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A sufficient condition for a triple of linear systems to be simultaneously stabilizable in a behavioral framework

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Abstract— **In this paper, we address simultaneous stabilization problem for a *triple* of linear systems within a behavioral framework. First, we provide a necessary and sufficient condition for a given pair of linear systems to be simultaneously stabilizable in a behavioral framework. Next, we show that a simultaneous stabilizer has a symmetric structure, which is also a self-standing interesting result from the theoretical points of view. By using these results, we give a new necessary and sufficient condition for a given pair of linear systems to be simultaneously stabilizable. This new condition is described in terms of the solvability of polynomial matrix equations consisting of polynomial matrices inducing kernel or image representations of the behaviors of given two behaviors. Finally, by using this new condition, we give a sufficient condition for a *triple* of linear systems to be simultaneously stabilizable in terms of the behaviors.**

I. INTRODUCTION

Simultaneous stabilization is to stabilize two or more plants by using one controller, which is one of the interesting and important issues in control and system theory. This problem was formulated and studied in [19], [20], and [21]. In these studies, a complete necessary and sufficient condition for a pair of given two systems to be simultaneous stabilizable was derived as the strong stabilizability (cf. [24]), which is the stabilizability of a plant by using a stable controller, of the augmented plant constructed by using given two plants. In [15] and [20], it was shown that the simultaneous stabilizability of two plants can be represented by using the condition on the poles and the zeros of transfer functions of given plants. In [16], a characterization of simultaneous stabilizers for two plants was given. For the case of that the number of given systems is three or more, it is well known that it is difficult to solve the simultaneous stabilization problem. Since this issue is deeply concerned with an interesting and meaningful nature on the notion of the stabilization from the system theoretic points of view (cf. [4] [5]), the simultaneous stabilization of three systems has attracted many attentions of researchers in the control and system theory from not only practical but also theoretical points of view. In the standard system theory, there are many studies related to the simultaneous stabilization problem. As representative works on this issue, we can cite the references [1], [2], [3], [4], [5], [7], [8], [9], [17], [19]. Particularly, as stated in [2], it was shown that the simultaneous stabilizability of three linear systems is rationally undecidable, that is to say, it is not possible to find necessary and sufficient conditions for the simultaneous stabilization of three linear systems in the sense that there exist no ways for checking the simultaneous stabilizability by algebraic, logical, and sign test operations. In [3], the difficulty of the simultaneous stabilization of three systems was deeply investigated with respect to interpolation problem. Although an important problem related to simultaneous stabilization stated in [2] and [3] has been solved in [17], it is still difficult to obtain a condition for a triple of linear systems to be simultaneously stabilizable.

By the way, J.C. Willems has proposed the behavioral approach, which provides a new viewpoint for dynamical system theory(cf. [22]). In this approach, a control is regarded as an "interconnection"(cf.[23], [14]) which is a generalization of the concept of "control" from a broader perspective. Thus, it is to be expected that the behavioral approach provides new and meaningful insights for an important theoretical issue like the simultaneous stabilization problem.

From these reasons, this paper addresses the simultaneous stabilization problem for a triple of linear systems within a behavioral framework. First, we review the simultaneous stabilization problem for a pair of linear systems and clarify the symmetric structures of simultaneous stabilizers in a behavioral framework as stated in [11] and [12]. And then we give a new necessary and sufficient condition for a pair of linear systems to be simultaneous stabilizable, and also give a new sufficient condition. The later is completely described in terms of the behaviors independently from the mathematical representations. Next, by using this sufficient condition, we characterize the simultaneous stabilizers for a pair of linear systems satisfying some assumptions. Finally, by using these results, we give a sufficient condition for a triple of linear systems to be a simultaneously stabilizable with in a behavioral framework. The sufficient condition presented here depends on not mathematical models but the behaviors. We also provide a representation of simultaneous stabilizers for a triple of linear systems satisfying this sufficient condition.

[Notations]: Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and complex numbers, respectively. Let $\mathbb{R}^{\mathtt{q}}$ and $\mathbb{R}^{\mathtt{p}\times\mathtt{q}}$ denote the set of real vectors of size q and the set of real matrices of size $p \times q$, respectively. Let $\mathbb{R}[\xi]$ denote the set of polynomials of real coefficients and $\mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ denote the matrix version of them of size $p \times q$. In the case of $p \geq q$ ($p \leq q$), let $\mathbb{R}^{\mathtt{p}\times\mathtt{q}}_{\mathbb{C}}[\xi]$ denote the set of polynomial matrices satisfying $\{A(\xi) \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ s.t. $A(\lambda)$ is full column (row,

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respectively) rank for all $\lambda \in \mathbb{C}\}$. Let $\mathbb{R}_{\mathbb{H}}^{\mathtt{q}\times\mathtt{q}}[\xi]$ denote the set of polynomial matrices whose determinants have no roots on the closed right half-plane. We call an element of $\mathbb{R}_{\mathbb{H}}^{\mathtt{q}\times\mathtt{q}}[\xi]$ *Hurwitz*. Note that a unimodular matrix on $\mathbb{R}^{\mathtt{q}\times\mathtt{q}}[\xi]$ is also Hurwitz. We denote the set of the trajectories such that $\lim_{t\to\infty} w(t) = 0$ (i.e., the set of asymptotic stable trajectories) of size $\mathtt{q}$ with $\mathcal{S}^{\mathtt{q}}$. Finally, we assume that the solution of a differential equation appearing in this paper is included in the infinitely differentiable functions.

