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Observers for Interval Systems using Set and Trajectory-based Approaches

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Abstract— **In this paper, set and trajectory-based approaches to interval observation of uncertain systems are presented and compared. The kind of uncertain systems considered are those systems described by a discrete linear time-invariant model with parameters bounded in intervals. The aim of this paper is to study the viability of using set-based approaches coming from the interval analysis community to solve the interval observation problem. Set-based approaches are appealing because of a lower computational complexity compared to trajectory-based approaches but they suffer from the wrapping effect and do not preserve uncertain parameter time-invariance. On the other hand, trajectory-based approaches are immune to these problems but their computational complexity is higher. However, these two families of approaches are equivalent when the observer satisfies the isotonicity condition, which give criteria to tune the observer gain. Finally, these two families of interval observation philosophies will be presented, analysed and compared by using them in an example.**

I. INTRODUCTION

IN this paper, set and trajectory-based approaches to interval observation of uncertain systems are presented and compared. The kind of uncertain systems considered are those systems described by a discrete linear time-invariant model with parameters bounded in intervals that can be expressed in state space form

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x}_k + \boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{u}_k \\ \boldsymbol{y}_k &= \boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{x}_k \end{aligned} \qquad (1)$$

being $\boldsymbol{\theta}$ the vector of uncertain parameters with their values described by a region of confidence $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ of box type, i.e. by an interval for every uncertain parameter: $\theta_i \in \left[\underline{\theta}_i, \overline{\theta}_i\right]$. A system described by a model with parameters bounded in intervals is called an ***interval dynamic system***. When such a model is used for simulation or state observation, the evolution of system states at every time instant will not be described by a point in the state space but by a set [17]. In general, the exact set will be very complicated to be described on a computer [1]. Typically, this set is approximated by, for example, a box, a polytope or an ellipsoid. This type of simulation or state observation is known as ***set-membership***. In case that the approximate set is the interval hull of the exact set, then, we talk about ***interval simulation*** or ***state observation***. A recent survey of this field can be found in [17]. In the literature, algorithms can be classified according to if they compute the approximate set of estimated states using one step-ahead iteration based on previous approximate sets (***set-based approaches***) [1][4][14], or a set of point-wise trajectories generated by selecting particular values of $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ using heuristics or optimisation (***trajectory-based approaches***) [15][16][18]. In the first case, the set of states $\boldsymbol{X}(k)$ is approximated at each iteration and some propagation algorithm is used to produce the approximate set of states $\boldsymbol{X}(k+1)$. This approach is affected by several problems (specially, in case that the approximate set is the interval hull): the wrapping effect, range evaluation of an interval function (in this case, the state space function*)* and the uncertain parameter time dependency that will be reviewed in *Section III*. However, in the second case, the interval hull of $\boldsymbol{X}(k)$ is built following real trajectories generated by selecting particular values of $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Consequently, this approach overcomes the wrapping effect and preserves the uncertain parameter time dependency, but the problem of the interval function (in this case the trajectory function $\boldsymbol{x}(k, \boldsymbol{u}, \boldsymbol{x_o}, \boldsymbol{\theta})$) range evaluation still remains. However, set-based approaches present a lower computationally complexity than trajectory-based approaches being their main interest.

The aim of this paper is to study the viability of using set based approaches coming from the interval analysis community to solve the interval observation problem. Set-based approaches are appealing because of a lower computational complexity but they suffer from the wrapping effect and do not preserve uncertain parameter time-invariance. On the other hand, trajectory based approaches are immune to this problem but the computational complexity is higher. Additionally, it will be shown that selecting the observer gain in such a way that the interval observer verifies the isotonicity condition, both approaches provide the same results. To the knowledge of the authors, algorithms for interval observation presented in this paper

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have never been compared, being another contribution of this paper.

