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Pole Placement of Time-Varying State Space Representations

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Abstract— **The pole sets of linear differential SISO systems are considered in the paper by using polynomial factorizations and state-space formalism. A state-feedback control algorithm is then constructed, which guarantees the stability of the closed-loop system. A similar methodology is used to form a stable time-varying observer to the system.**

I. INTRODUCTION

Controller design for linear time-varying differential systems is generally a difficult problem, because of the fundamental problems related to the analysis of such systems. The classical theory does not provide much help because the concepts of poles and zeros do not carry over to time-varying systems. This is basically a consequence of the fact that the solution of the system equations cannot generally be solved in closed form, i.e. the state transition matrix cannot usually be expressed in terms of elementary functions. There have been attempts to define time-varying poles, and based on them to design appropriate controller algorithms, but these methods have not received a general acceptance.

It is well-known that the eigenvalues of the system matrix calculated pointwise in time do not provide enough information regarding stability, see e.g. [1]. A considerable contribution to the theory of time-varying poles (or more specifically pole sets) was introduced in [2], where factorizations of operator polynomials were used to define the pole sets. Based on this analysis conditions for the stability of the system were obtained. A similar method by using polynomial algebra was also investigated in [3], and in the time-varying case in [4].

Recently there has been another approach to the problem, in which state-space techniques and state transformations were used to study the stability of the system [5], [6]. The well-known theory of Lyapunov transformations [7] has been used because of its stability preserving characteristics in the state transformation. In [8] it was shown that any time-varying system matrix of a continuous linear state-space representation can be changed into a constant matrix, but the needed state transformation depends on the state-transition matrix, which is generally impossible to solve analytically. Hence it is not possible to know, whether the transformation is a Lyapunov transformation or not. The topic has further been discussed in [9].

In this paper a novel concept of pole-placement design in the case of time-varying linear differential single input - single output (SISO) systems is discussed. Starting from a canonical realization of a general input-output model a time-variable state feedback controller is designed. The idea is that the closed loop equations are transformed into a form from which the pole sets can be calculated. These sets can be chosen arbitrarily by the designer, and the tuning coefficients of the controller can then be calculated. A fundamental result is that the state transformation turns out to be a Lyapunov transformation implying that stability is preserved in the design.

The paper is organized as follows. In Section II a linear time-varying input output differential system is considered by using algebraic methods, especially skew polynomial algebra. In Section III the general concept of time-varying poles is discussed, whereafter the results are linked to time-varying state transformations in Section IV. Based on the developed formalism a stabilizing state-feedback controller algorithm is presented in Section V, and an observer is designed correspondingly in Section VI. An example is presented in Section VII. Conclusions are given in Section VIII.

II. TIME-VARYING LINEAR SYSTEMS

Time-varying linear single-input-single-output (SISO) differential systems can be described by models of the form

$$\sum_{i=0}^{n} a_i(t)\frac{d^i y(t)}{dt^i} = \sum_{j=0}^{n} b_j(t)\frac{d^j u(t)}{dt^j} \tag{1}$$

or shortly by

$$a(p)y = b(p)u \tag{2}$$

where $u, y \in X$ are (real- or complex-valued) input and output signals on a time set T, p is the *differential operator* on X , and $a(p), b(p)$ are polynomials in p with coefficients from a suitable space K of (real- or complex-valued) functions on T. The existence and uniqueness of the solutions as well as the *realizability* of the models are difficult mathematical questions depending on the signal and coefficient spaces but they are not considered in this paper. Many analysis and design methods presume that the coefficients are differentiable at least once but often several times. Therefore for methodology development it is easiest to assume that the coefficients are infinitely differentiable functions [4].

The multiplication of these time-varying polynomials defined by composition of operators is not commutative because of the property

$$pa = ap + \frac{da}{dt} \tag{3}$$

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Therefore this multiplication makes the polynomials to *(left) skew polynomials*.

The set of skew polynomials is a noncommutative ring with coefficient space K as a subring. Most of the concepts and properties of ordinary polynomials can be generalized to skew polynomials. However, for stronger algebraic structures the coefficient ring K should be a field which is difficult to satisfy in the time-varying case without *extension* of coefficients and signals to corresponding *fractions* with nonzero coefficients as denominators [4].