II. PRELIMINARIES

In this section, we prepare notions and definitions required in the following sections, briefly. For more details, see the references [22], [23], [18] and so on.

A. The basics of behavioral system theory

A dynamical system Σ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with the time axis $\mathbb{T}$, the signal space $\mathbb{W}$ and the behavior $\mathfrak{B}$. Consider a dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{q}}, \mathfrak{B})$. Moreover, assume that Σ is linear, time-invariant and differentiable. Then, a dynamical system Σ is representable by

$$R_N \frac{d^N w}{dt^N} + \cdots + R_1 \frac{dw}{dt} + R_0 w = 0 \tag{1}$$

where $R_i \in \mathbb{R}^{\bullet\times\mathtt{q}}, i = 0, \cdots, N$. This is called a *kernel representation* of Σ and the variable w is called a manifest variable. A kernel representation is written as $R(\frac{d}{dt})w = 0$ by using a polynomial matrix $R := R_0 + R_1\xi + \cdots + R_N\xi^N \in \mathbb{R}^{\bullet\times\mathtt{q}}[\xi]$. Throughout this paper, we assume that a system is linear, time-invariant and differentiable.

There are many kernel representations for the behavior of a system Σ. Particularly, we call a kernel representation $R(\frac{d}{dt})w = 0$ *minimal* if R has normal full row rank. Let $\rho(\mathfrak{B})$ denote the size of rows of a minimal kernel representation of $\mathfrak{B}$ and note that $\rho(\mathfrak{B})$ is independent from representations of $\mathfrak{B}$. $\rho(\mathfrak{B})$ is called the output cardinality of $\mathfrak{B}$.

A dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{q}}, \mathfrak{B})$ is said to be *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ there exist $w \in \mathfrak{B}$ and $T_1, T_2 (\in \mathbb{R})$ such that $w(t) = w_1(t)$ for $t \le T_1$ and $w(t) = w_2(t)$ for $t > T_2$. Σ is controllable if and only if a minimal kernel representation is induced by an element of $\mathbb{R}_{\mathbb{C}}^{\rho(\mathfrak{B})\times\mathtt{q}}[\xi]$. The controllability of a system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{q}}, \mathfrak{B})$ is also equivalent to saying that $\mathfrak{B}$ can be described by

$$w = M_L \frac{d^L \ell}{dt^L} + \cdots + M_1 \frac{d\ell}{dt} + M_0 \ell \tag{2}$$

where $M_i \in \mathbb{R}^{\mathtt{q}\times\bullet}, i = 0, \cdots, L$. This is called an *image representation* of Σ and ℓ is called a latent variable. Similarly to kernel representations, we use the notation $w = M(\frac{d}{dt})\ell$ by using a polynomial matrix $M := M_0 + M_1\xi + \cdots + M_L\xi^L \in \mathbb{R}^{\mathtt{q}\times\bullet}[\xi]$. Moreover, there are many image representations for the behavior of a controllable system Σ. In addition, ℓ is said to be *observable* from w if $w = 0$ implies $\ell = 0$. A latent variable ℓ in $w = M(\frac{d}{dt})\ell$ is observable from w if and only if $M \in \mathbb{R}_{\mathbb{C}}^{\mathtt{q}\times(\mathtt{q}-\rho(\mathfrak{B}))}[\xi]$.

A dynamical system is said to be *autonomous* if for w_1 and $w_2 \in \mathfrak{B}$ $w_1(t) = w_2(t) (\forall t < 0) \Rightarrow w_1(t) = w_2(t) (\forall t \ge 0)$. A system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{q}}, \mathfrak{B})$ is autonomous if and only if a minimal kernel representation of $\mathfrak{B}$ is induced by a non-singular polynomial matrix $R \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}[\xi]$. We also refer to the roots of $\det(R)$ as the pole of $\mathfrak{B}$ in this paper.

A dynamical system is said to be stable if $w \in \mathfrak{B}$ implies $w(t) \to 0$ as $t \to \infty$. As easily seen, the autonomy of a system is a necessary condition for the system to be stable. A system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{q}}, \mathfrak{B})$ is stable if and only if a minimal kernel representation of $\mathfrak{B}$ is induced by a non-singular Hurwitz polynomial matrix $R \in \mathbb{R}_{\mathbb{H}}^{\mathtt{q}\times\mathtt{q}}[\xi]$.

B. Behavioral system synthesis

Consider controllable systems $\Sigma_P = (\mathbb{R}, \mathbb{R}^{\mathtt{q}}, \mathfrak{B}_P)$ and $\Sigma_C = (\mathbb{R}, \mathbb{R}^{\mathtt{q}}, \mathfrak{B}_C)$. Each of the behaviors $\mathfrak{B}_P$ and $\mathfrak{B}_C$ are described by minimal kernel representations induced by polynomials $R \in \mathbb{R}^{\rho(\mathfrak{B}_\mathrm{P})\times\mathtt{q}}[\xi]$ and $C \in \mathbb{R}^{\rho(\mathfrak{B}_\mathrm{C})\times\mathtt{q}}[\xi]$, respectively. Then, the behavior of the interconnection of Σ_P and Σ_C , say $\mathfrak{B}_P \cap \mathfrak{B}_C$, is described by

$$\begin{bmatrix} R\left(\frac{d}{dt}\right) \\ C\left(\frac{d}{dt}\right) \end{bmatrix} w = 0. \tag{3}$$

The interconnection of Σ_P and Σ_C is said to be a regular if $\rho(\mathfrak{B}_P) + \rho(\mathfrak{B}_C)$ is equal to $\rho(\mathfrak{B}_P \cap \mathfrak{B}_C)$. The set of the behavior of systems interconnecting to a plant are described by not only proper- but also non proper rational functions. Thus, the set of the achievable behaviors via regular interconnection is broader than that via well-known (regular) feedback interconnection driven by CPU.