The structure of rest of the paper is the following: in *Section II*, interval observers are introduced and reformulated as interval simulators. In *Section III*, problems to be considered in interval observation are presented. In *Section IV* and *Section V*, set and trajectory-based approaches to interval observation are presented, respectively. In *Section VI*, the selection of the observer gain for the resulting interval observer avoids the wrapping effect and the problem of uncertain parameter time-invariance is discussed. In *Section VII*, set and trajectory- based approaches are compared using a test example discussing the effect of the observer gain. Finally, in *Section VIII*, the main conclusions are presented.

II. INTERVAL OBSERVATION

A. Interval Simulation

Definition 1. *The* ***solution set*** *of a system, whose model is described by (1) for the time interval [0,N] consists of* $\boldsymbol{X}(0,N)=\left\{\boldsymbol{x}(k,\boldsymbol{u},\boldsymbol{x}_o,\boldsymbol{\theta}):k\in[0,N],\ \boldsymbol{\theta}\in\boldsymbol{\Theta},\boldsymbol{x}_o\in\boldsymbol{X}_o\right\}$, *where* $\boldsymbol{x}(k,\boldsymbol{u},\boldsymbol{\theta},\boldsymbol{x}_o)$ *denotes the solution of (1) at time k for some vector of parameters* $\boldsymbol{\theta}\in\boldsymbol{\Theta}$ *and some initial condition* $\boldsymbol{x}_o\in\boldsymbol{X}_o$ *at time k=0. The set of values for a fixed time interval* [0,N] *will be referred to as the* ***reachability set*** *at time k and denoted by*

$$\boldsymbol{X}(k)=\left\{\boldsymbol{x}(k,\boldsymbol{u},\boldsymbol{x}_o,\boldsymbol{\theta}):\boldsymbol{\theta}\in\boldsymbol{\Theta},\boldsymbol{x}_o\in\boldsymbol{X}_o\right\} .$$

Herein, it will be assumed that the uncertain system is stable for all $\boldsymbol{\theta}\in\boldsymbol{\Theta}$. This assumption will allow ***X(k)*** to be a bounded region for each $k\in[0,\infty)$.

Definition 2. The ***interval simulation*** of a system, whose model is described by (1), for the time interval [0,N] consists in computing the interval hull of the reachability set ***X(k)***, i.e., the smallest interval vector containing it: □***X***(k) $=\left[\underline{\boldsymbol{x}}(k),\overline{\boldsymbol{x}}(k)\right]$, where □ is used to denote the ***interval hull*** of ***X***, for all k∈[0,N]. The sequence of interval vectors □***X(k)*** with k∈[0,N] will be called the ***interval solution*** or ***envelopes*** of (1) .

B. Interval State Observation

Let the model for the state observer of the system described by (1) be formulated as

$$\begin{aligned}\hat{\boldsymbol{x}}_{k+1}&=\boldsymbol{A}(\boldsymbol{\theta})\hat{\boldsymbol{x}}_k+\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{u}_k+\boldsymbol{K}(\boldsymbol{y}_k-\hat{\boldsymbol{y}}_k)\\ \hat{\boldsymbol{y}}_k&=\boldsymbol{C}(\boldsymbol{\theta})\hat{\boldsymbol{x}}_k\end{aligned}\qquad(2)$$

where: $\boldsymbol{\theta}\in\boldsymbol{\Theta}$ is the vector of uncertain parameters and ***K*** the observer gain, that has been designed to stabilise it. This observer is known as an ***interval observer***. Using (2) as the expression of the estimator model, it can be rearranged as a discrete-time system with two inputs that can be reorganised as:

$$\hat{\boldsymbol{x}}_{k+1}=(\boldsymbol{A}-\boldsymbol{K}\boldsymbol{C})\hat{\boldsymbol{x}}_k+\begin{bmatrix}\boldsymbol{B} & \boldsymbol{K}\end{bmatrix}\begin{bmatrix}\boldsymbol{u}_k\\ \boldsymbol{y}_k\end{bmatrix}=\boldsymbol{A}_o\hat{\boldsymbol{x}}_k+\boldsymbol{B}_o\boldsymbol{u}_k^o\qquad(3)$$

where: $\boldsymbol{A}_o=\boldsymbol{A}-\boldsymbol{K}\boldsymbol{C}$, $\boldsymbol{B}_o=\begin{bmatrix}\boldsymbol{B} & \boldsymbol{K}\end{bmatrix}$ and $\boldsymbol{u}_k^0=\begin{bmatrix}\boldsymbol{u}_k & \boldsymbol{y}_k\end{bmatrix}^t$. Then, the problem of interval state observation can be formulated as a problem of interval simulation.