In particular, this holds for the *division algorithms*. For instance, the *right division algorithm*

$$\begin{aligned} a(p) &= q(p)b(p) + r(p) \\ \deg(r(p)) &< \deg(b(p)) \end{aligned} \tag{4}$$

is satisfied for *all* $a(p), b(p) \neq 0$ only if the coefficient ring K is a field. This is important because the division algorithm is needed for manipulation of skew polynomial matrices used in descriptions of multivariable systems.

In multivariable (MIMO) case the *input-output description* is

$$A(p)y = B(p)u \tag{5}$$

where $A(p)$ and $B(p)$ are skew polynomial matrices with full column rank and $A(p)$ square. Obviously, the system

$$\begin{aligned} S &= \{(u,y) | A(p)y = B(p)u\} \\ &= (\ker[A(p) \mid -B(p)])^{-1} \end{aligned} \tag{6}$$

where $(\cdot)^{-1}$ means the converse operation, is uniquely determined by the *generator* $[A(p)| - B(p)]$. On the other hand, there can be infinitely many generators for the same system. Two generators determine the same system if they are *row equivalent* as polynomial matrices i.e. they can be obtained from each other by premultiplication with a *unimodular matrix* [3], [4].

The system can also be *decomposed* to a *state space representation* (or *description*).

$$S = \left\{ (u,y) \,|\, \exists x \begin{bmatrix} px = Ax + Bu \\ y = Cx + Du \end{bmatrix} \right\} \tag{7}$$

where A, B, C, D are matrices over K of time-varying coefficients. Conversely, the input-output description $[A(p)| - B(p)]$ can be obtained from the state space representation by bringing the equations in (7) to a row equivalent upper triangular form

$$\begin{bmatrix} A_1(p) & A_2(p) \\ 0 & A(p) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} B_1(p) \\ B(p) \end{bmatrix} u \tag{8}$$

by unimodular *elementary row operations*. The state space representation is *completely observable* if $A_1(p)$ is unimodular which in the so-called canonical upper triangular form means that $A_1(p) = I$, [3], [4].

III. Poles of time-varying systems

The *(output) modes* and *poles* are defined by means of solutions of first order differential equations

$$(p - \lambda)y = 0 \Leftrightarrow y(t) = y_0 e^{\int_0^t \lambda(t)dt} \tag{9}$$

where y_0 is a *constant*.

Putting the mode to the equation $a(p)y = 0$ leads to

$$a(p) y_0 e^{\int_0^{(.)} \lambda(t)dt} = a^S(\lambda) y_0 e^{\int_0^{(.)} \lambda(t)dt} = 0 \tag{10}$$

where $\lambda \mapsto a(p)^S(\lambda) \widehat{=} a^S(\lambda)$ is a *skew polynomial function* $K \to K$ associated with $a(p)$ [5]. The skew polynomial functions have the following properties

$$(a_0)^S(\lambda) = a_0 \tag{11}$$
$$(p)^S(\lambda) = \lambda \tag{12}$$
$$(a(p) + b(p))^S(\lambda) = a^S(\lambda) + b^S(\lambda) \tag{13}$$
$$(a(p)b(p))^S(\lambda) = (a(p)b^S(\lambda))^S(\lambda) \tag{14}$$

Now the poles (pole functions) can be solved from

$$a^S(\lambda) = 0 \tag{15}$$

This is a *nonlinear differential equation* of order $(n-1)$ and the existence of its solutions depends on the chosen coefficient ring and initial values.

Using the right division algorithm of skew polynomials it can be proven [5]:

Proposition 1: $a^S(\lambda) = 0$ if and only if $(p - \lambda)$ is a right factor of $a(p)$.

This means that all *modes are related to linear right factors*. For multi-variable systems described by (5) the output modes

$$y(t) = y_0 e^{\int_0^t \lambda(t)dt} \tag{16}$$

where y_0 's are *constant vectors*, satisfy the equation

$$A^S(\lambda) y_0 = 0 \tag{17}$$

where $A^S(\lambda)$ is defined in the obvious way. Now the problem is to find a λ and a $y_0 \neq 0$ such that for *all t*

$$A^S(\lambda)(t) y_0 = 0 \tag{18}$$

Note that only in few special cases (e.g. if $A(p)$ is diagonal) λ can be obtained from $\det A^S(\lambda)(t) = 0$.

