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Minimum Order Transfer Function: the Interpolation Approach

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Abstract— **This paper presents an algorithm for obtaining the minimum order MISO transfer function model for the use in a model-based predictive controller. The source model can be either a non-minimal ARX model, a state-space model or any interconnection of linear models of mixed state-space and transfer function representations. The algorithm is based on polynomial interpolation theory, representing polynomials by their values on a set of points in the complex plane. Using this theory, we can find the minimum order from the dimension of the null space of a particular matrix. Finding the minimum order model is equivalent to finding a specific base of the null space. A novel feature of the presented approach is using a set of complex interpolation nodes obtained by mapping the standard set of real Chebyshev nodes by a bilinear transform.**

I. INTRODUCTION

THIS paper addresses the problem of obtaining minimum order multi-input/single-output ARX model of the form

$$y(k) = -\sum_{i=1}^{s} a_i y(k-i) + \sum_{r=1}^{m}\sum_{i=1}^{s} b_{ri} u_r(k-i-n_{dr}) + e(k) \quad (1)$$

where y is the output variable; u_r is the input variable; e is Gaussian white noise. Further, n_{dr} is the time delay associated with r-th input. This model is used in model-based predictive controller (MPC) described in [3]. It is very simple as far as process model representation is concerned, and economical from the point of view of the on-line computational effort and data storage. On the other hand, its noise model may not be realistic; it performs poorly under a significant measurement noise, i.e. a situation where an output error model or a state-space model with process and output noises would be appropriate. However, it is very costly to change the engine of a commercial MPC. To enhance the ARX-based MPC, Kalman filter is used as an incremental improvement. Its function is twofold: first, it is used for estimating unmeasured disturbances, to improve control performance. Second, it is used to modify internal data of the predictor (the past output values $y(k-1),\dots,$ $y(k-s)$) to obtain predictions that are the same as if they were obtained from the state-space model with a more complete noise model. The integration of Kalman filter and an ARX-based MPC is described in the recent paper [2].

This integration requires that both the state-space and the ARX models are input-output equivalent in the noise-free case. The stochastic part of (1), represented by $e(k)$, becomes irrelevant, because noises are handled by Kalman filter. The natural choice of the master model is the state-space one, which is able to reflect the physical structure of the plant. In particular, it can capture the multi-input multi-output (MIMO) nature of the process with cross-channel coupling, whereas the model used in MPC is a collection of multi-input-single-output (MISO) models. When obtaining MISO transfer function models from MIMO state-space ones, the issue of model minimality arises.

Minimizing order of model representation is an old problem. In the state-space, a classical staircase form of Rosenbrock is well known, see [6]. This algorithm is suitable for low-to-medium size problems. Modern, balanced realization based methods (e.g., [7]) are numerically reliable even for high orders, but have limitations, in particular, in handling unstable non-minimal modes.

As far as reducing order in the transfer matrix framework is concerned, an approach based on manipulation with the coefficients of the full order transfer matrix is given (in a wider context of minimum-order solution of rational matrix equations) in [5] and references therein. A related problem of finding a greatest common divisor in polynomial matrix fractions is treated in [8] using Sylvester matrix.

A modern approach to numerical problems involving polynomial matrices is based on polynomial matrix interpolation of Antsaklis and Gao [1]. This approach is applied here to reduce MISO transfer matrices; solving this problem by polynomial interpolation is new to the best of our knowledge. This approach does not require the non-minimal transfer matrix coefficients, but its values evaluated on a set of selected interpolation nodes in the complex plane. Therefore, we can skip the cumbersome computation of the full-order transfer matrix and obtain the interpolation values directly from the state-space data. In a similar fashion, the full order system can be represented as an interconnection of linear sub-systems, each of them given either by a transfer function or by a state-space model.

This paper further suggests some improvements to enhance numerical stability. First, the Chebyshev polynomial basis is used, which is a standard approach. Second, the Chebyshev nodes, which are distributed in the [-1,1] interval, are mapped to a set of points in the complex plane via the bilinear transform. Thus the advantages of the Chebyshev basis are preserved, whereas a new set of interpolation nodes is obtained which can well capture the

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properties of the interpolated model. By exploiting symmetries in the interpolation set and the problem structure, only real matrices are involved in the computation. In this way, we are able to achieve a high accuracy for mid-sized problems (the order of the target (minimum-order) system being around 15) for a fixed set of interpolation points. If we allow changing this interpolation set (by changing parameters of the bilinear transformation) we are able to a handle systems whose order is, after reduction, well over twenty. The proposed algorithm is more effective in order reduction than the state-space staircase algorithm as well as the `mineral` function of the Control Toolbox of Matlab ([12]).