In order to stabilize the plant, the controller must be designed so as to satisfy that $\begin{bmatrix} R^{\mathrm{T}} & C^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ must be an element of $\mathbb{R}_{\mathbb{H}}^{\mathtt{q}\times\mathtt{q}}[\xi]$. Then, Σ_c is said to be a *stabilizer* for Σ_p. Throughout this paper, let $\Omega_r(\mathfrak{B}_P) \subseteq \mathbb{R}^{(\mathtt{q}-\rho(\mathfrak{B}_\mathrm{P}))\times\mathtt{q}}[\xi]$ denote the set of polynomial matrices inducing minimal kernel representations of the stabilizers for $\Sigma_P = (\mathbb{R}, \mathbb{R}^{\mathtt{q}}, \mathfrak{B}_P)$ via regular interconnections.

Concerned with stabilization, we cite the following lemmas obtained in [14] by Kuijper. The first lemma is a generalization of doubly coprime factorizations of polynomial matrices (cf. Lemma 6.3.9 in [10]).

Lemma 1: Consider $R \in \mathbb{R}_{\mathbb{C}}^{\mathtt{p}\times\mathtt{q}}[\xi]$. Then there exist $M \in \mathbb{R}_{\mathbb{C}}^{\mathtt{q}\times(\mathtt{q}-\mathtt{p})}[\xi]$, $N \in \mathbb{R}_{\mathbb{C}}^{\mathtt{q}\times\mathtt{p}}[\xi]$, $Q \in \mathbb{R}_{\mathbb{C}}^{(\mathtt{q}-\mathtt{p})\times\mathtt{q}}[\xi]$ such that

$$\begin{bmatrix} R \\ Q \end{bmatrix} \begin{bmatrix} N & M \end{bmatrix} = \begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & I_{\mathtt{q}-\mathtt{p}} \end{bmatrix} \tag{4}$$

holds. □

The next lemma is a parameterization of all stabilizers in a behavioral sense.

Lemma 2: Let $\Sigma_P = (\mathbb{R}, \mathbb{R}^{\mathtt{q}}, \mathfrak{B}_P)$ be a controllable plant. Let $R(\xi) \in \mathbb{R}_{\mathbb{C}}^{\rho(\mathfrak{B}_\mathrm{P})\times\mathtt{q}}[\xi]$ induce a minimal kernel representation of $\mathfrak{B}_P$. Then, all of the elements of $\Omega_r(\mathfrak{B}_P)$ can be parameterized by

$$C := \begin{bmatrix} F & B \end{bmatrix} \begin{bmatrix} R \\ Q \end{bmatrix} \tag{5}$$

where $B \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\rho(\mathfrak{B}_\mathtt{P}))\times(\mathtt{q}-\rho(\mathfrak{B}_\mathtt{P}))}[\xi]$ and $F \in \mathbb{R}^{(\mathtt{q}-\rho(\mathfrak{B}_\mathtt{P}))\times\rho(\mathfrak{B}_\mathtt{P})}[\xi]$ are free parameters.□

The next lemma is one of the equivalent conditions for the system described by $C(\frac{d}{dt})w = 0$ to be a stabilizer for the plant.

Lemma 3: Assume that $\Sigma_P = (\mathbb{R}, \mathbb{R}^\mathtt{q}, \mathfrak{B}_P)$ is controllable. Let M induce an observable image representation for $\mathfrak{B}$. Consider $C \in \mathbb{R}^{(\mathtt{q}-\rho(\mathfrak{B})\times \mathtt{q}}[\xi]$. Then $C \in \Omega_r(\mathfrak{B}_P)$ if and only if $CM \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\rho(\mathfrak{B}_\mathtt{P}))\times(\mathtt{q}-\rho(\mathfrak{B}_\mathtt{P}))}[\xi]$.□

The next lemma is a straightforward application of Lemma 3, which is used to describe the statements in our main theorems.

Lemma 4: Consider two controllable plants $\Sigma_1 = (\mathbb{R}, \mathbb{R}^\mathtt{q}, \mathfrak{B}_1)$ and $\Sigma_2 = (\mathbb{R}, \mathbb{R}^\mathtt{q}, \mathfrak{B}_2)$. Let R_1 and M_2 induce a minimal kernel representation of $\mathfrak{B}_1$ and an observable image representation of $\mathfrak{B}_2$, respectively. Assume that the output cardinalities of both behaviors are the same, i.e., $\rho(\mathfrak{B}_1) = \rho(\mathfrak{B}_2) =$:p. Then $\mathfrak{B}_1 \cap \mathfrak{B}_2 \subseteq \mathcal{S}^\mathtt{q}$ if and only if $R_1 M_2 \in \mathbb{R}_{\mathbb{H}}^{\mathtt{p}\times\mathtt{p}}[\xi]$. □

In order to state the final main result of this paper, we introduce the notion of complementary behavior $\mathfrak{B}^\perp$ such that

$$\mathfrak{B} \oplus \mathfrak{B}^\perp = (\mathbb{R}^\mathtt{q})^\mathbb{R}. \tag{6}$$

It is easy to see that Q and N induce a minimal kernel- and an observable image- representations of $\mathfrak{B}^\perp$, respectively.