The multiplication of (5) by a unimodular matrix does not change the poles, thus the modes and poles are *independent of descriptions* [5].

In terms of the state-space representations (7) the (internal) modes satisfy for *all t*

$$\begin{aligned} &\begin{bmatrix} pI - A & 0 \\ -C & I \end{bmatrix}^S (\lambda)(t) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ = &\begin{bmatrix} \lambda(t)I - A & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = 0 \end{aligned} \tag{19}$$

Thus the poles of the state space representation are such *pointwise eigenvalues* of $A(t)$ for which there exist *constant* *eigenvectors* x_0 with $C(t)x_0$ constant. Only those eigenvalues satisfying $C(t)x_0 = \text{constant} \neq 0$ are also poles of the system. Obviously, the internal poles are also poles of the state system described by

$$px = Ax + Bu \tag{20}$$

Suppose that the equations in (7) can be brought to upper triangular form

$$\begin{bmatrix} A_1(p) & A_2(p) \\ 0 & A(p) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} B_1(p) \\ B(p) \end{bmatrix} u \tag{21}$$

by elementary row operations (note that the right division algorithm is needed). Further, assume that λ is a pole of the system generated by (5) i.e. it holds that $A^S(\lambda)y_0 = 0$ with $y_0 \neq 0$. Then λ is also a pole of the state space representation only if the equation

$$A_1^S(\lambda)x_0 + A_2^S(\lambda)y_0 = 0 \tag{22}$$

has a non-zero constant solution x_0. All this means that in the time-varying case there are state-space representations the poles of which are different from the poles of the system even in the observable case, where $A_1(p)$ can be taken equal to I. The problem is to find such state-space representations, which have the same poles as the system itself. This question is discussed in the next section.

IV. Transformation of State space representations

Consider a *SISO* input-output differential system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{23}$$

where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are continuously differentiable matrix functions with suitable dimensions. The linear but possibly time-varying transformation

$$x(t) = P(t)z(t) \tag{24}$$

where $P(\cdot)$ is an invertible square matrix of the same dimension as $A(\cdot)$, is used to change the system representation (23) into the form

$$\begin{aligned} \dot{z}(t) &= E(t)z(t) + F(t)u(t) \\ y(t) &= G(t)z(t) + H(t)u(t) \end{aligned} \tag{25}$$

($z(t_0) = P^{-1}(t_0)x_0$) with

$$\begin{aligned} E(t) &= P^{-1}(t)[A(t)P(t) - \dot{P}(t)] \\ F(t) &= P^{-1}(t)B(t) \\ G(t) &= C(t)P(t) \\ H(t) &= D(t) \end{aligned} \tag{26}$$

It has been shown in [8] and [9] that the matrix $E(\cdot)$ of the target system can be chosen arbitrarily by choosing

$$P(t) = \Phi_A(t,t_0)P(t_0)\Phi_E^{-1}(t,t_0) \tag{27}$$

where $\Phi_A(\cdot,\cdot)$, $\Phi_E(\cdot,\cdot)$ are the state transition matrices related to $A(\cdot)$ and $E(\cdot)$, respectively.

To investigate the preservation of stability, the important concept of a *Lyapunov transformation* can be used. Results related to this theory can be found in the literature, see e.g. [7], [10], [1]. A definition used in [1] is: An $n \times n$ matrix $P(t)$ that is continuously differentiable and invertible at each t is called a Lyapunov transformation if there exist finite positive constants ρ and η such that for all t

$$\|P(t)\| \leq \rho, \quad |\det P(t)| \geq \eta \tag{28}$$

which is equivalent to finding a finite positive constant ρ such that

$$\|P(t)\| \leq \rho, \quad \left\|P^{-1}(t)\right\| \leq \rho \tag{29}$$

If a system matrix is changed into another one by a Lyapunov transformation, the stability of the original and target representations remain invariant. The key issue is then to determine, whether the matrix (27) is a Lyapunov-transformation matrix or not. As long as the transition matrices $\Phi_A(\cdot,\cdot)$ and $\Phi_E(\cdot,\cdot)$ are not known, there seems to be no general procedure to determine this.

