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Collective Motion of Ring-Coupled Planar Particles

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***Abstract*— We study stabilization of collective motion of N constant-speed, planar particles with less than all-to-all coupling. Our interest is in circular motions of the particles around the fixed center of mass of the group, as has been studied previously with all-to-all coupling. We focus on coupling defined by a ring, i.e., each particle communicates with exactly two other particles. The Kuramoto model of coupled oscillators, restricted to "ring" coupling, serves as our model for controlling the relative headings of the particles. Each phase oscillator represents the heading of a particle. We prove convergence to a set of solutions that correspond to symmetric patterns of the phases about the unit circle. The exponentially stable patterns are generalized regular polygons, determined by the sign of the coupling gain K.**

I. Introduction

The distributed control of collective motion of particles is motivated both by innovative engineering applications and by problems in modelling complex biological dynamics. A distributed control system can be much more robust to individual agent malfunctions than a centralized control system. In an engineering application that involves interactions among many agents, such as adaptive ocean sampling with autonomous underwater vehicles [1] or formation flying in unmanned aerial vehicles [2], it is desirable for the rest of the system to continue operation even if a few agents get lost or are damaged. If the entire controller resides on a single agent or on a fixed outpost, the malfunction of that agent or outpost renders the entire system useless.

Models of distributed control systems are also useful in modelling biological systems such as the dynamics of animal groups. Many animals that display coordinated grouping behaviors, such as swarming honeybees, schooling fish, or flocking birds, do not have a well-defined leader. Models that take into account the numerous interactions between individual animals have been shown to display some of the same behaviors exhibited in nature [3].

However, in both engineering applications and in biological models, the number of interactions required for every agent to communicate with every other agent gets very large as the number of agents increases. At these large numbers, scaling of performance can become a problem and it may be impractical to expect every agent to interact with every other agent. For example, in engineering applications, communication constraints may limit the number of communication links and in case of large numbers of agents, complex electronics may be necessary to avoid communication interference. In biological models, it may be quite unreasonable to expect that a single animal such as a fish can effectively process the relative positions of hundreds of its neighbors.

In this paper, we consider a kinematic model of N particles that move at constant speed in the plane. The control is distributed in that each particle can control its own heading using feedback of its own sensor measurements. This model is used in [2], [4], [5] to study collective motion of planar particles, assuming all-to-all coupling among the particles, i.e., assuming that each particle can sense the relative heading and relative position of every other particle in the group.

In [2], [4], [5], control laws are proven to stabilize parallel motions of the particles and circular motions of the particles about the fixed center of mass of the group. In [4], [5], an analogy with phase models of coupled oscillators is exploited to prove stabilization: each oscillator phase represents the heading of one of the particles. The control law consists of two terms, one that controls relative headings and one that controls relative spacing of particles. The model with only the relative heading control term is the Kuramoto model of coupled oscillators with identical natural frequencies [6]. The relative spacing control term depends both on relative headings and relative positions.

In [7], the analogy with oscillator models is exploited further to stabilize the *splay state formation*, a highly symmetric formation defined by circular motion of all particles on a circle of prescribed radius with all particles evenly spaced on the circle. All-to-all coupling is assumed and higher harmonics of the phase differences in the coupling function are used. This work is generalized in [8] where, for example, stabilizing control laws are proven for all symmetric, circular patterns.

In this paper, motivated by communication constraints described above, we study stabilization of circular motions of particles in symmetric patterns (such as the splay state formation) with less than all-to-all coupling and without higher harmonics in the coupling function. In particular, we consider the coupling (communication) topology defined by a *ring* such that each particle is coupled to exactly two other particles (see Fig. 1). As in previous work, the control law

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is the sum of an alignment term to control relative headings and a spacing term to control relative positions. This model is described in Section II.

In Section III, we discuss how the model with only alignment control looks like the Kuramoto model of coupled oscillators restricted to ring coupling. In Section IV, we prove that all solutions converge to a set of symmetric patterns of the phases about the unit circle. These patterns are described by generalized regular polygons (see also [9]) as presented in Section V. The sign of the coupling strength parameter K determines which of the symmetric patterns are exponentially stable. In Section VI, we add a relative spacing control term and demonstrate with a simulation the particles converging to a circular motion in a symmetric pattern about the fixed group center of mass.