C. The simultaneous stabilizability of two systems

As for the simultaneous stabilization problem for a pair of linear systems, a necessary and sufficient condition and a parameterization has already been studied by the author in the behavioral setting in [11] and [12]. Particularly, the next theorem obtained in these references is a necessary and sufficient condition for the simultaneous stabilizability, which also plays a crucial role in this paper.

Theorem 5: Consider two controllable systems $\Sigma_1 = (\mathbb{R}, \mathbb{R}^\mathtt{q}, \mathfrak{B}_1)$ and $\Sigma_2 = (\mathbb{R}, \mathbb{R}^\mathtt{q}, \mathfrak{B}_2)$. Assume that the output cardinality of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are the same. Define $\mathfrak{B}_{12}$ described by the following new kernel representation

$$R_2(\tfrac{d}{dt}) \left[\begin{array}{cc} N_1(\frac{d}{dt}) & M_1(\frac{d}{dt}) \end{array}\right] w = 0. \tag{7}$$

Then, Σ_1 and Σ_2 are simultaneously stabilizable if and only if there exists $C_n^{(12)} \in \mathbb{R}^{(\mathtt{q}-\mathtt{p})\times\mathtt{q}}[\xi]$ and $C_d^{(12)} \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]$ such that

$$\left[\begin{array}{cc} C_n^{(12)} & C_d^{(12)} \end{array}\right] \in \Omega_r(\mathfrak{B}_{12}). \tag{8}$$

Proof: See [12]. ■

From this theorem, we can see that the simultaneous stabilizability of two plants is equivalent to the stabilizability of an augmented plant with a certain class of the stabilizers. Roughly speaking, $C_d^{(12)}(\xi)$ corresponds to the "denominator" of the controller, so the above condition is equivalent to the stabilizability by using a "stable" (in the sense that the denominator is Hurwitz) controller. Thus, the above condition is also equivalent to the *strong* stabilizability of the augmented plant, which is a generalization of the well-known result obtained in [20] in the standard system theory.

III. PROBLEM FORMULATION

Let $R_i \in R_{\mathbb{C}}^{\rho(\mathfrak{B}_\mathtt{i})\times\mathtt{q}}[\xi]$ and $M_i \in R_{\mathbb{C}}^{\mathtt{q}\times(\mathtt{q}-\rho(\mathfrak{B}_\mathtt{i}))}[\xi]$ induce a minimal kernel- and an observable image- representations of $\mathfrak{B}_i$ $(i = 1,2,3)$, respectively. Moreover, let $N_i \in \mathbb{R}_{\mathbb{C}}^{\mathtt{q}\times\rho(\mathfrak{B}_\mathtt{i})}[\xi]$ and $Q_i \in \mathbb{R}_{\mathbb{C}}^{(\mathtt{q}-\rho(\mathfrak{B}_\mathtt{i}))\times\mathtt{q}}[\xi]$ denote polynomial matrices satisfies Eq.(4), for R_i and M_i, $i = 1,2,3$ throughout this paper.

Now we are ready to state the problem we attack here in the following.

Problem 6: First, let $\Sigma_i = (\mathbb{T}, \mathbb{R}^\mathtt{q}, \mathfrak{B}_i)$ $(i = 1,2,3)$ denote three controllable plants. Assume that $\rho(\mathfrak{B}_1) = \rho(\mathfrak{B}_2) = \rho(\mathfrak{B}_3)$ (=: p). Moreover, we assume that $2\mathtt{p} = \mathtt{q}$.

(a). Find a sufficient condition for the existence of a controller that stabilizes three plants Σ_i $(i = 1,2,3)$ via regular interconnection.

(b). Find a representation of the controllers stabilizing three plants Σ_i $(i = 1,2,3)$ that satisfy the above sufficient condition .□

Throughout this paper, we call $\Sigma_C = (\mathbb{R}, \mathbb{R}^\mathtt{q}, \mathfrak{B}_C)$ with $\mathfrak{B}_C = \mathrm{Ker}(C)$ induced by $C(\xi) \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ a simultaneous stabilizer for $\mathfrak{B}_1$ $\mathfrak{B}_2$ and $\mathfrak{B}_3$ if

$$\left[\begin{array}{c} R_1 \\ C \end{array}\right], \left[\begin{array}{c} R_2 \\ C \end{array}\right], \left[\begin{array}{c} R_3 \\ C \end{array}\right] \in \mathbb{R}_{\mathbb{H}}^{\mathtt{q}\times\mathtt{q}}[\xi] \tag{9}$$

hold. The set of polynomial matrices inducing minimal kernel representations of simultaneous stabilizers is denoted with $\Omega_r(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)$. Similarly, let $\Omega_r(\mathfrak{B}_i, \mathfrak{B}_j)$ denote the set of polynomial matrices inducing minimal kernel representations of simultaneous stabilizers for $\mathfrak{B}_i$ and $\mathfrak{B}_j$, $i \neq j$.