We assume that there is no uncertainty in the source (full order) model. Order reduction of uncertain systems is beyond the scope of this paper; results in this direction can be found in [9], [10], [11]. Finally we remark that rich theory has been developed on the Nevanlinna-Pick interpolation and its use in control, see e.g. [13]. There, additional conditions are imposed on the interpolating functions, namely stability and positive realness, which are not required in the present context.

II. INTERPOLATION FRAMEWORK FOR TRANSFER MATRIX REDUCTION

An equivalent representation (in the deterministic part) to model (1) is the transfer matrix in Z-transform given by

$$G(z) = \frac{\left[\sum_{i=1}^{s} b_{1i} z^{s+n_D - n_{d1} - i} \ \vdots \ \cdots \ \vdots \ \sum_{i=1}^{s} b_{mi} z^{s+n_D - n_{dm} - i}\right]}{z^{s+n_D} + \sum_{i=1}^{s} a_i z^{s+n_D - i}}, \quad (2)$$

$n_D = \max\{n_{d1}, \ldots, n_{dr}\}$. This can also be written as

$$G(z) = [G_1(z) \ \cdots \ G_m(z)] = \frac{1}{a(z)}[b_1(z) \ \cdots \ b_m(z)]. \quad (3)$$

To simplify the notation, we shall consider polynomials in the standard form $a(z) = \sum_{i=0}^{n} \overline{a}_i z^i$ and $b_r(z) = \sum_{i=0}^{n} \overline{b}_{ri} z^i$, $n = s + n_D$, which differs from the form in (2) by the indexing convention. Transfer functions can be represented equivalently in the interpolation framework as introduced in [1]: For any complex z_i, the Z-transform images of the inputs and the output evaluated at z_i denoted as $u_1^i, \ldots, u_m^i$ and y^i, respectively, satisfy the equation

$$[b_1(z_i) \ \cdots \ b_m(z_i) \ \vdots \ -a(z_i)]\begin{bmatrix} u_1^i \\ \vdots \\ u_m^i \\ \overline{y^i} \end{bmatrix} = 0. \quad (4)$$

This can be expressed in terms of the coefficients as

$$[\overline{b}_{10} \ldots \overline{b}_{1n} \ \vdots \ \cdots \ \vdots \ \overline{b}_{m0} \ldots \overline{b}_{mn} \ \vdots \ -\overline{a}_0 \ldots -\overline{a}_n]\begin{bmatrix} V_n(z_i)u_1^i \\ \vdots \\ V_n(z_i)u_m^i \\ V_n(z_i)y^i \end{bmatrix} = 0, \quad (5)$$

where $V_n(z) = \begin{bmatrix} 1 & z & \cdots & z^n \end{bmatrix}^T$. A condition under which a rational-function matrix (with coefficients defined over the field of complex numbers) can be represented by a finite set of interpolation triplets is stated in the following theorem:

Theorem 1 ([1]). *Let* $d = (m+1)n+m$ *and let d triplets* $\left\{z_i, [u_1^i, \ldots, u_m^i], y^i\right\}_{i=1}^{d}$ *be given so that the matrix*

$$M = \begin{bmatrix} V_n(z_1)u_1^1 & V_n(z_2)u_1^2 & \cdots & V_n(z_d)u_1^d \\ \vdots & \vdots & & \vdots \\ V_n(z_1)u_m^1 & V_n(z_2)u_m^2 & \cdots & V_n(z_d)u_m^d \\ V_n(z_1)y^1 & V_n(z_2)y^2 & \cdots & V_n(z_d)y^d \end{bmatrix} \quad (6)$$

is of full column rank. Then the $1 \times m$ *rational-function matrix* $G(z)$ *satisfying the interpolation condition*