II. Particle Model with Ring Coupling

We investigate the dynamics of a system of N particles that move in the plane at constant (unit) speed governed by the following system of equations [2]:

$$\begin{aligned} \dot{r}_k &= e^{i\theta_k} &(1)\\ \dot{\theta}_k &= u_k, &(2)\end{aligned}$$

where $k = 1,\ldots,N$. Here $r_k = x_k + iy_k \in \mathbb{C} \approx \mathbb{R}^2$ is the position of the k^{th} particle, $\theta_k \in S^1$ is the heading of the k^{th} particle, and u_k is the steering control input to the k^{th} particle. The (unit) velocity of the k^{th} particle is given by $e^{i\theta_k} = \cos\theta_k + i\sin\theta_k$.

Define relative position variables $r_{kj} = r_k - r_j$ and relative heading variables $\theta_{kj} = \theta_k - \theta_j$. Then, as in [4], [5], [7], [8] we consider a control u_k that is the sum of two terms, the first an alignment term that depends only on relative headings and the second a spacing term that depends only on relative positions and relative headings. In the previous works, all-to-all coupling is assumed and the control takes the form:

$$\begin{aligned} u_k &= u_k^{\text{align}}(\theta_{kj}) + u_k^{\text{spac}}(r_{kj},\theta_{kj})\\ &= \frac{K}{N}\sum_{j=1}^{N}\sin\theta_{kj} + u_k^{\text{spac}}(r_{kj},\theta_{kj}) \qquad (3)\end{aligned}$$

where K is a scalar coupling strength parameter. Note that in the case without the spacing term, the controlled heading dynamics become, for $k = 1,\ldots,N$,

$$\dot{\theta}_k = \frac{K}{N}\sum_{j=1}^{N}\sin\theta_{kj}$$

which is the Kuramoto model for coupled phase oscillators with identical (trivial) natural frequencies [6].

For $K < 0$, with an appropriate spacing term, stabilization was proved for parallel motion of the particles with some average inter-vehicle relative spacing. For $K > 0$, circular motion of the particles was proved with prescribed radius about the fixed center of mass of the particle group. The spacing term used in [7], [8] is

$$u_k^{\text{spac}} = -\omega_0(1 + \kappa < \tilde{r}_k, \dot{r}_k >) \qquad (4)$$

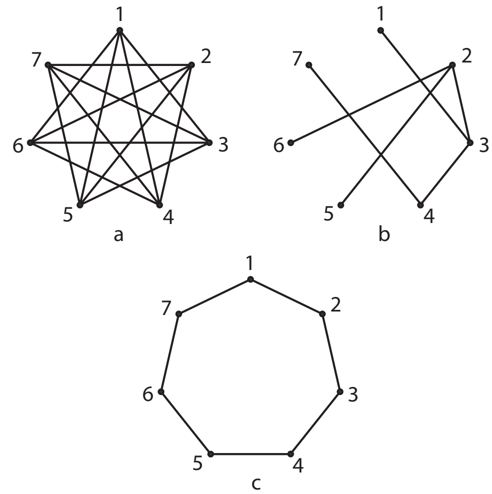

Fig. 1. Examples of communication topologies for $N = 7$. Each black dot is a particle and each line-segment is a bidirectional communication link. a: all-to-all. b: connected. c: ring.

where $\kappa > 0$ is a gain, $\omega_0 = 1/\rho_0$ and ρ_0 is the constant desired radius of the circle of rotation. The vector $\tilde{r}_k = r_k - R$ is the position of the k^{th} particle relative to the position of the center of mass of the group $R = \frac{1}{N}\sum_{j=1}^{N} r_j$. The inner product is $< z_1, z_2 >= \text{Re}\{z_1\bar{z}_2\}$, $z_1, z_2 \in \mathbb{C}$.

The average linear momentum of the group, defined as

$$p_\theta = \frac{1}{N}\sum_{k=1}^{N}\dot{r}_k = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k},$$

plays a key role. Parallel motion is characterized by $|p_\theta| = 1$, its maximal value, and circular motion is characterized by $|p_\theta| = 0$, its minimal value. It was observed in [4], [5] that p_θ is equal to the complex order parameter for the phase model, defined as the centroid of the phases. This can be interpreted as parallel motion, or maximal group linear momentum, corresponding to synchronization of the phases (headings) and circular motion, or zero linear momentum, corresponding to a distribution of phases (headings) around the unit circle with zero centroid.