III. PROBLEMS TO BE CONSIDERED IN INTERVAL OBSERVATION

A. Wrapping effect

The problem of wrapping is related to the use of a crude approximation (its interval hull) of the interval observer solution set and its iteration using one-step ahead recursion of the state space observer function. This problem does not appear if instead the estimated trajectory function is used. On the other hand, when using the one-step ahead recursion approach, at each iteration, the true solution set is wrapped into a superset feasible to construct and to represent on a computer (in this paper, its interval hull □). Since the overestimation of the wrapped set is proportional to its radius, a spurious growth of the enclosures can result if the composition of wrapping and mapping is iterated [7]. This wrapping effect can be completely unrelated to the stability properties of the observer, and even stable observers are shown to exhibit exponentially fast growing enclosures that are useless for practical purposes. Not all the interval observers exhibit this problem. It has been shown that those that are monotone with respect to states do not present this problem. This kind of observers (systems) is known as ***isotonic*** [3] or ***cooperative*** [5]. In case of linear discrete-time systems, a system is isotonic if all the elements of the system matrix ***A*** are positive [3].

B. Range evaluation of an interval function

Many approaches to interval observation need to evaluate the range of an interval function at any iteration, in order to determine the interval hull for systems states. One possibility for evaluating the function range of the function is to use interval arithmetic [10]. But, although the ranges of basic interval arithmetic operations are exactly the ranges of the corresponding real operations, this is not the case if the operations are composed. This phenomenon is termed as interval dependence or multi-incidence problem [10].

C. Temporal variance on uncertain parameters

An additional issue that should be taken into account is how uncertain parameter time-invariance is assured since set-based approaches are based on one step-ahead algorithms do not preserve it. Instead they consider implicitly uncertain parameters as a time varying [4][14]. This is because the relation between parameters and states is not preserved at each iteration since every parameter contained in the parameter set $\boldsymbol{\Theta}$ is combined with every state in the state set $\boldsymbol{X}(k)$ when determining the new state set $\boldsymbol{X}(k+1)$. On the other hand, trajectory-based approaches allow considering that uncertain parameters are unknown but bounded in their confidence intervals and they cannot vary at each time step [15][16].

IV. Set Based Approaches To Interval Observation

A. Moore's algorithm [10]

This algorithm, when applied to a linear discrete-time system as (3), computes the interval for estimated states $[\hat{\boldsymbol{x}}(k+1)]$ at time $k+1$ using as initial condition the interval for estimated states $[\hat{\boldsymbol{x}}(k)]$ at time k. It is based on computing the natural interval extension of the state space function by replacing each occurrence of $\hat{\boldsymbol{x}}(k)$ and $\boldsymbol{\theta}$ by its corresponding interval and each standard function by its interval evaluation (***absolute algorithm***) [10]:

$$[\hat{\boldsymbol{x}}(k+1)] = \boldsymbol{A}_o([\boldsymbol{\theta}])[\hat{\boldsymbol{x}}(k)] + \boldsymbol{B}_o([\boldsymbol{\theta}])\boldsymbol{u}_o(k) \quad (4)$$