$y^i = G(z_i)\begin{bmatrix} u_1^i & \cdots & u_m^i \end{bmatrix}^T$ *is uniquely determined by the given set of interpolation triplets.*

This theorem states that the complex-valued rational function $G(z)$ can be obtained from a given interpolation set, provided a rank condition is satisfied. Let us denote

$$\overline{b}_i := [\overline{b}_{i0} \ \cdots \ \overline{b}_{in}], \quad \overline{a} := [\overline{a}_0 \ \cdots \ \overline{a}_n]. \quad (7)$$

Then the desired vector $[\overline{b}_1 \ \cdots \ \overline{b}_m \ \vdots \ -\overline{a}]$ is the rank-one left kernel space of matrix M. This subspace parameterizes all representations of one transfer function, which differ by a constant multiplicative factor of the numerator and denominator polynomials. If we wish to obtain the transfer function in a normalized form, such as that with $a_n = 1$, the problem can be re-formulated as a solution of a matrix equation, see [1].

Now the question is how to choose the right set of interpolation points to obtain the full rank matrix. We formulate a lemma regarding the minimum order of SISO transfer functions. First, let us condense the notation as $V_n^i := V_n(z_i)$. From now on, we shall assume that n is the order of the known non-minimal representation of $G(z)$.

Lemma 1. *Let a set of n triplets* $\left\{z_i, u^i, y^i\right\}_{i=1}^{2n+1}$ *be given so that* $z_i \neq z_j \ \forall i,j$, *none of the interpolation points* z_i *is a pole of* $G(z)$, $y^i = G(z_i)u^i$ *and* $u^i \neq 0$ *for* $i = 1, \ldots, 2n+1$. *Let the* $2(n+1) \times 2n+1$ *matrix M be given as*

$$M = \begin{bmatrix} V_n^1 u^1 & \cdots & V_n^{2n+1}u^{2n+1} \\ V_n^1 y^1 & \cdots & V_n^{2n+1}y^{2n+1} \end{bmatrix}. \quad (8)$$

The minimum order of the transfer matrix satisfies

$n_0 = \text{Rank}(M) - n - 1$.

Proof: First, in the single input case we shall drop the input index r. Notice, that in this case the rank of matrix M does not depend on the choice of u^i; with no loss of generality, we can consider $u^i = a(z^i)$. Then some simple algebraic manipulations yield

$$\begin{bmatrix} V_n^1 a(z_1) & \cdots & V_n^{2n+1} a(z_{2n+1}) \\ \hline V_n^1 b(z_1) & \cdots & V_n^{2n+1} b(z_{2n+1}) \end{bmatrix} = S \begin{bmatrix} V_{2n}^1 & \cdots & V_{2n}^{2n+1} \end{bmatrix}, \quad (9)$$

where

$$S := \begin{bmatrix} \overline{a}_0 & \overline{a}_1 & \cdots & \overline{a}_n & 0 & \cdots & 0 \\ 0 & \overline{a}_0 & \cdots & \overline{a}_{n-1} & \overline{a}_n & & \\ & & \ddots & & & \ddots & \\ 0 & \cdots & 0 & \overline{a}_0 & \overline{a}_1 & \cdots & \overline{a}_n \\ \hline \overline{b}_0 & \overline{b}_1 & \cdots & \overline{b}_n & 0 & \cdots & 0 \\ 0 & \overline{b}_0 & \cdots & \overline{b}_{n-1} & \overline{b}_n & & \\ & & \ddots & & & \ddots & \\ 0 & \cdots & 0 & \overline{b}_0 & \overline{b}_1 & \cdots & \overline{b}_n \end{bmatrix}. \quad (10)$$

We can see that the matrix $\begin{bmatrix} V_{2n}^1 & \cdots & V_{2n}^{2n+1} \end{bmatrix}$ in (9) is the $(2n+1) \times (2n+1)$ Vandermonde matrix which is invertible whenever the interpolation points are distinct. Hence, the rank of M is the same as the rank of S in (10); the latter is the Sylvester matrix. It is well known from linear system theory, that polynomials $a(z)$ and $b(z)$ are coprime, whenever S is of full rank. Moreover, it is proven in [8] that $\text{Rank}(S) = n + 1 + n_0$. ∎