The interesting point to notice is that a triangularization procedure exists, which transforms the system matrix into an upper triangular form, see [11], [6]. For the transformation matrix it holds that $\det P(t) = 1$, which implies that $P(\cdot)$ is a Lyapunov transformation, if its elements are bounded.

To apply, consider again the SISO input-output differential system described by (1). The system has a state space representation (23) with

$$\begin{gathered} A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0(t) & -a_1(t) & \cdots & \cdots & -a_{n-1}(t) \end{bmatrix} \\ B(t) = \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix} \\ C(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \quad D(t) = \gamma_0(t) \end{gathered} \tag{30}$$

and

$$\begin{cases} \gamma_0(t) = b_n(t), \\ \gamma_i(t) = b_{n-i}(t) \\ -\sum\limits_{k=0}^{i-1}\sum\limits_{j=0}^{i-k} \frac{(n+j-i)!}{j!(n-i)!} a_{n-i+k+j}(t) \frac{d^j\gamma_k(t)}{dt^j} \end{cases} \tag{31}$$

($i = 1,2,\cdots,n$), [12].

This representation is observable and has a constant output matrix $C(t)$ but, in general, it does not satisfy the property (c.f. the previous section)

$$x_0 = -A_2^S(\lambda)y_0 = \begin{bmatrix} 1 \\ p \\ \vdots \\ p^{n-1} \end{bmatrix}^S (\lambda)y_0 = \text{const} \tag{32}$$

so that its poles are not guaranteed to be equal to the poles of the system itself.

Suppose next that $a(p)$ can be factored to

$$a(p) = (p - p_1)(p - p_2)\cdots(p - p_n) \tag{33}$$

where p_n is a pole of the system. Then there exists a transformation (24) such that the system representation (23) is changed into the form (25) where

$$\begin{aligned} E(t) &= P^{-1}(t)[A(t)P(t) - \dot{P}(t)] \\ &= \begin{bmatrix} p_n(t) & 1 & 0 & \cdots & 0 \\ 0 & p_{n-1}(t) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & p_1(t) \end{bmatrix} \\ F(t) &= P^{-1}(t)B(t) \\ G(t) &= C(t)P(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \\ H(t) &= D(t) \end{aligned} \tag{34}$$

$P(t)$ can be chosen as

$$P(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ p_n(t) & 1 & 0 & \cdots & 0 & 0 \\ x_{31}(t) & p_{n-1}(t)+p_n(t) & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & 1 & \vdots \\ x_{n1}(t) & x_{n2}(t) & x_{n3}(t) & \cdots & p_2(t)+p_3(t)+\cdots+p_n(t) & 1 \end{bmatrix} \tag{35}$$

where the entries $x_{ij}(t)$ represent appropriate expressions containing the $p_i(t)$'s and their derivatives.

The resulting state space representation is observable and has a constant output matrix $C(t)$. It further satisfies

$$\begin{aligned} z_0 &= -\tilde{A}_2^S(p_n)y_0 \\ &= \begin{bmatrix} 1 \\ p-p_n \\ (p-p_{n-1})(p-p_n) \\ \vdots \\ (p-p_1)(p-p_{n2})\cdots(p-p_n) \end{bmatrix}^S (p_n)y_0 \\ &= \text{constant} \end{aligned} \tag{36}$$

Therefore this state space representation has exactly the same poles than the system itself.

Consider again the SISO system described by (2) with left coprime $a(p)$ and $b(p)$. Its transfer matrix can be factored into *right factorization*

$$a(p)^{-1}b(p) = d(p)c(p)^{-1} \tag{37}$$

using elementary column operations to the generator $[a(p)|-b(p)]$ for construction of a greatest common left divisor of $a(p)$ and $b(p)$. This gives

$$[a(p)|-b(p)]\underbrace{\begin{bmatrix} q_1(p) & q_2(p) \\ q_3(p) & q_4(p) \end{bmatrix}}_{Q(p)} = [1\,|\,0] \tag{38}$$

so that

$$a(p)q_2(p) - b(p)q_4(p) = 0 \tag{39}$$

resulting in $q_2(p) = d(p)$ and $q_4(p) = c(p)$.