However, as explained in *Section III.B* replacing real numbers in a function by intervals often leads to large overestimations that derive in an interval for estimated states $[\hat{\boldsymbol{x}}(k+1)]$ that always increases, even if the true solution contracts. A better approach is to apply the interval ***mean-value theorem*** [10] to equation (4) (***relative algorithm***):

$$\begin{aligned}[\hat{\boldsymbol{x}}(k+1)] &= \hat{\boldsymbol{x}}_c(k+1) \\ &\quad + \boldsymbol{A}_o([\boldsymbol{\theta}])([\hat{\boldsymbol{x}}(k)] - \hat{\boldsymbol{x}}_c(k))\end{aligned} \quad (5)$$

where $\hat{\boldsymbol{x}}_c(k+1) = \boldsymbol{A}_o(\hat{\boldsymbol{\theta}})\hat{\boldsymbol{x}}_c(k) + \boldsymbol{B}_o(\hat{\boldsymbol{\theta}})\boldsymbol{u}_o(k)$ with $\hat{\boldsymbol{x}}_c(k+1)$, $\hat{\boldsymbol{x}}_c(k)$ and $\hat{\boldsymbol{\theta}}$ being the mid-points respectively of intervals $[\hat{\boldsymbol{x}}(k+1)]$, $[\hat{\boldsymbol{x}}(k)]$ and $[\boldsymbol{\theta}]$.

However, this method suffers from the wrapping effect in some ill-conditioned systems, as for example, those with eigenvalues with very different magnitudes [11].

B. Lohner's algorithm [9]

In cases where Moore's algorithm is ill-conditioned, equation (5) should be modified according to [9]:

$$[\hat{\boldsymbol{x}}(k+1)] = \hat{\boldsymbol{x}}_c(k+1) + \boldsymbol{A}([\boldsymbol{\theta}])\boldsymbol{\Gamma}_k[\boldsymbol{r}(k)] \quad (6)$$

and then: $[\boldsymbol{r}(k+1)] = \boldsymbol{\Gamma}_{k+1}^{-1}\boldsymbol{A}([\boldsymbol{\theta}])\boldsymbol{\Gamma}_k[\boldsymbol{r}(k)]$

with: $[\boldsymbol{r}(0)] = [\boldsymbol{x}_o] - \hat{\boldsymbol{x}}_o$, $\boldsymbol{A}_0 = \boldsymbol{I}$ and $\hat{\boldsymbol{x}}_c(k+1)$, $\hat{\boldsymbol{x}}_c(k)$ and $\hat{\boldsymbol{\theta}}$ being the mid-points of their corresponding intervals as in (5). $\boldsymbol{\Gamma}_{k+1}$ is chosen as the Q-factor from the QR-factorisation of the mid-point of $\boldsymbol{A}([\boldsymbol{\theta}])\boldsymbol{\Gamma}_k$. Lohner's algorithm can avoid the wrapping effect in many systems but Kühn [7] has discovered some cases where this approach fails.

C. Neumaier's algorithm [12]

Instead of using the interval hull of the set of estimated states, Neumaier [12] proposes to use the smallest ellipsoid containing it. An ellipsoid is a set of the form:

$$\mathcal{E}(\boldsymbol{z}, \boldsymbol{L}, \boldsymbol{r}) = \{\boldsymbol{z} + \boldsymbol{L}\boldsymbol{\xi} \mid \boldsymbol{\xi} \in \Re^n, \|\boldsymbol{\xi}\| \le r, r > 0\} \quad (7)$$

where $z \in \Re^n$ is the centre, $\boldsymbol{L} \in \Re^{n\times n}$ is the axis matrix and $r \in \Re$ is the radius. Neumaier's algorithm generalises for an uncertain system described as (3), the property that for a linear certain system, given the ellipsoid enclosing the set of possible states at time k such that $\hat{\boldsymbol{x}}_k \in \mathcal{E}(\boldsymbol{z}_k, \boldsymbol{L}_k, r)$, then the enclosing ellipsoid at time $k+1$ $\hat{\boldsymbol{x}}_{k+1} \in \mathcal{E}(\boldsymbol{z}_{k+1}, \boldsymbol{L}_{k+1}, r)$ can be constructed by propagating separately the centre and axis matrix according to:

$$\begin{aligned}\boldsymbol{z}_{k+1} &= \boldsymbol{A}\boldsymbol{z}_k + \boldsymbol{B}\boldsymbol{u}_k \\ \boldsymbol{L}_{k+1} &= \boldsymbol{A}\boldsymbol{L}_k\end{aligned} \quad (8)$$

being implicitly relative. Then, the interval simulation can be generated by computing the interval hull of the ellipsoid $\mathcal{E}(\boldsymbol{z}_k, \boldsymbol{L}_k, \boldsymbol{r})$ at each time instant according to

$$[\hat{\boldsymbol{x}}_k] = z_k + [-r, r]\|\boldsymbol{L}_{ti\bullet}\| \quad (9)$$

where $i\bullet$ represent the i^{th} row of the matrix $\boldsymbol{L}_t$.