IV. MAIN RESULTS

In order to find the answers for the above questions in Problem 6, we start with reviewing the simultaneous stabilizability of two plants.

A. Symmetric structure of simultaneous stabilizers for two systems

First, we can see that the following corollary can be derived from Theorem 5 directly.

Corollary 7: Consider $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Define $\mathfrak{B}_{21}$ described by the following new kernel representation

$$R_1(\tfrac{d}{dt}) \left[\begin{array}{cc} N_2(\frac{d}{dt}) & M_2(\frac{d}{dt}) \end{array}\right] w = 0. \tag{10}$$

Then, Σ_1 and Σ_2 are simultaneously stabilizable if and only if there exists $C_n^{(21)}(\xi) \in \mathbb{R}^{(\mathtt{q}-\mathtt{p})\times\mathtt{q}}[\xi]$ and $C_d^{(21)}(\xi) \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]$ such that

$$\left[\begin{array}{cc} C_n^{(21)} & C_d^{(21)} \end{array}\right] \in \Omega_r(\mathfrak{B}_{21}). \tag{11}$$

By using Theorem 5 and Corollary 7, we show that simultaneous stabilizers for two plants has a symmetric structure between $\Omega_r(\mathfrak{B}_{12})$ and $\Omega_r(\mathfrak{B}_{21})$ as follows.

Theorem 8: Let $\mathfrak{B}_{12}$ and $\mathfrak{B}_{21}$ denote the behaviors described by Theorem 5 and Corollary 7, respectively. Let $[C_n^{(12)}\ \ C_d^{(12)}]$ and $[C_n^{(21)}\ \ C_d^{(21)}]$ induce stabilizers for $\mathfrak{B}_{12}$ and $\mathfrak{B}_{21}$ with Hurwitz $C_d^{(12)}$ and $C_d^{(21)}$, respectively. Then,

$$\begin{bmatrix} C_n^{(12)} & C_d^{(12)} \end{bmatrix} \begin{bmatrix} R_1 \\ Q_1 \end{bmatrix} M_2 = C_d^{(21)} \tag{12}$$

$$\begin{bmatrix} C_n^{(21)} & C_d^{(21)} \end{bmatrix} \begin{bmatrix} R_2 \\ Q_2 \end{bmatrix} M_1 = C_d^{(12)}. \tag{13}$$

Proof: The arguments for these two relations are analogous, so we focus on Eq.(12) here. The outline of the proof is the following. Since $[C_n^{(12)}\ \ C_d^{(12)}] \in \Omega_r(\mathfrak{B}_{12})$, it follows from Lemma 3 that

$$[C_n^{(12)}\ \ C_d^{(12)}] \begin{bmatrix} R_1 \\ Q_1 \end{bmatrix} M_2 =: B^{(12)} \tag{14}$$

is Hurwitz. Define

$$[C_n^{(12)}\ \ C_d^{(12)}] \begin{bmatrix} R_1 \\ Q_1 \end{bmatrix} N_2 =: F^{(12)}. \tag{15}$$

Apposing Eq.(14) and Eq.(15) yields

$$[C_n^{(12)}\ \ C_d^{(12)}] \begin{bmatrix} R_1 \\ Q_1 \end{bmatrix} [N_2\ \ M_2] = [F^{(12)}\ \ B^{(12)}], \tag{16}$$

which implies

$$[C_n^{(12)}\ \ C_d^{(12)}] = [F^{(12)}\ \ B^{(12)}] \begin{bmatrix} R_2 \\ Q_2 \end{bmatrix} [N_1\ M_1]. \tag{17}$$

On the other hand, $C_d^{(12)}$ is Hurwitz, thus

$$[F^{(12)}\ \ B^{(12)}] \begin{bmatrix} R_2 \\ Q_2 \end{bmatrix} M_1 \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]. \tag{18}$$

It follows from Lemma 3 that this fact implies that $[F^{(12)}\ \ B^{(12)}]$ is included in $\Omega_r(\mathfrak{B}_{21})$. Moreover, note that $B^{(12)} \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]$. This means that $[F^{(12)}\ \ B^{(12)}]$ induces also a strong stabilizer for $\mathfrak{B}_{21}$. By regarding $B^{(12)}$ as the "denominator" $C_d^{(21)}$ of the strong stabilizer for $\mathfrak{B}_{21}$, we see that Eq.(12) holds. ∎

We can regard the roots of the determinant of each left hand side in Eq.(13) and Eq.(12) as the pole of the interconnected systems (See Lemma 3). We can also regard $C_d^{(12)}$ and $C_d^{(21)}$ as the 'denominators' of the strong stabilizers for $\mathfrak{B}_{12}$ and $\mathfrak{B}_{21}$, respectively. That is, the above theorem says that the roots of the denominator of the strong stabilizer for $\mathfrak{B}_{12}$ and the mode of the interconnected systems consisting $\mathfrak{B}_{21}$ and its strong stabilizer are the same (vice versa).