Now, let us outline the procedure of obtaining the transfer function coefficients from the above interpolation matrix, assuming $\text{Rank}(M) = n + 1 + n_0$. First, with no loss of generality, we assume $u^i = 1$ for $i = 1,...,2n+1$ -- this is only a column scaling. Therefore, we have $y^i = G(z_i)$. The following equation holds

$$\begin{bmatrix} \overline{b} & -\overline{a} \end{bmatrix} M = 0. \quad (11)$$

Post-multiply this equation by the inverse Vandermonde matrix $\begin{bmatrix} V_{2n}^1 & \cdots & V_{2n}^{2n+1} \end{bmatrix}^{-1}$ to get

$$\begin{bmatrix} \overline{b} & -\overline{a} \end{bmatrix} \begin{bmatrix} I_{n+1} & 0 \\ P & Q \end{bmatrix} = 0, \quad (12)$$

where

$$\begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} V_n^1 y^1 & \cdots & V_n^{2n+1} y^{2n+1} \end{bmatrix} \begin{bmatrix} V_{2n}^1 & \cdots & V_{2n}^{2n+1} \end{bmatrix}^{-1}. \quad (13)$$

Denominator coefficients thus belong to the left null space of the $(n+1) \times n$ matrix Q whose rank is n_0. Let the singular value decomposition of Q be

$$Q = \underbrace{\begin{bmatrix} \overline{u}_1 & \cdots & \overline{u}_{n+1} \end{bmatrix}}_{U} \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_{n0} & & 0 \\ \hline & & & 0 & \\ & & & & \ddots \end{bmatrix} V^*. \quad (14)$$

The asterisk denotes the conjugate transpose. The left null space is spanned by the rows of matrix N given by

$$N = \begin{bmatrix} \overline{u}_{n0+1} & \cdots & \overline{u}_{n+1} \end{bmatrix}^*. \quad (15)$$

Obtaining $\overline{a}$ belonging to this subspace such that its last $n - n_0$ elements are zero is straightforward: a numerically robust way is using a modified QR factorization of N

$$N = W \begin{bmatrix} n_{11} & \cdots & & n_{1,n+1} \\ \vdots & & \cdot^{\cdot} & \\ n_{n+1-n0,1} & \cdots & n_{n+1-n0,n0+1} & 0 \end{bmatrix}, \quad (16)$$

where the lowest order numerator coefficients are in the last row of the factorized matrix. The numerator coefficients are obtained from the denominator ones uniquely by

$$\overline{b} = \overline{a} P, \quad (17)$$

which concludes the procedure for the SISO case.

Extending this procedure to MISO cases is done as follows: first, define for each input u_r matrix M_r as in (8). Set $u_r^i = 1$ for all $r = 1,...,m$, $i = 1,...,2n+1$ and let $y_r^i = G_r(z_i)$, where $G_r(z)$ denotes r^{th} column of $G(z)$. Let, as in the SISO case,

$$M_r = \begin{bmatrix} V_n^1 & \cdots & V_n^{2n+1} \\ \hline V_n^1 y_r^1 & \cdots & V_n^{2n+1} y_r^{2n+1} \end{bmatrix}. \quad (18)$$

Partition this matrix as

$$M_r = \begin{bmatrix} M_{ar} \\ \hline M_{br} \end{bmatrix} = \begin{bmatrix} I_{n+1} & 0 \\ \hline P_r & Q_r \end{bmatrix} \begin{bmatrix} V_{2n}^1 & \cdots & V_{2n}^{2n+1} \end{bmatrix}. \quad (19)$$

Let us build the $(m+1)(n+1) \times m(2n+1)$ matrix $\overline{M}$ as

$$\overline{M} = \begin{bmatrix} M_{a1} & & 0 \\ & \ddots & \\ 0 & & M_{am} \\ \hline M_{b1} & \cdots & M_{bm} \end{bmatrix} \quad (20)$$

The result corresponding to that of Lemma 1 to MISO systems now can be stated:

Lemma 2. *Let matrix $\overline{M}$ be given by (18)--(20). The minimum order of the $1 \times m$ transfer matrix is* $n_0 = \text{Rank}(M) - m(n+1)$.