The advantage of using ellipsoids instead of parallelepipeds as in Moore's and Lohner's algorithm is that the rotation of the state space of the interval system is implicit being not necessary to make additionally computations. The disadvantage is that the algorithm for computing with ellipsoids is more complicated than those of parallelepipeds and in general the wrapping effect when uncertain parameters are considered is not avoided [12].

D. Kühn's algorithm [7]

Kühn's algorithm [7] is based on approximating the set of system states using zonotopes. A zonotope $\mathcal{Z}$ of order m is the Minkowski sum

$$\mathcal{Z} = \mathcal{P}^1 + \cdots + \mathcal{P}^m \quad (10)$$

of m parallelepipeds $\mathcal{P}^i$. The order m is a measurement for the geometrical complexity of the zonotopes. It can be chosen freely and is a performance parameter for the Kühn's algorithm. Given the zonotope $\mathcal{Z}_{k-1}$ enclosing the set of estimated states $\hat{\boldsymbol{X}}(k-1)$ by system observer (3), then the set of estimated states $\hat{\boldsymbol{X}}(k)$ is enclosed by the following zonotope

$$\mathcal{Z}_k = \mathcal{R}(\boldsymbol{E}_k + \boldsymbol{T}_k \mathcal{Z}_{k-1}) \quad (11)$$

where $\boldsymbol{T}_k$ are square matrices and $\boldsymbol{E}_k$ are intervals such that

$$\boldsymbol{f}(\mathcal{Z}_{k-1}) = \boldsymbol{A}_o(\boldsymbol{\theta}) + \boldsymbol{B}_o(\boldsymbol{\theta})\boldsymbol{u}_{k-1} \subseteq \boldsymbol{E}_k + \boldsymbol{T}_k \mathcal{Z}_{k-1} \quad (12)$$

and the reduction operator $\mathcal{R}$ is defined in the following way: let $\mathcal{Z} = \mathcal{P}^0 + \mathcal{P}^1 + \cdots + \mathcal{P}^m$ be a $m+1$ zonotope and $1 \le \ell \le m$ be the largest integer such that:

$$diam(\mathcal{P}^0 + \mathcal{P}^1 + \cdots + \mathcal{P}^{\ell-1}) \ge diam(\mathcal{P}^\ell) \quad (13)$$

or $\ell = 1$ otherwise, then:

$$\mathcal{R}(\mathcal{Z}) := \square\ (\mathcal{P}^0 + \mathcal{P}^1 + \cdots + \mathcal{P}^\ell) + \mathcal{P}^{\ell+1} + \cdots + \mathcal{P}^m \quad (14)$$

According to [1], the set of estimated states $\hat{\boldsymbol{X}}(k)$ is a zonotope when the state function $\boldsymbol{f}$ is linear and only uncertainty in initial conditions is considered. Then, Kühn's algorithm provides a good solution to the enclosure of $\hat{\boldsymbol{X}}(k)$. However, only including uncertainty in parameters, the set $\hat{\boldsymbol{X}}(k)$ becomes a more complex than a zonotope [1]. In this case even approximating this set using subpavings [8] and algorithms to propagate them [7], the wrapping effect could no avoided at a reasonable computing time. However, Kühn's algorithm can manage parameter uncertainty bigger than in the case of Lohner's and Neumaier's algorithms, as it will be shown in the test example.