The right factorization corresponds to the series composition of the system described by $c(p)z = u$ followed by the system described by $y = d(p)z$.

The system described by

$$(p^n + c_{n-1}p^{n-1} + \cdots + c_0)z = u \tag{40}$$

has a state space representation (cf. 30)

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c u \\ z &= C_{zc}x_c + D_{zc}u \end{aligned} \tag{41}$$

with

$$\begin{aligned} A_c &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & \cdots & \cdots & -c_{n-1} \end{bmatrix} \\ B_c &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ C_{zc} &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \quad D_{zc} = 0 \end{aligned} \tag{42}$$

(time variable t dropped out for brevity). When this is composed with the system described by

$$y = (d_np^n + d_{n-1}p^{n-1} + \cdots + d_0)z \tag{43}$$

the output equation is changed to the form

$$y = C_cx_c + D_cu \tag{44}$$

with

$$\begin{aligned} C_c &= \begin{bmatrix} d_0 - d_nc_0 & d_1 - d_nc_1 & \cdots & d_{n-1} - d_nc_{n-1} \end{bmatrix} \\ D_c &= d_n \end{aligned}$$

This kind of representation is said to be of *controllability canonical form.*

Consider again a SISO system described by

$$\begin{aligned} &(p^n + a_{n-1}p^{n-1} + \cdots + a_0)y \\ &= (b_np^n + b_{n-1}p^{n-1} + \cdots + b_0)u \end{aligned} \tag{45}$$

Write here the left skew polynomials $a(p)$ and $b(p)$ as *right skew polynomials*

$$\begin{aligned} &(p^n + p^{n-1}\tilde{a}_{n-1} + \cdots + \tilde{a}_0)y \\ &= (p^n\tilde{b}_n + p^{n-1}\tilde{b}_{n-1} + \cdots + \tilde{b}_0)u \end{aligned} \tag{46}$$

and regroup the terms

$$\begin{aligned} &p(p(p(\cdots p(p(y - \tilde{b}_nu) + \tilde{a}_{n-1}y \\ &-\tilde{b}_{n-1}u)\cdots) + \tilde{a}_2y - \tilde{b}_2u) + \tilde{a}_1y - \tilde{b}_1u) \\ &= \tilde{b}_0u - \tilde{a}_0y \end{aligned} \tag{47}$$

Choosing the expressions in parentheses on the left hand side as state variables

$$x_o = \begin{bmatrix} y - \tilde{b}_nu \\ py - p\tilde{b}_n + \tilde{a}_{n-1}y - \tilde{b}_{n-1}u \\ \vdots \\ p^{n-1}y - p^{n-1}\tilde{b}_n + \cdots + \tilde{a}_1y - \tilde{b}_1u \end{bmatrix} \tag{48}$$

leads to a state space representation

$$\begin{aligned}\dot{x}_o &= A_o x_o + B_o u \\ y &= C_o x_o + D_o u\end{aligned} \tag{49}$$

with

$$A_o = \begin{bmatrix} -\tilde{a}_{n-1} & 1 & 0 & \cdots & 0 \\ -\tilde{a}_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\tilde{a}_1 & 0 & 0 & \cdots & 1 \\ -\tilde{a}_0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$
$$B_o = \begin{bmatrix} \tilde{b}_{n-1} - \tilde{a}_{n-1}\tilde{b}_n \\ \tilde{b}_{n-2} - \tilde{a}_{n-2}\tilde{b}_n \\ \vdots \\ \tilde{b}_0 - \tilde{a}_0\tilde{b}_n \end{bmatrix} \tag{50}$$
$$C_o = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \quad D_o = \tilde{b}_n$$

This is said to be of *observability canonical form*. The output matrix C_o is constant but the equation $x_{o0} + A^S_{o2}(\lambda)y_0 = 0$ has not in general a constant solution x_{o0}, so that the poles are not the same than the system has.

However, the observability canonical form can be brought to the form (25) directly using the same kind of lower triangular transformation matrix P_o than (35) above, with entries containing p_i's in factorization (33) and their derivatives.