By using Theorem 5 and the above interpretation, we can see that

$$[C_n^{(12)}\ \ C_d^{(12)}] = [C_n^{(21)}\ \ C_d^{(21)}] \begin{bmatrix} R_2 \\ Q_2 \end{bmatrix} [N_1\ M_1]. \tag{19}$$

where $[C_n^{(12)}\ \ C_d^{(12)}]$ and $[C_n^{(21)}\ \ C_d^{(21)}]$ are strong stabilizers for $\mathfrak{B}_{12}$ and $\mathfrak{B}_{21}$, respectively. These relations can be also written by

$$[C_n^{(12)}\ \ C_d^{(12)}] \begin{bmatrix} R_1 \\ Q_1 \end{bmatrix} = [C_n^{(21)}\ \ C_d^{(21)}] \begin{bmatrix} R_2 \\ Q_2 \end{bmatrix}. \tag{20}$$

From the observations stated above, both of the left and the right hand side of Eq.(20) are included in $\Omega_r(\mathfrak{B}_1, \mathfrak{B}_2)$.

Theorem 8 and Eq.(20), we obtain the following new necessary and sufficient condition for a pair of linear systems to be simultaneously stabilizable.

Theorem 9: The following three statements are equivalent.

1. $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are simultaneously stabilizable.
2. There exist H_a, $H_b \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]$, and F_a, $F_b \in \mathbb{R}^{(\mathtt{q}-\mathtt{p})\times\mathtt{p}}[\xi]$ such that

$$\begin{aligned} \begin{bmatrix} H_a & H_b \end{bmatrix} &\begin{bmatrix} Q_1M_2 & -I \\ -I & Q_2M_1 \end{bmatrix} \\ &= \begin{bmatrix} F_a & F_b \end{bmatrix} \begin{bmatrix} -R_1M_2 & 0 \\ 0 & -R_2M_1 \end{bmatrix} \end{aligned} \tag{21}$$

3. There exist H_a, $H_b \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]$, and F_a, $F_b \in \mathbb{R}^{(\mathtt{q}-\mathtt{p})\times\mathtt{p}}[\xi]$ such that

$$\begin{aligned} \begin{bmatrix} H_a & H_b \end{bmatrix} &\begin{bmatrix} -Q_1N_2 & 0 \\ 0 & -Q_2N_1 \end{bmatrix} \\ &= \begin{bmatrix} F_a & F_b \end{bmatrix} \begin{bmatrix} R_1N_2 & -I \\ -I & R_2N_1 \end{bmatrix}. \end{aligned} \tag{22}$$

Proof: (2 ⇒ 1): Assume that Eq.(21) is solvable, which is equivalently to saying that

$$\begin{aligned} H_aQ_1M_2 + F_aR_1M_2 = H_b \\ H_bQ_2M_1 + F_aR_2M_1 = H_a \end{aligned} \tag{23}$$

are solvable. The first equation of Eq.(23) is also equivalent to

$$\begin{bmatrix} F_a & H_a \end{bmatrix} \begin{bmatrix} R_1 \\ Q_1 \end{bmatrix} M_2 = H_b. \tag{24}$$

It follows from $H_b \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]$ and Theorem 5 that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are simultaneously stabilizable. (Of course, using the second equation of Eq.(23) yields the same conseqence.)

(1 ⇒ 2): Assume that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are simultaneously stabilizable. Then, from Theorem 8, putting $C_d^{(12)} = H_a$, $C_d^{(21)} = H_b$, $C_n^{(12)} = F_a$ and $C_n^{(21)} = F_b$ yield the solvability of Eq.(21).

The proof of the equivalence of 1 and 3 is analogous with the above argument, so we omit it here. ∎

It is easy to see that Theorem 9 provides alternative representations of the strong stabilizability of $\mathfrak{B}_{12}$, or equivalently, that of $\mathfrak{B}_{21}$. In the simultaneous stabilization problem, we require the stability of the interconnected system and the "denominator" of the stabilizer. Thus, for any H_a and $H_b \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]$, if there exist F_a and $F_b \in \mathbb{R}^{(\mathtt{q}-\mathtt{p})\times\mathtt{p}}[\xi]$ satisfying Eq.(21) or equivalently Eq.(22), then the pair of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is simultaneously stabilizable. From this points of view, consider the condition under which Eq.(21) is solvable for any H_a and $H_b \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]$. Then we obtain the following sufficient condition for a pair of linear systems to be simultaneously stabilizable.

Theorem 10: Consider $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Assume that

$$\mathfrak{B}_1 \cap \mathfrak{B}_2 = \{0\}. \tag{25}$$

Then the pair of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is simultaneously stabilizable. Moreover, for any H_a and $H_b \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]$, the following polynomial matrix

$$\begin{bmatrix} (H_aQ_1M_2 - H_b)(R_1M_2)^{-1} & H_a \end{bmatrix} \begin{bmatrix} R_1 \\ Q_1 \end{bmatrix} \tag{26}$$

or equivalently,

$$\begin{bmatrix} (H_bQ_2M_1 - H_a)(R_2M_1)^{-1} & H_b \end{bmatrix} \begin{bmatrix} R_2 \\ Q_2 \end{bmatrix} \tag{27}$$

is included in $\Omega(\mathfrak{B}_1, \mathfrak{B}_2)$.

