aliasing is avoided, the optimal spatially *continuous* response can be obtained from a *discrete* set of actuators.

Lemma 1: If there exists a σ_u such that $|\hat{B}_2(\lambda)| = 0$ for $|\lambda| > \sigma_u$, then the control input, $u(x,t)$ generated by the $\mathcal{H}_2$ controller in (15) is also bandlimited, so that $|\hat{u}(\lambda,t)| = 0$ for $|\lambda| > \sigma_u$

Proof: From (10), $\hat{u}(\lambda,t) = -\hat{B}_2^*(\lambda)\hat{P}_1(\lambda)\hat{\psi}_K(\lambda,t)$, so $\hat{u}(\lambda,t) = 0$ for values of λ where $|\hat{B}_2(\lambda)| = 0$, because $\hat{B}_2(\lambda)$ is a multiplicative operator. ■

Lemma 2: For an array of equally spaced, spatially discrete actuators positioned at $\{x_n^{(a)} = nd_a : n \in \mathbb{Z}\}$, if the following conditions are satisfied

(C1) $|\hat{B}_2(\lambda)| = 0$ for $|\lambda| > \sigma_u$
(C2) the separation between the actuators is chosen so that $\pi/d_a > \sigma_u$

when the input signals $\{d_a u_n^{(a)}(t) : n \in \mathbb{Z}\}$ are applied to each of the actuators, where $u_n^{(a)}(t) = u(x_n^{(a)}, t)$, then

$$\hat{B}_2(\lambda)\hat{u}(\lambda,t) = d_a\hat{B}_2(\lambda)\hat{u}_d(\lambda,t) \qquad (17)$$

Proof: $u_n^{(a)}(t)$ can be regarded as the set of signals obtained by sampling $u(x,t)$ at $x = x_n^{(a)}$. The discrete Fourier transform of the sampled signal is [10]

$$\hat{u}_d(\lambda,t) = \frac{1}{d_a}\sum_{m=-\infty}^{\infty}\hat{u}\left(\lambda + m\frac{2\pi}{d_a}, t\right) \qquad (18)$$

Condition (C2) shows that $\hat{B}_2(\lambda)$ is bandlimited and hence from Lemma 1, $\hat{u}(\lambda,t)$ is also bandlimited. The separation between the actuators satisfies $\pi/d_a > \sigma_u$, so there will be no overlap between the repeated copies of $\hat{u}(\lambda,t)$ in the summation in (18). Because $\hat{B}(\lambda)$ is a multiplicative operator and $|\hat{B}_2(\lambda)| = 0$ for $|\lambda| > \sigma_u$, then

$$\hat{B}_2(\lambda)\hat{u}_d(\lambda,t) = \frac{1}{d_a}\hat{B}_2(\lambda)\sum_{m=-\infty}^{\infty}\hat{u}\left(\lambda + 2m\frac{\pi}{d_a}, t\right) \qquad (19)$$

$$= \frac{1}{d_a}\hat{B}_2(\lambda)\hat{u}(\lambda,t) \qquad (20)$$

which can be rearranged to give (17) ∎

These results show that the term $\hat{B}_2(\lambda)\hat{u}(\lambda,t)$ on the right hand side of (6) can be replaced by $d_a\hat{B}_2(\lambda)\hat{u}_d(\lambda,t)$, which means that the spatially continuous control action can be generated by an array of discrete actuators, where the input to each actuator is $\{d_a u(x_n^{(a)},t) : n \in \mathbb{Z}\}$.

In a similar manner, the following lemmas show that subject to certain restrictions, an array of discrete sensors can be used in the place of a continuous, spatially distributed sensor.