Only a sketch of the proof: From the assumption, there holds

$$\begin{bmatrix} \overline{b}_1 & \cdots & \overline{b}_m & | & -\overline{a} \end{bmatrix} \overline{M} = 0. \quad (21)$$

This is equivalent to

$$\begin{bmatrix} \overline{b}_1 & \cdots & \overline{b}_m \end{bmatrix} = \overline{a} \begin{bmatrix} P_1 & \cdots & P_m \end{bmatrix}, \overline{a} \begin{bmatrix} Q_1 & \cdots & Q_m \end{bmatrix} = 0. \text{ (22)}$$

Further, it is clear from the above development that the fact that an order-n_0 realization of $G(z)$ exists is equivalent to the existence of vectors

$$\overline{a}_k = \begin{bmatrix} \overline{a}_0 & \cdots & \overline{a}_{n-k} & \underbrace{0 \cdots 0}_{k} \end{bmatrix}, \ k = 0,...,n-n_0 \text{ (23)}$$

such that

$$\overline{a}_k \in \bigcap_{r=1}^{m} \text{Null}(Q_r), \ \ k = 0,...,n-n_0. \quad \text{(24)}$$

This in turn implies that $\text{Rank}\left(\begin{bmatrix} Q_1 & \cdots & Q_m \end{bmatrix}\right) = n_0$. ∎

Hence, the procedure for obtaining the numerator and denominator coefficients is the same as that of the SISO system, for $Q := \begin{bmatrix} Q_1 & \cdots & Q_m \end{bmatrix}$. As we do not need the right singular vectors contained in the V matrix, then a column-wise recursive procedure can be used; a possible implementation is as follows:

$$\hat{Q}_1 = Q_1; \ \hat{Q}_1 = U_1 S_1 V_1^*;$$
$$\hat{Q}_{k+1} = \begin{bmatrix} S_k(1:n+1,1:n) & U_k^* \hat{Q}_k \end{bmatrix}. \quad \text{(25)}$$

Notice that matrix $\overline{M}$ has $m(2n+1)$ columns, as opposed to $d = (m+1)n + m$ columns of matrix M in (6). It would be possible to modify this procedure so that we would deal with lower dimensional matrix. However, decoupling the computation of $\overline{a}$ and $\overline{b}_r$ would be slightly more complicated and sometimes resulting in the loss of accuracy due to the propagation of round-off errors. On the other hand, the computation of the left null space via singular value decomposition is numerically robust and the accuracy does not degrade with the number of columns.

III. TRANSFORMATIONS FOR ENHANCING NUMERICAL STABILITY

In the order reduction procedure outlined above, there are two main causes of the loss of numerical stability. First, it is the poorly conditioned Vandermonde matrix, and second, a possible collision of the plant poles with the interpolation nodes. These issues are addressed in this section.

A. Chebyshev polynomial basis

Vandermonde matrices tend to be ill-conditioned even for medium orders. Therefore, polynomial bases different from $1, z, z^2, ...$ are used. A typical choice is using Chebyshev polynomials $t_i(z)$, see [1], [4], generated recursively as

$$t_0(z) = 1, \ t_1(z) = z, \ t_n(z) = 2z \cdot t_{n-1}(z) - t_{n-2}(z) \quad \text{(26)}$$

These polynomials are orthogonal with respect to a suitably defined inner product. Further, these polynomials map the interval [-1,1] to [0,1]. The zeros of $t_k(z)$ are distributed between -1 and 1, as follows:

$$z_i = \cos\left(\frac{(2i-1)\pi}{2k}\right), \ i = 1,...,k \quad \text{(27)}$$

Let us denote $T_n(z) = \begin{bmatrix} t_o(z), t_1(z), ..., t_n(z) \end{bmatrix}^T$. Then, the $(n+1) \times (n+1)$ matrix $\begin{bmatrix} T_n(z_1), ..., T_n(z_{n+1}) \end{bmatrix}$ is nonsingular if and only if the points $z_1, ..., z_{n+1}$ are distinct. Again, we shall use the condensed notation $T_n^i = T_n(z_i)$. In this basis, the interpolation matrices in (18) become

$$M_{tr} = \begin{bmatrix} T_n^1 & \cdots & T_n^{2n+1} \\ \hline T_n^1 y_r^1 & \cdots & T_n^{2n+1} y_r^{2n+1} \end{bmatrix}$$
$$= \begin{bmatrix} I_{n+1} & 0 \\ \hline \overline{P}_{tr} & \overline{Q}_{tr} \end{bmatrix} \begin{bmatrix} T_{2n}^1 & \cdots & T_{2n}^{2n+1} \end{bmatrix}. \quad \text{(28)}$$