V. STATE FEEDBACK

Suppose that the system under consideration is described by a controllability canonical state space representation (41), (44)

$$\begin{aligned}\dot{x}_c &= A_c x_c + B_c u \\ y &= C_c x_c + D_c u\end{aligned} \tag{51}$$

The problem is to construct a state feedback law

$$u = -Kx + r \tag{52}$$

where $K = [k_1\, k_2 \cdots k_n]$ is a time-varying feedback matrix and r is a reference signal. The closed loop system has then the state space representation

$$\begin{aligned}\dot{x}_c &= (A_c - B_c K)x_c + B_c r \\ y &= (C_c - D_c K)x_c + D_c r\end{aligned} \tag{53}$$

of controllability canonical form with

$$\begin{aligned}&A_c - B_c K \\ &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 - k_1 & -c_1 - k_2 & \cdots & \cdots & -c_{n-1} - k_n \end{bmatrix}\end{aligned} \tag{54}$$

Suppose then that the desired closed loop dynamics is given by a state space representation of the form (25) which contains the desired poles p_n of the closed loop system. The transformation P (35) can be used to bring this to the same form than (53), (54)

$$\begin{aligned}\dot{x}_c &= A_{cl} x_c + B_{cl} u \\ y &= C_{cl} x_c + D_{cl} u\end{aligned} \tag{55}$$

with

$$A_{cl} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{cl,0} & -a_{cl,1} & \cdots & \cdots & -a_{cl,n-1} \end{bmatrix} \tag{56}$$

Comparing the matrices $A_c - B_c K$ and A_{cl} gives the equations for solving

$$k_i = a_{cl,i-1} - c_{i-1}, \quad i = 1, \ldots, n \tag{57}$$

VI. OBSERVER

For construction of a state observer, suppose that the system under consideration is described by an observability canonical state space representation (49)

$$\begin{aligned}\dot{x}_o &= A_o x_o + B_o u \\ y &= C_o x_o + D_o u\end{aligned} \tag{58}$$

The problem is to construct a full state observer

$$\begin{aligned}\dot{\hat{x}}_o &= A_o \hat{x}_o + B_o u \\ &+ L(y - C_o \hat{x}_o - D_o u)\end{aligned} \tag{59}$$

where $L = [l_1\, l_2 \cdots l_n]^T$ is a time-varying weighting matrix. The behavior of the errors $\tilde{x} = x - \hat{x}$ satisfies the equation

$$\dot{\tilde{x}}_o = (A_o - LC_o)\tilde{x}_o \tag{60}$$

with

$$\begin{aligned}&A_o - LC_o \\ &= \begin{bmatrix} -\tilde{a}_{n-1} - l_1 & 1 & 0 & \cdots & 0 \\ -\tilde{a}_{n-2} - l_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\tilde{a}_1 - l_{n-1} & 0 & 0 & \cdots & 1 \\ -\tilde{a}_0 - l_n & 0 & \cdots & \cdots & 0 \end{bmatrix}\end{aligned} \tag{61}$$

Suppose then that the desired error dynamics is given by a state space representation of the form (25) which contains the desired poles p_n of the dynamics. A transformation P_o similar to P (35) can be used to bring this to the same form than (61)

$$\dot{\tilde{x}}_o = A_e \tilde{x}_o \tag{62}$$

with

$$A_e = \begin{bmatrix} -\tilde{a}_{e,n-1} & 1 & 0 & \cdots & 0 \\ -\tilde{a}_{e,n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\tilde{a}_{e,1} & 0 & 0 & \cdots & 1 \\ -\tilde{a}_{e,0} & 0 & \cdots & \cdots & 0 \end{bmatrix} \tag{63}$$

Comparing the matrices $A_o - LC_o$ and A_e gives the equations for solving

$$l_i = \tilde{a}_{e,n-i} - \tilde{a}_{n-i}, \quad i = 1, \ldots, n \tag{64}$$

VII. EXAMPLE

Consider the two-dimensional system

$$\ddot{y}(t)+a_1(t)\dot{y}(t)+a_0(t)y(t)=b(t)u(t) \tag{65}$$

which has a realization

$$\begin{aligned} A(t)&=\begin{bmatrix} 0 & 1 \\ -a_0(t) & -a_1(t) \end{bmatrix}, \quad B(t)=\begin{bmatrix} 0 \\ b(t) \end{bmatrix} \\ C(t)&=\begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D(t)=0 \end{aligned} \tag{66}$$