Lemma 3: If there exists a σ_s, such that

(C3) $|\hat{C}_2(\lambda)| = 0\ \forall|\lambda| > \sigma_s$

(C4) $|\hat{D}_{21}(\lambda)| = 0\ \forall|\lambda| > \sigma_s$ and/or $|\hat{w}(\lambda)| = 0\ \forall|\lambda| > \sigma_s$

then $\hat{y}(\lambda,t)$ will be bandlimited, with $|\hat{y}(\lambda,t)| = 0\ \forall|\lambda| > \sigma_s$

Proof: $|\hat{C}_2(\lambda)\hat{\psi}(\lambda,t)| = 0\ \forall|\lambda| > \sigma_s$, since $\hat{C}_2(\lambda)$ is a multiplicative operator and by condition (C3), $|\hat{C}_2(\lambda)| = 0\ \forall|\lambda| > \sigma_s$. From condition (C4), $|\hat{D}_{21}(\lambda)\hat{w}(\lambda,t)| = 0\ \forall|\lambda| > \sigma_s$, hence using (8), $\hat{y}(\lambda,t)$ will be bandlimited ∎

Lemma 4: For an array of equally spaced, spatially discrete sensors positioned at $\{x_n^{(s)} = nd_s : n \in \mathbb{Z}\}$, if conditions (C3) and (C4) are satisfied, together with

(C5) $\pi/d_s > \sigma_s$

then

$$\hat{P}_2(\lambda)\hat{C}_2^*(\lambda)\hat{y}(\lambda,t) = d_s\hat{P}_2(\lambda)\hat{C}_2^*(\lambda)\hat{y}_d(\lambda,t) \tag{21}$$

Proof: If $y_n^{(s)}(t)$ are the set of discrete signals obtained by sampling $y(x,t)$ at $x = x_n^{(s)}$, then its discrete Fourier transform is [10]

$$\hat{y}_d(\lambda,t) = \frac{1}{d_s}\sum_{m=-\infty}^{\infty}\hat{y}\left(\lambda + m\frac{2\pi}{d_s},t\right) \tag{22}$$

From conditions (C3) and (C4), then using Lemma 3, $\hat{y}(\lambda,t)$ will be bandlimited and since the separation between the sensors satisfies $\pi/d_s > \sigma_s$, there will be no overlap between the repeated copies of $\hat{y}(\lambda,t)$ in the summation in (22). $\hat{C}_2^*(\lambda)$ is a multiplicative operator, so using condition (C3)

$$\hat{P}_2(\lambda)\hat{C}_2^*(\lambda)\hat{y}_d(\lambda,t) = \frac{1}{d_s}\hat{P}_2(\lambda)\hat{C}_2^*(\lambda)\sum_{m=-\infty}^{\infty}\hat{y}\left(\lambda + 2m\frac{\pi}{d_s},t\right) \tag{23}$$

$$= \frac{1}{d_s}\hat{P}_2(\lambda)\hat{C}_2^*(\lambda)\hat{y}(\lambda,t) \tag{24}$$

Rearranging this expression gives (21). ∎

The term $\hat{P}_2(\lambda)\hat{C}_2^*(\lambda)\hat{y}(\lambda,t)$ on the right hand side of (9) can be replaced by $d_a\hat{B}_2(\lambda)\hat{u}_s(\lambda,t)$, which means that the spatially continuous measurement signal can be replaced by the set of measurements from an array of discrete sensors, $\{y(x_n^{(s)},t) : n \in \mathbb{Z}\}$.

Lemma 5: Provided that conditions (C1) to (C5) are satisfied, there is no difference between the response of the spatially continuous system in (1) to (3), together with the spatially continuous, optimal $\mathcal{H}_2$ control in (14) and (15), and the system with a discrete array of identical actuators positioned at $\{x_n^{(a)} : n \in \mathbb{Z}\}$

$$\frac{\partial}{\partial t}\psi(x,t) = [A\psi](x,t) + [B_1 w](x,t) + d_a[B_2 u](x_n^{(a)},t) \tag{25}$$

$$z(x,t) = [C_1\psi](x,t) + [D_{12}u](x,t) \tag{26}$$

$$y(x,t) = [C_2\psi](x,t) + [D_{21}w](x,t) \tag{27}$$

with

$$[B_2 u](x_n^{(a)},t) = \sum_{n=-\infty}^{\infty} B_2\left(x - x_n^{(a)}\right)u_n^{(a)}(t) \tag{28}$$