V. Trajectory Based Approaches To Interval Observation

A. *Puig's algorithm [15]*

At any time instant k, the observer state region $\hat{\boldsymbol{X}}_k$ will be bounded by its interval hull $\square\hat{\boldsymbol{X}}_k = \left[\underline{\hat{\boldsymbol{x}}}_k, \overline{\hat{\boldsymbol{x}}}_k\right]$ where:

$$\overline{\hat{\boldsymbol{x}}}_k = max\left[\boldsymbol{A}_o^k(\boldsymbol{\theta})\hat{\boldsymbol{x}}(0) + \sum_{j=0}^{k-1} \boldsymbol{A}_o^{k-1-j}(\boldsymbol{\theta})\boldsymbol{B}_o(\boldsymbol{\theta})\boldsymbol{u}_o(j)\right] \quad (15)$$

subject to: $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $\hat{\boldsymbol{x}}_o \in \hat{\boldsymbol{X}}_0$ and $\underline{\hat{\boldsymbol{x}}}_k$ is computed substituting *max* for *min*, in the previous optimisation problem. In deriving Eq. (15), it has been assumed time-invariant uncertain parameters. This is why this approach is known as ***time-invariant*** [15]. At the same time that time invariance is preserved, the wrapping effect is avoided due to the fact that uncertainty is not propagated from step to step but instead always from the initial state. This approach yields the accurate time-invariant interval observation without any conservatism, assuming that the previous optimisation problems could be solved with infinite precision and the global optimum could be determined. However, in practice it only could be solved with a given precision. On the other hand, one of the main drawbacks of this approach, besides its high computational complexity, is that the objective function is a polynomial with degree increasing by one at every iteration [16]. So, the amount of computation needed is increasing with time being impossible to operate over a large time interval. Then, some kind of approximation should be introduced to make the approach more tractable. If the observer given by Eq. (3) is asymptotically stable, any transients settle to negligible values in a finite-time, more precisely in t_s/T samples, being t_s the observer settling time and T the sampling time. This assumption implies that the outputs of the observer at time k depend only on the inputs that occurred during the last t_s/T samples. Therefore, for any time k, it is possible to approximate algorithm (15) using a sliding window of the length L of the order of the settling time measured in number of samples [16]:

$$\overline{\hat{\boldsymbol{x}}}_k = max\left[\boldsymbol{A}_o^L(\boldsymbol{\theta})\hat{\boldsymbol{x}}_{k-L} + \sum_{j=k-L}^{k-1} \boldsymbol{A}_o^{k-1-j}(\boldsymbol{\theta})\boldsymbol{B}_o(\boldsymbol{\theta})\boldsymbol{u}_j\right] \quad (16)$$

subject to:

$$\boldsymbol{\theta} \in \boldsymbol{\Theta} \text{ and } \hat{\boldsymbol{x}}_{k-L} \in \square\hat{\boldsymbol{X}}_{k-L}$$

and $\underline{\hat{\boldsymbol{x}}}_k$ is computed substituting *max* for *min*, in the previous optimisation problem. Of course, with this approximation parameter time-invariance is only guaranteed inside the sliding window. This is why this approach is called ***almost time-invariant*** [15].

B. *Kolev's algorithm [6]*

Kolev proposed an algorithm that provides an ***inner solution*** for the interval observation problem by solving the optimisation problems (15) involved in previous algorithm but subject to: $\boldsymbol{\theta} \in V(\boldsymbol{\Theta})$ and $\hat{\boldsymbol{x}}_o \in V(\hat{\boldsymbol{X}}_o)$ where: $V(\boldsymbol{\Theta})$ and $V(\hat{\boldsymbol{X}}_o)$ denotes the set of vertices of the uncertain parameters and initial states sets, respectively. This is why this algorithm is also known as a ***vertices algorithm***. According to Nickel [13], the inner solution coincides with the interval hull of the solution set for some systems, those without the wrapping effect that verify that their state function is ***isotonic*** with respect to all state variables [3]. For such systems, set and trajectory-based approaches will provide the same results.