Proof: The outline of the proof is the following. First, from Lemma 4, it is easy to see that $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \{0\}$ is equivalent to the unimodularity of R_1M_2, or equivalently, that of R_2M_1. This implies that $(R_1M_2)^{-1}$ and $(R_2M_1)^{-1}$ are also polynomial matrices. From these facts , for arbitrary H_a and $H_b \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]$, there exist polynomial matrices F_a and F_b satisfying Eq.(21). Thus, from Theorem 9, the pair of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is simultaneously stabilizable. Moreover, F_a and F_b satisfying Eq.(21) for arbitrary H_a and $H_b \in \mathbb{R}_{\mathbb{H}}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]$ can be described by

$$\begin{aligned} F_a &= (H_aQ_1M_2 - H_b)(R_1M_2)^{-1} \\ F_b &= (H_bQ_2M_1 - H_a)(R_2M_1)^{-1}. \end{aligned} \tag{28}$$

Notice that $[F_a \ H_a]$ ($[F_b \ H_b]$) inducing the strong stabilizer for $\mathfrak{B}_{12}$ ($\mathfrak{B}_{21}$, respectively). Thus Eq.(26) or equivalently, Eq.(27) is included in $\Omega_r(\mathfrak{B}_1, \mathfrak{B}_2)$. ■

The above theorem says that if there exists no trajectory except trivial zero trajectory in the interconnected behavior then these two systems are simultaneously stabilizable.

Another possibility for Eq.(21) to be solvable happens in the case of that R_1M_2 and R_2M_1 are scalar polynomials, which requires that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ can be described by single-input single-output scalar transfer functions. By using this observation, we also obtain the following theorem.

Theorem 11: Assume

$$\mathfrak{B}_1 \cap \mathfrak{B}_2 \subseteq \mathcal{S}^2 \tag{29}$$

Then the pair of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is simultaneously stabilizable. Moreover, for any H_a and $H_b \in \mathbb{R}_{\mathbb{H}}[\xi]$, the following polynomial matrix

$$\begin{bmatrix} H_aQ_1M_2 - H_b & R_1M_2H_a \end{bmatrix} \begin{bmatrix} R_1 \\ Q_1 \end{bmatrix} \tag{30}$$

or equivalently,

$$\begin{bmatrix} H_bQ_2M_1 - H_a & R_2M_1H_b \end{bmatrix} \begin{bmatrix} R_2 \\ Q_2 \end{bmatrix} \tag{31}$$

is included in $\Omega_r(\mathfrak{B}_1, \mathfrak{B}_2)$.

Proof: The outline of the proof is the following. First, from Lemma 4, we see that Eq.(29) implies R_1M_2 and R_2M_1 are Hurwitz. Moreover, we see that $R_1M_2 = R_2M_1$. Define $B := R_1M_2 = R_2M_1$. Thus, Eq.(21) is rewritten by

$$\begin{bmatrix} H_a & H_b \end{bmatrix} \begin{bmatrix} Q_1M_2 & -I \\ -I & Q_2M_1 \end{bmatrix} = B \begin{bmatrix} F_a & F_b \end{bmatrix}. \tag{32}$$

Since the right hand side of Eq.(32) has the left coprime factor B, the left hand side also has to include B as the left coprime factor. For H_a and $H_b \in \mathbb{R}_{\mathbb{H}}[\xi]$ described by $H_a = BH'_a$ and $H_b = BH'_b$ for any H'_a and $H'_b \in \mathbb{R}_{\mathbb{H}}[\xi]$, there exist polynomial matrices F_a and F_b satisfying Eq.(21). Thus, from Theorem 9, the pair of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is simultaneously stabilizable. Then Eq.(32) is described by

$$B \begin{bmatrix} H'_a & H'_b \end{bmatrix} \begin{bmatrix} Q_1M_2 & -I \\ -I & Q_2M_1 \end{bmatrix} = B\begin{bmatrix} F_a & F_b \end{bmatrix}. \tag{33}$$

and we see that $F_a = H'_aQ_1M_2 - H'_b$ and $F_b = H'_bQ_2M_1 - H'_a$ is a solution of Eq.(21) for arbitrary H'_a and H'_b. Regard the above H'_a and H'_b as H_a and H_b in the statement of the theorem, respectively. From Theorem 9, we conclude that Eq.(30) or equivalently Eq.(31) is included in $\Omega_r(\mathfrak{B}_1, \mathfrak{B}_2)$. ■

B. A sufficient condition for a triple of linear systems to be simultaneously stabilizable

Now we are ready to provide a sufficient condition of the simultaneous stabilizability of three plants. The results we present here is based on the previous subsection.

Assume that $\mathfrak{B}_1 \cap \mathfrak{B}_2 \subseteq \mathcal{S}^2$. It follows from Theorem 11 that for any $H_a^{(12)}$ and $H_b^{(12)} \in \mathbb{R}_{\mathbb{H}}[\xi]$,

$$\begin{bmatrix} H_a^{(12)}Q_1M_2 - H_b^{(12)} & R_1M_2H_a^{(12)} \end{bmatrix} \begin{bmatrix} R_1 \\ Q_1 \end{bmatrix} \tag{34}$$

or equivalently,

$$\begin{bmatrix} H_b^{(12)}Q_2M_1 - H_a^{(12)} & R_2M_1H_b^{(12)} \end{bmatrix} \begin{bmatrix} R_2 \\ Q_2 \end{bmatrix} \tag{35}$$

is included in $\Omega_r(\mathfrak{B}_1, \mathfrak{B}_2)$. Moreover, assume also that $\mathfrak{B}_1 \cap \mathfrak{B}_3 \subseteq \mathcal{S}^2$. Similarly to the case of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, for any $H_a^{(13)}$ and $H_b^{(13)} \in \mathbb{R}_{\mathbb{H}}[\xi]$,

$$\begin{bmatrix} H_a^{(13)}Q_1M_3 - H_b^{(13)} & R_1M_3H_a^{(13)} \end{bmatrix} \begin{bmatrix} R_1 \\ Q_1 \end{bmatrix} \tag{36}$$

or equivalently,

$$\begin{bmatrix} H_b^{(13)}Q_3M_1 - H_a^{(13)} & R_3M_1H_b^{(13)} \end{bmatrix} \begin{bmatrix} R_3 \\ Q_3 \end{bmatrix} \tag{37}$$

is included in $\Omega_r(\mathfrak{B}_1, \mathfrak{B}_3)$.