and $u_n^{(a)}(t) = u(x_n^{(a)},t)$, where $u(x,t)$ is obtained from the controller that uses the measurements $y_n^{(s)}(t) = y(x_n^{(s)},t)$ from a discrete array of identical sensors positioned at $\{x_n^{(s)} : n \in \mathbb{Z}\}$

$$\frac{\partial}{\partial t}\psi_K(x,t) = [A_K\psi_K](x,t) + d_s L(x) \star y(x_n^{(s)},t) \tag{29}$$

$$u(x,t) = F(x) \star \psi_K(x,t) \tag{30}$$

with

$$L(x) \star y(x_n^{(s)},t) = \sum_{n=-\infty}^{\infty} L(x - x_n^{(s)})y(x_n^{(s)},t) \tag{31}$$

where $\hat{L}(\lambda) = \hat{P}_2(\lambda)\hat{C}_2^*(\lambda)$ and $\hat{F}(\lambda) = -\hat{B}_2^*(\lambda)\hat{P}_1(\lambda)$, with $\hat{P}_1(\lambda)$ and $\hat{P}_2(\lambda)$ being obtained from the Riccati equations in (12) and (13).

Proof: The Fourier transforms of the spatially continuous system and the continuous optimal controller are given in (6) to (8) and (9) to (10). Substituting (17) and (21) and applying the inverse transform leads to system and controller given in (25) to (27) and (29) to (30) ∎

Remark 1. Although these results have been developed for an $\mathcal{H}_2$ controller, because the corresponding expressions for $\hat{L}(\lambda)$ and $\hat{F}(\lambda)$ for the $\mathcal{H}_\infty$ controller have the same form, it is readily shown that an $\mathcal{H}_\infty$ controller can be implemented using discrete arrays of actuators and sensors.

Remark 2. The separation between actuators that avoids aliasing is determined solely by the bandwidth of the actuators' spatial response. This gives the maximum allowable separation and there is no benefit in reducing the separation by including extra actuators in the array. The same argument applies to the separation of sensors. This result has been used in the design of actuator and sensor arrays for the cross-directional control web processes [11], [12], [13].

Remark 3. Note that if there is one sensor associated with each actuator, so that $d_a = d_s$, the scaling terms in (25) and (29) can be cancelled and the sampled values can be used directly without scaling.

III. Example

A. Actuator Array

Consider the dynamics of heat transfer in an infinite homogeneous medium with distributed control [2]

$$\frac{\partial\psi}{\partial t} = c\frac{\partial^2\psi}{\partial x^2} + [Bu](x,t) \tag{32}$$

where $\psi(x,t)$ is the temperature profile in the medium and the A operator is $c\frac{\partial^2}{\partial x^2}$, which is spatially invariant [1], [2]. The temperature profile in the medium is controlled by adjusting the input, $u(x,t)$, to a distributed heat source, which takes the form of a convolution operator in (4) where $B(x)$ denotes the spatial "footprint" of the distributed actuator. In this example, the footprint is taken to have a gaussian shape, as shown by the solid line in Fig. 1. The temperature profile throughout the medium is regulated by the $\mathcal{H}_2$ controller in (14) and (15). Suppose that at time t, the control signal generated by controller takes the form shown in Fig. 2, giving $[Bu](x,t)$ as shown in Fig. 3, which is obtained by convoluting $u(x,t)$ with the spatial footprint in Fig. 1. The Fourier transform of the spatial footprint is shown in Fig. 4 and it can be seen that $\hat{B}(\lambda)$ is bandlimited, so that $|\hat{B}(\lambda)| = 0$ for $|\lambda| > 180$ rad m^{-1}. Choosing $d_a = 0.011\mathrm{m}$ ensures that $\pi/d_a = 286$ rad m^{-1}, which is greater than $\sigma_a = 180$ rad m^{-1}, so aliasing is avoided. The location of the responses of the individual actuators at this separation is shown by the dashed lines in Fig. 1. Sampling $u(x,t)$ at the location of the actuators to obtain $u_n^{(a)}(t)$, the input signals to each of the actuators generates the response shown by the solid line in Fig. 5, which is identical to the spatially continuous $[Bu](x,t)$ in Fig. 3. However, if the separation between the actuators is increased to $d_a = 0.022\mathrm{m}$ so that $\pi/d_a = 143$ rad m^{-1}, which is less than σ_a, then aliasing occurs and the discrete array is not able to generate $[Bu](x,t)$ without error, as shown by the dashed line in Fig. 5.