The transformation matrix

$$P(t)=\begin{bmatrix} 1 & 0 \\ p_2(t) & 1 \end{bmatrix} \tag{67}$$

gives the input-output representation

$$\begin{aligned} E(t)&=\begin{bmatrix} \bar{p}_2(t) & 1 \\ 0 & \bar{p}_1(t) \end{bmatrix}, \quad F(t)=\begin{bmatrix} 0 \\ b(t) \end{bmatrix} \\ G(t)&=\begin{bmatrix} 1 & 0 \end{bmatrix}, \quad H(t)=0 \end{aligned} \tag{68}$$

where

$$\begin{aligned} \bar{p}_2(t)&=p_2(t) \\ \bar{p}_1(t)&=-p_2(t)-a_1(t) \end{aligned} \tag{69}$$

and

$$-p_2^2(t)-a_0(t)-a_1(t)p_2(t)-\dot{p}_2(t)=0 \tag{70}$$

using an arbitrary initial condition. The system structure corresponds to the polynomial factorization

$$\begin{aligned} & p^2+a_1(t)p+a_0(t) \\ &=(p-\bar{p}_1(t))(p-\bar{p}_2(t)) \end{aligned} \tag{71}$$

where $\bar{p}_2(t)$ is the *pole* of the system. If $p_2(t)$ is a bounded function, $P(t)$ is a Lyapunov transformation.

The controllability and observability canonical forms become

$$\begin{aligned} \dot{x}_c&=\begin{bmatrix} 0 & 1 \\ -a_0-a_1\frac{\dot{b}_0}{b_0}-\frac{\ddot{b}_0}{b_0} & -a_1-2\frac{\dot{b}_0}{b_0} \end{bmatrix}x_c+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u \\ y&=\begin{bmatrix} b_0 & 0 \end{bmatrix}x_c \end{aligned} \tag{72}$$

$$\begin{aligned} \dot{x}_o&=\begin{bmatrix} -a_1 & 1 \\ -a_0+\dot{a}_1 & 0 \end{bmatrix}x_o+\begin{bmatrix} 0 \\ b_0 \end{bmatrix}u \\ y&=\begin{bmatrix} 1 & 0 \end{bmatrix}x_o \end{aligned} \tag{73}$$

Using the state feedback the closed-loop system is

$$\begin{aligned} \dot{x}_c&=\begin{bmatrix} 0 & 1 \\ -a_0-a_1\frac{\dot{b}_0}{b_0}-\frac{\ddot{b}_0}{b_0}-k_1 & -a_1-2\frac{\dot{b}_0}{b_0}-k_2 \end{bmatrix}x_c \\ &\quad+\begin{bmatrix} 0 \\ 1 \end{bmatrix}r \\ y&=\begin{bmatrix} b_0 & 0 \end{bmatrix}x_c \end{aligned} \tag{74}$$

Given the desired closed loop pole sets p_1, p_2 it follows

$$\begin{aligned} a_{cl,1}&=-(p_1+p_2) \\ a_{cl,0}&=p_1p_2-\dot{p}_2 \end{aligned} \tag{75}$$

Finally, the controller parameters can be calculated, and the result is

$$\begin{aligned} k_1&=-a_0-a_1\frac{\dot{b}_0}{b_0}-\frac{\ddot{b}_0}{b_0}-p_1p_2+\dot{p}_2 \\ k_2&=-a_1-2\frac{\dot{b}_0}{b_0}+p_1+p_2 \end{aligned} \tag{76}$$

VIII. CONCLUSIONS

The pole sets of linear time-varying differential systems have been defined by using polynomial factorizations and state-space realizations. The concept of Lyapunov transformation has been utilized to show that the poles determine the stability of the system. Time-variable canonical forms are used to design state-feedback control algorithms, which can be proved to give a stable closed-loop system. Time-variable observer has also been designed in an analogous manner.

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