However, the control law (3) requires that every particle can communicate with every other particle. In this paper we consider a control law with coupling (or communication) topology that is not all-to-all but is still *connected*. By connected, it is meant that every particle is linked, via at least one path of arbitrary length, to every other particle. The connected topology that we consider is a *ring* coupling topology defined such that each particle communicates (in both directions) with exactly two other particles. In particular, the k^{th} particle communicates only with the $(k-1)^{th}$ particle and the $(k+1)^{th}$ particle, for $k = 1,\ldots,N$. All indices are taken modulo N, i.e., particle $N+1$ is identified with particle 1 and particle 0 is identified with particle N. See Figure 1.

III. Ring-Coupled Alignment Control Law

Using notation from algebraic graph theory, the communication topology for a network can be represented as a matrix.

The incidence matrix B of a graph with N nodes and e edges is defined in [10] as the $N \times e$ matrix where $B_{ij} = -1$ if edge j leaves node i and $B_{ij} = 1$ if edge j enters node i. $B_{ij} = 0$ if edge j does not connect to node j. Note that the incidence matrix defines the orientation of each edge.

The incidence matrix B for the ring topology defined in Section II is a square matrix since $N = e$. We arbitrarily choose an orientation for each edge, although the edges are really undirected, and compute

$$B = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & & 0 & 0 \\ \vdots & & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{5}$$

B has rank $N-1$ and the one-dimensional null space is spanned by the vector $\mathbf{1}_e^T = (1, 1, \ldots, 1) \in R^N$.

The alignment control law that we propose is the alignment control in (3) restricted to the ring coupling: any term $\sin\theta_{kj}$ in (3) is set to zero if there is no edge between k and j. This restricted alignment control derives from the potential defined in [11] (see also [12]) and can be written as

$$u^{\text{align}} = \frac{K}{N} B \sin(B^T\theta), \tag{6}$$

where $u^{\text{align}} = (u_1^{\text{align}}, \ldots, u_N^{\text{align}})$ and $\theta = (\theta_1, \ldots, \theta_N)$. Define the N-dimensional vector $\phi = B^T\theta$. $\phi = (\phi_1, \ldots, \phi_N)$ is the vector of relative headings between pairs of particles that communicate. The term $\sin(B^T\theta)$ denotes the N-dimensional vector with i^{th} element equal to $\sin\phi_i$. Although B signifies a directed graph, the presence of both B and B^T in Eq. (6) results in an undirected structure so that

$$\dot{\theta}_k = \frac{K}{N}\left[\sin(\theta_k - \theta_{k+1}) + \sin(\theta_k - \theta_{k-1})\right], \tag{7}$$

for all $k = 1, \ldots, N$, as desired. Before including the spacing control term, the controlled system becomes

$$\dot{\theta} = \frac{K}{N} B \sin(B^T\theta). \tag{8}$$

This is the Kuramoto model of coupled phase oscillators (with identical, trivial natural frequencies) restricted to the ring topology.

IV. Stability Analysis

In this section we study stability of the equilibrium solutions of the system with ring coupling and alignment control only as defined by (5) and (8). The configuration space of the system (8) is the N torus, T^N. We first compute the equilibrium solutions of the system (8). These correspond to values of $\theta \in T^N$ such that $\sin\phi$ is in the null space of B. That is, θ is a fixed point of (8) if and only if

$$\sin\phi = \alpha\mathbf{1}_e, \tag{9}$$

$$\mathbf{1}_e^T\phi = 2\pi n, \tag{10}$$

both hold where $\phi = B^T\theta$, $\alpha \in [-1, 1]$ and n is any integer. Eq. (10) ensures that any solution is compatible with ring coupling. Given any solution $\theta \in T^N$ and constant angle $\beta \in S^1$, it holds that $\theta + \beta\mathbf{1}_e \in T^N$ is also a solution of (8), i.e., there is an S^1 set of solutions corresponding to θ. Equivalently, because the control law depends only on relative headings, the dynamics are invariant to a rotation of the group of particles as a whole, i.e., there is an S^1 symmetry. The direction of this symmetry is given by the vector $\mathbf{1}_e$.