VI. Designing the Observer Gain to Avoid the Wrapping Effect and Time-invariance Problems

Given a non-isotonic interval system described by (1), an interval observer described by (2) could be designed to fulfil the condition of isotonicity if all the elements of the observer matrix $\boldsymbol{A_0}$ satisfy $a_{ii}^o \geq 0$. This implies that all the coefficients of the system matrix $\boldsymbol{A}$ should be cancelled through the corresponding observer gain. Such requirement also implies which measurements should be available in order the isotonic interval observer could be designed. In case of an isotonic observer is designed through appropriate selection of the observer gain, the wrapping effect is not present [3][5]. Consequently, a simple iterative scheme as Moore's algorithm will work providing the same results than Puig's Algorithm starting from the initial state (i.e, infinite window length). That means, the optimisation problem (15) can be simplified since computations must not be referred to the initial state but only to previous iteration (i.e, window length *L=1*). Such equivalence establishes that in case isotonicity condition is fulfilled, set and trajectory based approaches to interval observation produce the same results, as already noticed in *Section V.B*. Then, in this case, interval observation using any of the (set or trajectory based) algorithms presented in this paper will provide the same results. Additionally, as stated in *Section III,* considering either a trajectory or set-based approach to interval observation parameter time-invariance is or not, respectively, preserved. In [2], the relation between the observations produced preserving or not uncertain parameter time-invariance for the same interval observer is presented. In particular, it is established that $IO_{invariant} \subseteq IO_{varying}$ where: *IO* means interval observation. This means that interval observing an uncertain time-invariant system using the time-varying approach can be very conservative. However, in case of isotonic discrete-time observer a set-based (therefore time-varying) interval observation based on one-step recursion and a trajectory based (therefore time-invariant) interval observation based on recursion (15) or even vertices algorithm presented in *Section V.B* will provide the same interval observation [3]. This result can be interpreted intuitively: interval hull of states at any iteration can be computed independently from uncertain parameters and states since parameters and states are decoupled because of the isotonicity condition. In this case, not preserving the relation between parameters and states is not important at all.

VII. Set Versus Trajectory-Based Approaches on a Test Examples

An example based on a linear interval system proposed in Neumaier [12] is considered:

$$\begin{aligned} x_1(k) &= \theta_1 x_1(k-1) + \theta_1 x_2(k-1) + \theta_2 \\ x_2(k) &= -\theta_1 x_1(k-1) + \theta_1 x_2(k-1) + \theta_3 \end{aligned} \qquad (17)$$

with uncertain initial conditions: $x_1(0) \in [-1,1]$, $x_2(0) \in [-1,1]$ and parameters: $\theta_1 \in [0.4, 0.5]$, θ_2, $\theta_3 \in [-10^{-12}, 10^{-12}]$. In order to show the effectiveness in propagating state uncertainty, first previous algorithms will be tested when applied to solve the interval observation problem in the hardest conditions, i.e., when observers gain $\boldsymbol{K}$ is equal to zero (interval simulation). Later, the observer gain will be appropriately tuned such that the resulting observer will satisfy the condition of isotonicity [5] discussed in last section. System given by Eq. (17) suffers from the wrapping effect because it does not fulfil the isotonicity condition [3]. So, the naive approach based on the absolute Moore's algorithm will fail. Fig. 1 shows the results of the application of algorithms presented in this paper. It can be observed that algorithms which use set propagation, except Moore's algorithms (absolute and relative), avoid the instabilisation due to the wrapping effect, but just provide an outer solution with a certain degree of conservatism depending on the kind of geometry used to approximate the real state set. Neumaier's and Kühn's algorithms provide a better approximation because of the use of ellipsoids and zonotopes (of order *m*=5), respectively, than Lohner's algorithm which uses parallepipeds. Additionally, an inner solution is obtained taking the vertex solution provided by Kolev's algorithm providing in this case a good approximation of the exact solution derived using Puig's algorithm referring all the computations to initial state. If the parameter uncertainty is increased in such way that $\theta_1 \in [0.4, 0.7]$, all set-based algorithms would fail, while only trajectory based approaches (Kolev's and Puig's algorithms) could avoid the increase of uncertainty due to the wrapping effect. Finally, the observer gain matrix will be selected in such a way that the resulting observer satisfies the isotonicity condition In particular, in this case, the observer gains should satisfy