Now, consider the situation under which a stabilizer described by Eq.(34) for $\mathfrak{B}_1$ and $\mathfrak{B}_2$ also stabilizes $\mathfrak{B}_3$. Note that we can choose arbitrary Hurwitz polynomials $H_a^{(12)}$, $H_b^{(12)}$, $H_a^{(13)}$,$H_b^{(13)}$. Thus, it is possible to take

$$R_1M_2H_a^{(12)} = R_1M_3H_a^{(13)} \tag{38}$$

from the assumption that R_1M_2 and R_1M_3 are Hurwitz polynomials. At the current point, $H_b^{(12)}$ and $H_b^{(13)}$ can be still assigned arbitrarily.

Next, assume that Q_1M_2 and Q_1M_3 are also Hurwitz polynomials, which are equivalent to $\mathfrak{B}_1^{\perp} \cap \mathfrak{B}_2 \subseteq \mathcal{S}^2$ and $\mathfrak{B}_1^{\perp} \cap \mathfrak{B}_3 \subseteq \mathcal{S}^2$, respectively. Under this additional assumptions, we can also take

$$H_b^{(13)} = -H_a^{(12)} Q_1 M_2, \quad H_b^{(12)} = -H_a^{(13)} Q_1 M_3 \qquad (39)$$

which implies that

$$H_a^{(12)} Q_1 M_2 - H_b^{(12)} = H_a^{(13)} Q_1 M_3 - H_b^{(13)}.$$

Of course, even if the role of $\mathfrak{B}_1$ is replaced with that of $\mathfrak{B}_2$ or $\mathfrak{B}_3$ in the above argument, we can apply the similar argument. Consequently, we have the following result.

Theorem 12: If at least one of the following conditions holds

$$\{\mathfrak{B}_1 \cap \mathfrak{B}_j \subseteq \mathcal{S}^2\} \wedge \{\mathfrak{B}_1^{\perp} \cap \mathfrak{B}_j \subseteq \mathcal{S}^2\} \quad j = 2,3 \qquad (40)$$
$$\{\mathfrak{B}_2 \cap \mathfrak{B}_j \subseteq \mathcal{S}^2\} \wedge \{\mathfrak{B}_2^{\perp} \cap \mathfrak{B}_j \subseteq \mathcal{S}^2\} \quad j = 1,3 \qquad (41)$$
$$\{\mathfrak{B}_3 \cap \mathfrak{B}_j \subseteq \mathcal{S}^2\} \wedge \{\mathfrak{B}_3^{\perp} \cap \mathfrak{B}_j \subseteq \mathcal{S}^2\} \quad j = 1,2 \qquad (42)$$

then, $\mathfrak{B}_1$,$\mathfrak{B}_2$ and $\mathfrak{B}_3$ are simultaneously stabilizable. Moreover, the following polynomial matrices induces simultaneous stabilizers in each cases for arbitrary $B \in \mathbb{R}_{\mathbb{H}}$;

a. Eq.(40):

$$BR_1 \begin{bmatrix} M_3Q_1M_2 + M_2Q_3M_1 & M_3R_1M_2 \end{bmatrix} \begin{bmatrix} R_1 \\ Q_1 \end{bmatrix}$$

b. Eq.(41):

$$BR_2 \begin{bmatrix} M_1Q_2M_3 + M_3Q_1M_2 & M_1R_2M_3 \end{bmatrix} \begin{bmatrix} R_2 \\ Q_2 \end{bmatrix}$$

c. Eq.(42):

$$BR_3 \begin{bmatrix} M_2Q_3M_1 + M_1Q_2M_3 & M_2R_3M_1 \end{bmatrix} \begin{bmatrix} R_3 \\ Q_3 \end{bmatrix}$$

Proof: From the previous discussions, it is easy to see that the above representations induces simultaneous stabilizers in each cases by using algebraic manipulations, so we omit the detailed proof here. ■

Although the above result is a sufficient condition for the simultaneous stabilizability of three systems, it is worthwhile to notice that the condition can be characterized in terms of the behaviors.

V. Concluding remarks

In this paper, we have studied simultaneous stabilization problem for a triple of linear systems in a behavioral framework. We have provided some new results on simultaneous stabilization of two systems. By using these results, we have derived a sufficient condition for a triple of linear systems to be simultaneously stabilizable in a behavioral framework. More detailed discussions and observations will be shown in the recent works by the authors in [13].

One of the further studies is to consider the system theoretic interpretation of the obtained result here. Moreover, we derive a less conservative sufficient condition of the simultaneous stabilizability of three plants within a behavioral framework. The other direction is to consider the simultaneous stabilizability for more than three plants.

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