Fig. 2. Example control signal, $u(x,t)$

Fig. 3. $[Bu](x,t)$ generated by control signal in Fig. 2

Fig. 1. Spatial footprint of individual actuator (solid line) together with footprints of adjacent actuators (dotted lines).

The analysis of the previous section assumed that both $\hat{u}(\lambda,t)$ and $\hat{y}(\lambda,t)$ are bandlimited in the spatial frequency domain. In the case of $\hat{u}(\lambda,t)$, this is a reasonable assumption, since $|\hat{u}(\lambda,t)| = 0$ for values of λ where $|\hat{B}_2(\lambda)| = 0$. Hence the spatial "bandwidth" of $\hat{u}(\lambda,t)$ is determined by the bandwidth of $\hat{B}_2(\lambda)$. B_2 is taken to be a convolution operator and in some cases, such as a point heat source or an injector into a flow stream, the spatial footprint of the operator may be highly localized, which means that the bandwidth of $\hat{B}_2(\lambda)$ will be large. For actuation mechanisms where $B_2(x)$ has bounded support, such that $|B_2(x)| = 0$ for $|x| > \delta$, the bandwidth will be infinite. However, the spatial footprint is likely to be continuous and as a result for any given ϵ, it is possible to find σ_u such that $|\hat{B}_2(\lambda)| \leq \epsilon$ for $|\lambda| > \sigma_u$ [14]. By making ϵ sufficiently small and choosing the actuator spacing such that $d < \pi/\sigma_u$, the spatial aliasing associated with the actuator array will be negligible.

B. Sensor Array

The measurements from the sensors are given by sampling $y(x,t)$ in (3) at $\{x_n^{(s)} = nd_s : n \in \mathbb{Z}\}$. The C_2 operator is taken to be a convolution operator and a common example is a sensor that averages $\psi(x,t)$ over the spatial region of width h, such that

$$[C_2\psi](x,t) = \frac{1}{h}\int_{x-h/2}^{x+h/2} \psi(\eta,t)d\eta \qquad (33)$$

This form of sensor is typical in web processes [11], [15] and in flow processes [1]. Transforming this operator into the spatial frequency domain gives

$$\hat{C}_2(\lambda) = 2\frac{\sin(\lambda h/2)}{\lambda h} \qquad (34)$$

The spatial frequency roll-off of this response is proportional to $\frac{1}{\lambda}$ and if the bandwidth, defined as the value of σ_s such that $|\hat{C}_2(\lambda)| \leq \epsilon$ for $|\lambda| > \sigma_s$ for a given ϵ, will be large. As a consequence, the sensor separation that avoids aliasing, which is given by $d_s = \frac{\pi}{\sigma_s}$, will be very much larger than

Fig. 4. Plot of $|B(\lambda)|$

Fig. 5. Response generated by array of actuators with $d_a = 0.011$ m (solid line) together with response generated by array when $d_a = 0.022$ m (dashed line)

$2h$. For example, if ϵ is taken to be 0.1 (corresponding to 10% of the peak value), then $d_s < \pi h/20$. This means that if the array consists of sensors that average the spatial variation over a set of adjacent zones, each of width h, then a high degree of spatial aliasing will occur. To avoid aliasing, it is important that the spatial footprint of the operator $C_2(x)$ is sufficiently wide to extend beyond the neighboring sensors in the array. In practical web processes, the alternative approach is adopted where more sensors than actuators are used. Each sensor measures the average value property over a "zone", where the width of the zone is equal to the separation between sensors [11], [15], [16]. Although this will introduce aliasing into the sampled signal, because the spatial sampling frequency of the actuators is greater than the corresponding sampling frequency of the actuators, the aliasing is concentrated at frequencies above the spatial bandwidth of the actuators' response.