Moreover, if the interpolation nodes are as in (27) for $k = 2n+1$, this matrix satisfies

$$\begin{bmatrix} T_{2n}^1, ..., T_{2n}^{2n+1} \end{bmatrix}^{-1} = \begin{bmatrix} T_{2n}^1, ..., T_{2n}^{2n+1} \end{bmatrix}^T \text{diag}\left(2n, \underbrace{n, ..., n}_{2n}\right). \text{(29)}$$

Hence, if the Chebyshev polynomial basis is used, and the zeros of $(n+1)$-th order Chebyshev polynomial are used as interpolation nodes, the decomposition of M_{tr} in (28) avoids using the matrix inverse thus eliminating the major source of numerical instability.

The Chebyshev interpolation nodes are real numbers between -1 and 1; they are distributed more densely near the interval borders than in the middle of the interval. Their realness is a significant advantage, because all matrices involved are real as well. On the other hand, these fixed nodes may become close to the plant poles, making the problem ill-conditioned. A way of moving the interpolation nodes to locations where a collision with plant poles is unlikely, while preserving the advantage of the Chebyshev basis, is in the next subsection.

B. Bilinear transform of Chebyshev nodes

We shall use the bilinear transform to map the complex plane in the z-domain, to a complex plane in the p-domain and vice versa as

$$p = \left(\frac{-j}{\alpha}\right)\frac{z/r-1}{z/r+1}, \quad z = r\frac{1+j\alpha p}{1-j\alpha p} \quad \text{(30)}$$

where j denotes the imaginary unit, α and r are positive real parameters. The real interval [-1,1] in the p-domain is mapped to a circle arc in the z-domain with the center in the origin, radius r and phase bounded by $\pm \arctan\left(2\alpha/(1-\alpha^2)\right)$. Two sets of interpolation nodes, obtained by mapping Chebyshev roots for order 20 by (30), for two pairs of parameters is in Figure 1. A typical choice for the radius is the number slightly larger than the expected spectral radius of the plant. The choice of parameter α shall be discussed later. Note that (28) requires $2n+1$ interpolation nodes. In this case, one interpolation node would always be a real one equal to r, possibly in a

proximity to a plant pole at, or close to 1. To avoid this situation, we shall use an even number of interpolation points, at the cost of increasing the number of columns of the interpolation matrix by m (the number of inputs). Let

$$r_{i0} + r_{i1}z + \cdots + r_{in}z^n = (z+1)^{n-i}(z-1)^i, \quad i = 0,1,\ldots,n, \qquad (31)$$

$$\bar{r}_i = [r_{i0} \quad \cdots \quad r_{in}]; \quad R = [\bar{r}_1 \quad \cdots \quad \bar{r}_n]^T. \qquad (32)$$

Let $\bar{a}_p$ be the denominator coefficients of the transfer function computed in the p-domain. Then, the corresponding vector in the z-domain is computed as

$$\bar{a} = \bar{a}_p \cdot \mathrm{diag}\left\{(-j/\alpha)^i\right\}_{i=0}^n R \cdot \mathrm{diag}\left\{1/r^i\right\}_{i=0}^n \qquad (33)$$

The same transformation holds also for the numerator polynomials $\bar{b}_1,\ldots,\bar{b}_m$.

Using a set of complex interpolation nodes implies complexity of the interpolation matrices as well as the intermediate polynomials in the p-domain. However, the target polynomials must be real and therefore it follows from (33) that in the p-domain, the even coefficients are real and the odd ones are imaginary. The real representation of the numerator polynomial is then

$$\hat{\bar{a}}_p = \begin{bmatrix} \mathrm{Re}(a_{p0}) & \mathrm{Im}(a_{p1}) & \mathrm{Re}(a_{p2}) & \cdots \end{bmatrix}. \qquad (34)$$