$$\begin{aligned} a_{11}^o &= \theta_1 - k_{11} \geq 0 \rightarrow k_{11} \leq \theta_1 \\ a_{12}^o &= \theta_1 - k_{12} \geq 0 \rightarrow k_{12} \leq \theta_1 \\ a_{21}^o &= -\theta_1 - k_{21} \geq 0 \rightarrow k_{21} \leq -\theta_1 \\ a_{22}^o &= \theta_1 - k_{22} \geq 0 \rightarrow k_{22} \leq \theta_1 \end{aligned} \qquad (18)$$

assuming that the two states are measured ($y_1(k) = x_1(k)$ and $y_2(k) = x_2(k)$, i.e., $C = [1 \quad 0; 0 \quad 1]$). Additionally, the observer should be stable for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and with a dynamic faster than that of the observed system. Selecting $k_{21} = -\theta_1$ in order to the resulting observer satisfies isotonicity condition, and selecting the observer poles for the nominal model $p_{1,2}^0 = 0.2$, faster than the nominal system poles located at $p_{1,2} = 0.45 \pm 0.45j$, the following values for the rest of observer gains are: $k_{11} = k_{22} = 0.25, k_{12} = 0$. The stability of the interval observer has been checked using interval tools presented in Jaulin [8]. In Figure 2, results of designed isotonic interval observer are presented with uncertain initial conditions: $x_1(0)\in[-1,1]$, $x_2(0)\in[-1,1]$ using any of the methods presented in this paper. This allows verifying that if the observer gain is selected to fulfil the isotonicity condition all algorithms produce the same results.

VIII. CONCLUSIONS

Results on previous example have allowed showing the difficulty of interval observing/simulating a system when uncertain parameters are present, even it the system is linear. After analysing those results, we can conclude that although set-based approaches look appealing because their lower complexity compared with trajectory-based approaches, in many cases they can derive in unstable observations because of the wrapping effect. This seems to reinforce the use of trajectory-based approaches, but still in this case the computational complexity limits their applicability where real-time computations are required. Reached this point, the need to design the observer gain such that the isotonicity condition is satisfied seems a possible solution. In this case, set-based approaches will not suffer from the wrapping effect and will provide the same results as trajectory based approaches. Additionally, the problem of uncertain parameter time-invariance is solved. As a future research, the idea of selecting the observer gain to fulfil the isotonicity condition should be further investigated in order to better understand the limits of applicability.

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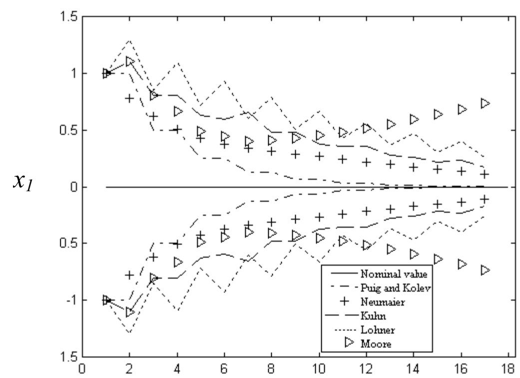

Figure 1. Algorithms results comparison

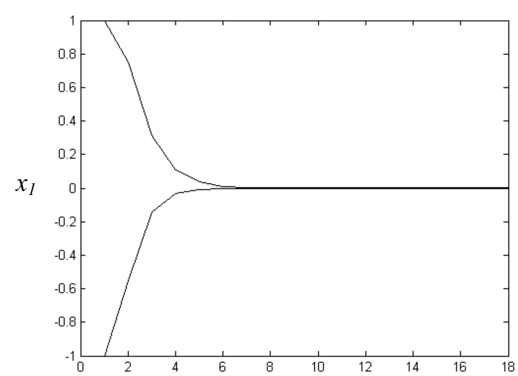

Figure 2. Isotonic interval observer