Even if the separation of the sensors can be chosen to match the bandwidth of $\hat{C}_2(\lambda)$, aliasing may still occur if the term $\hat{D}_{21}(\lambda)\hat{w}(\lambda,t)$ contains spatial frequency components above σ_s. Typically, this term may include the effect of sensor noise, so care needs to be taken in the sensor design to ensure that the spatial bandwidth of either $\hat{w}(\lambda,t)$ or $\hat{D}_{21}(\lambda)$ does not exceed σ_s.

IV. SAMPLING ERRORS

A. Aliasing

Wherever possible, the spacing between the actuators and the sensors should be chosen to avoid aliasing. However, the constraints of the process may mean that this is not possible, for example when the measurement noise contains components that have frequency components above the spatial bandwidth of the sensors. One approach is to obtain an upper bound on the magnitude of the aliasing error can be obtained [17]. By considering the aliasing error as a perturbation signal a controller can be designed, which ensures that the system is robust to this perturbation. However, this approach is likely to be highly conservative and a more direct approach to incorporating the effect of spatial aliasing into the control design is achieved by noting that the spatial frequency component of the sampled $\hat{y}_s(\lambda,t)$ for $\lambda > 0$ is aliased by the component, $\hat{y}_s(2\sigma_s - \lambda, t)$. When $\lambda < 0$, the aliasing component occurs at $\hat{y}_s(-2\sigma_s - \lambda, t)$. Here it is assumed that the aliasing is not so bad that higher aliasing frequencies (e.g. $4\sigma_s - \lambda$) are significant. Since each spatial frequency is aliased by a single discrete frequency, the effect of aliasing can be regulated by considering a coupled system model, consisting of the responses at spatial frequencies, λ and $\tilde{\lambda} = 2\sigma_s - \lambda$ (for simplicity, it is assumed that $d_a = d_s$, so that $\sigma_a = \sigma_s = \sigma$)

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} \hat{\psi}(\lambda,t) \\ \hat{\psi}(\tilde{\lambda},t) \end{bmatrix} &= \begin{bmatrix} \hat{A}(\lambda) & 0 \\ 0 & \hat{A}(\tilde{\lambda}) \end{bmatrix}\begin{bmatrix} \hat{\psi}(\lambda,t) \\ \hat{\psi}(\tilde{\lambda},t) \end{bmatrix} \\
&+ \begin{bmatrix} \hat{B}_1(\lambda) & 0 \\ 0 & \hat{B}_1(\tilde{\lambda}) \end{bmatrix}\begin{bmatrix} \hat{w}(\lambda,t) \\ \hat{w}(\tilde{\lambda},t) \end{bmatrix} \\
&+ \begin{bmatrix} \hat{B}_2(\lambda) & \hat{B}_2(\lambda) \\ \hat{B}_2(\tilde{\lambda}) & \hat{B}_2(\tilde{\lambda}) \end{bmatrix}\begin{bmatrix} \hat{u}(\lambda,t) \\ \hat{u}(\tilde{\lambda},t) \end{bmatrix} \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} \hat{z}(\lambda,t) \\ \hat{z}(\tilde{\lambda},t) \end{bmatrix} &= \begin{bmatrix} \hat{C}_1(\lambda) & 0 \\ 0 & \hat{C}_1(\tilde{\lambda}) \end{bmatrix}\begin{bmatrix} \hat{\psi}(\lambda,t) \\ \hat{\psi}(\tilde{\lambda},t) \end{bmatrix} \\
&+ \begin{bmatrix} \hat{D}_{12}(\lambda) & 0 \\ 0 & \hat{D}_{12}(\tilde{\lambda}) \end{bmatrix}\begin{bmatrix} \hat{u}(\lambda,t) \\ \hat{u}(\tilde{\lambda},t) \end{bmatrix} \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} \hat{y}(\lambda,t) \\ \hat{y}(\tilde{\lambda},t) \end{bmatrix} &= \begin{bmatrix} \hat{C}_2(\lambda) & \hat{C}_2(\lambda) \\ \hat{C}_2(\tilde{\lambda}) & \hat{C}_2(\tilde{\lambda}) \end{bmatrix}\begin{bmatrix} \hat{\psi}(\lambda,t) \\ \hat{\psi}(\tilde{\lambda},t) \end{bmatrix} \\
&+ \begin{bmatrix} \hat{D}_{21}(\lambda) & 0 \\ 0 & \hat{D}_{21}(\tilde{\lambda}) \end{bmatrix}\begin{bmatrix} \hat{w}(\lambda,t) \\ \hat{w}(\tilde{\lambda},t) \end{bmatrix} \qquad (37)
\end{aligned}
$$