The interpolation matrices are, in the Chebyshev basis and the p-domain, expressed as

$$[P_{tr} \quad Q_{tr}] = \begin{bmatrix} T_n(p_1)G_i(z_1) & \cdots & T_n(p_{2n+2})G_i(z_{2n+2}) \end{bmatrix} \times$$

$$\times \begin{bmatrix} T_{2n+1}(p_1) & \cdots & T_{2n+1}(p_{2n+2}) \end{bmatrix}^T \cdot \mathrm{diag}\left(1, \underbrace{1/2,\ldots,1/2}_{2n+1}\right)(2n+1) \qquad (35)$$

for $r = 1,\ldots,m$. These matrices can be modified for the real representation (34) of the p-domain polynomials. Let us denote them as $\hat{P}_{tr}$ and $\hat{Q}_{tr}$; their elements are computed as

$$\hat{P}_{tr}(k,s) = \begin{bmatrix} t_{k-1}(p_1)\Psi_{k1} & \cdots & t_{k-1}(p_{n+1})\Psi_{k,n+1} \end{bmatrix} \begin{bmatrix} t_{s-1}(p_1) \\ \vdots \\ t_{s-1}(p_{n+1}) \end{bmatrix} \xi_s, \quad (36)$$

$$\hat{Q}_{tr}(k,s) = \begin{bmatrix} t_{k-1}(p_1)\Phi_{k1} & \cdots & t_{k-1}(p_{n+1})\Phi_{k,n+1} \end{bmatrix} \begin{bmatrix} t_{s-1}(p_{n+2}) \\ \vdots \\ t_{s-1}(p_{2n+2}) \end{bmatrix} \xi_s \quad (37)$$

for $k,s = 1,\ldots,n+1$, where

$$\Psi_{kj} = \begin{cases} \mathrm{Re}\left(G_i(z_j)\right), & \text{for } k+j \text{ odd} \\ (-1)^k \mathrm{Im}\left(G_i(z_j)\right), & \text{otherwise,} \end{cases} \qquad (38)$$

$$\Phi_{kj} = \begin{cases} \mathrm{Re}\left(G_i(z_j)\right), & \text{for } k+(n+1-j) \text{ odd} \\ (-1)^k \mathrm{Im}\left(G_i(z_j)\right), & \text{otherwise} \end{cases} \qquad (39)$$

for $j = 1,\ldots,n+1$ and ξ_s is equal to $2n+1$ for $s=1$ and $n+1/2$ otherwise.

This concludes the description of the minimum order ARX computation. We shall make a few comments on the choice

of bilinear transform parameter α; its optimal choice [meaning the value producing resulting best accuracy of the resulting model] depends on the distribution of poles and zeros and cannot be chosen a priori. It has been observed, that higher values of α (say 1.5--3) are better for estimating the minimal order. This order is estimated using the singular values in (14) as

$$n_0 = \arg\min_i\left\{\sigma_{i+1}/\sigma_i \mid \sigma_{i+1} < \varepsilon, i = 1,\ldots,n\right\} \qquad (40)$$

Estimating the minimum order correctly is essential—if it is overestimated, the interpolation procedure may produce spurious modes which may be arbitrary, even unstable. Hence it is better to underestimate than overestimate the order, which should be taken into account in choosing the tolerance ε. On the other hand, large values of α result in interpolation sets, which are crude around the positive real axis, resulting in lower accuracy in modeling the low frequency behavior. The way around it is to estimate the minimal order n_0 using the full interpolation set, for $\alpha = 2$ and tolerance $\varepsilon = 10^{-9}$; then, compute the minimum order coefficients using the reduced interpolation set of $2n_0 + 2$ points, the bilinear transform parameter $\alpha = 1$ and tolerance $\varepsilon = 10^{-12}$ (the risk of overestimating the order is now low). This setting works well for nearly all cases of systems of minimal order up to 15, which is sufficient in most applications. It works reasonably well in the vast majority of cases with the minimum order of 25; the failures are typically due to the fact, that several plant poles are clustered near the unit circle, and near an interpolation node. Then, an interactive parameter change of the bilinear transform parameters usually helps. This situation may also indicate a fast sampling rate; its reduction results in a more precise model conversion.
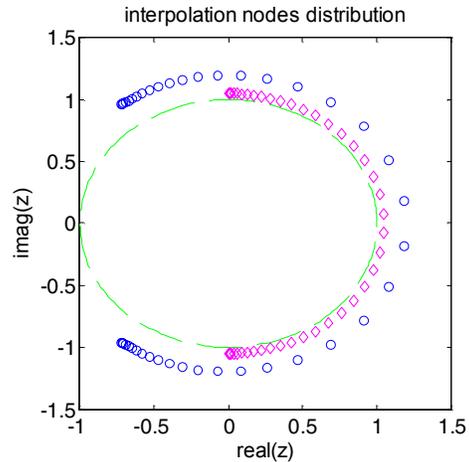
Figure 1 Mapping of Chebyshev nodes (order 20) by bilinear transform: (a) $r = 1.05$, $\alpha = 1$ (diamonds); (b), $r = 1.2$, $\alpha = 2$ (circles).

IV. EXAMPLE

As an example, we choose a transfer function of non-minimal order 10 with two inputs, given by

$$G_1(z) = \frac{10^{-3}(z-1)(z+5)(z-0.89)(z-0.1)(z-0.5\pm0.25j)}{z^2(z-1)(z-0.1)(z-0.7\pm0.5j)(z-0.89)(z-0.94)(z-0.5\pm0.05j)}$$

$$G_2(z) = \frac{-10^{-4}(z-2.3)(z-.5)(z-0.89)(z-0.1)(z-3\pm2.5j)}{z^2(z-1)(z-0.1)(z-0.7\pm0.5j)(z-0.89)(z-0.94)(z-0.5\pm0.05j)}$$

The minimum order is 8. The singular values of the Q-matrix are shown in Figure 2, where the minimum order is obvious. It can be seen in Figure 3 and Figure 4 that the match of the full/minimum order responses is perfect.
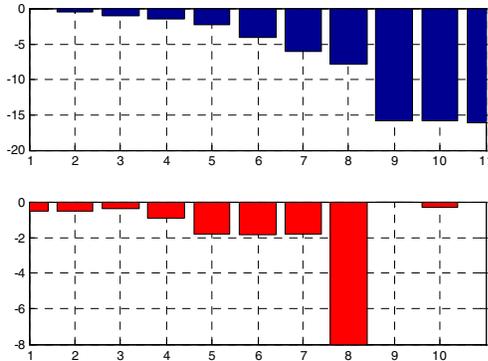
Figure 2 Singular value analysis of Q: $\log_{10}(\sigma_i)$ (top), $\log_{10}(\sigma_{i+1}/\sigma_i)$ (bottom).
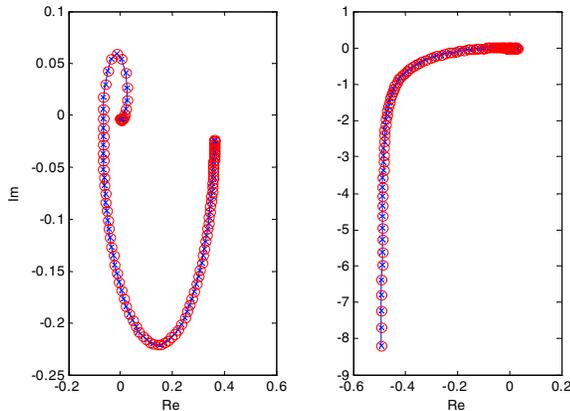
Figure 3 Nyquist plots: the original system (x), transfer matrix after cancellation (o).
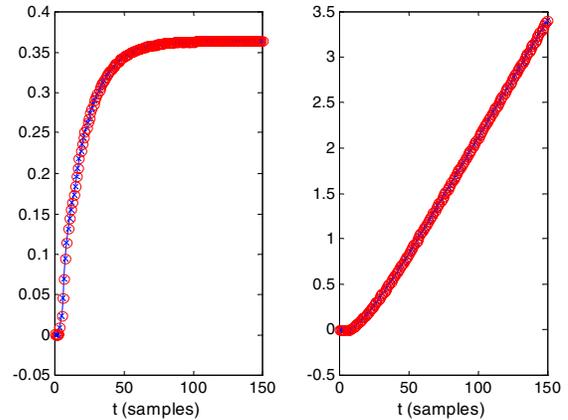
Figure 4 Step responses: the original system (x), transfer matrix after cancellation (o).

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