for values of $|\lambda| \leq \sigma$. Using this coupled model as the basis of the control design allows the effect of the aliasing component on the sampled actuator and sensor signals to be incorporated into the control design.

B. Finite Arrays

The replacement of the spatially continuous input response, $[B_2u](x,t)$, by the discrete convolution in (16) assumes that the input is generated by an infinite array of discrete actuators. In practice, there will be a finite number of actuators in the array, so that

$$d_a[B_2 u_s]^{(N)}N(x,t) = d_a \sum_{n=-N}^{N} B_2(x - x_n^{(a)})u(x_n^{(a)},t) \tag{38}$$

In order to minimize the difference $|[B_2 u_s](x,t) - [B_2 u_s]^N(x,t)|$, $|u(x,t)|$ must be small for $|x| > (N+1/2)d_a$. If $w(x,t)$ is localized in space, then $u(x,t)$ will also be spatially localized. However, for some distributed process, for example, the control of basis weight variations on a paper machine, there are significant disturbances near the edge of the machine [18]. The basis weight variations are controlled by an array of actuators at the headbox of the paper machine, where there are no actuators outside the edge of the sheet. Although the variations are localized to the width of the sheet, the spatially continuous control input, $u(x,t)$ will extend beyond the edges, so that extra actuators are required to provide this response. Under these circumstances, the best that can be done is to minimize the effect of truncating the actuator array.

This can be achieved by noting that (38) can be written as

$$d_a[B_2 u_s]^{(N)}(x,t) = d_a \sum_{n=-\infty}^{\infty} B_2(x - x_n^{(a)}) \left\{ m^{(N)}(x_n^{(a)}) u(x_n^{(a)},t) \right\} \tag{39}$$

where $m^{(N)}(x)$ is the boxcar windowing function [8]

$$m^{(N)}(x) = \begin{cases} 1 & \text{for} \quad |x| \leq (N+1/2)d_a \\ 0 & \text{for} \quad |x| > (N+1/2)d_a \end{cases} \tag{40}$$

In the spatial frequency domain

$$\hat{B}_2(\lambda)\hat{u}^{(N)}(\lambda,t) = \hat{B}_2(\lambda) \left\{ \hat{m}^{(N)}(\lambda) \star \hat{u}(\lambda,t) \right\} \tag{41}$$

so that N must be made sufficiently large to ensure that the effect of convoluting with $\hat{m}^{(N)}(\lambda)$ does not distort $\hat{u}(\lambda,t)$.

Remark 4. An important exception is when the actuators are arranged around the edge of a cylinder, such as in the extrusion of blown film [19], [4]. Under these circumstances, the system is periodic in x and a finite array can be used, provided that the aliasing conditions are satisfied [1].

V. CONCLUSION

This paper has considered the implementation of distributed control for spatially invariant systems using discrete arrays of identical actuators and sensors. By using results from sampling theorem, it is shown that the controller designed for spatially continuous input and measurement signals can be implemented on discrete arrays by sampling the underlying continuous signals at the locations of the actuators and sensors. However, the spacing between individual components of the actuator and sensor arrays need to be chosen to avoid aliasing in the spatial domain. The maximum allowable separation is determined by the spatial response of the B_2 and C_2 operators in the system model. The paper also shows that in practice, aliasing is likely to occur, primarily because the measurement signal, $y(x,t)$ cannot be guaranteed to be band limited in the spatial frequency domain. Under these circumstances, it may be better to explicitly include the effect of aliasing by design a controller using a system model that includes the effect of cross-coupling between aliased frequencies.

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