Now define an arbitrary angle $\phi_0 \in S^1$. Then, ϕ satisfies (9) if and only if

$$\phi_i \in \{\phi_0, \pi - \phi_0\}, \quad i = 1, \ldots, N. \tag{11}$$

Let $M \le N$ be the number of components of ϕ which are equal to ϕ_0. Then, an equilibrium of this form satisfies (10) if and only if

$$M\phi_0 + (N-M)(\pi - \phi_0) = 2n\pi \tag{12}$$

for some integer n. If $M \neq N/2$, then (12) is equivalent to

$$\phi_0 = \left(\frac{2n + M - N}{2M - N}\right)\pi \tag{13}$$

for some integer n. For example, if $N = M$ then ϕ_0 must satisfy

$$\phi_0 = n\frac{2\pi}{N} \tag{14}$$

for some integer n. If we identify ϕ_0 with $\phi_0 + 2m\pi$ where m is any integer, then the condition (13) yields a finite number of sets of equilibria, parametrized by the finite number of solutions ϕ_0 of (13) for each N and $M \neq N/2$.

In the case $M = N/2$ (which requires that N be even), then condition (12) becomes

$$\frac{N}{2}\pi = 2n\pi \tag{15}$$

for some integer n, which is equivalent to $N = 4n$. This means that there are equilibria for $M = N/2$ only in the case that N is an integer multiple of 4. We note that in this case there is a set of equilibria corresponding to every $\phi_0 \in S^1$.

Next we prove that all solutions of (8) converge to the set of equilibria defined above. In other work, where all-to-all coupling is assumed, we use $U_{\text{all-to-all}}(\theta) = N/2|p_\theta|^2$ in the Lyapunov function (see, for example, [7]). Note that

$$-K\frac{\partial U_{\text{all-to-all}}(\theta)}{\partial\theta_k} = \frac{K}{N}\sum_{j=1}^{N}\sin\theta_{kj},$$

which is the alignment control law (3) in the case of all-to-all coupling.

Here, with ring coupling, we use a generalization of the potential $U_{\text{all-to-all}}(\theta)$ for limited communication defined formally in [11] (see also the Lyapunov function used in [12]). We define

$$U(\theta) = \frac{K}{N}\mathbf{1}_e^T\cos B^T\theta = \frac{K}{N}\sum_{i=1}^{e}\cos\phi_i\,. \tag{16}$$

Then, the control (6) is a gradient control and the controlled ring-coupled system (8) is a gradient system since

$$u^{\text{align}} = -\frac{\partial U(\theta)}{\partial\theta} = \frac{K}{N}B\sin(B^T\theta). \tag{17}$$

Theorem 1: Consider the system (8) with $\theta \in T^N$ and ring coupling defined by the incidence matrix B of (5). All solutions θ in T^N converge to the set E of equilibria defined by (9)-(10). These fixed points are given by (11). Let M be the number of components of $\phi = B^T\theta$ which are equal to $\phi_0 \in S^1$. Then if $M \neq N/2$, ϕ_0 must satisfy (13) for some integer n. In the case that $M = N/2$ then N must be an integer multiple of 4 in which case $\phi_0 \in S^1$ in (11) can be arbitrary.

Proof: Consider the function $U(\theta)$ defined by (16). Then,

$$\dot{U}(\theta) = -\left\| \frac{K}{N} B \sin(B^T\theta) \right\|^2 \qquad (18)$$

For $K \neq 0$ this yields $\dot{U}(\theta) = -\frac{K^2}{N^2}\dot{\theta}^T\dot{\theta} \leq 0$. Since T^N is compact, by LaSalle's Invariance Principle all solutions converge to the largest invariant set for which $\dot{\theta} = 0$, i.e., the set of equilibria E. ∎

We next prove which equilibria in the set E are exponentially stable and which are unstable in the case $K > 0$. We do the same for the case $K < 0$.

Theorem 2: Consider the equilibria in the set E defined in Theorem 1. If $K \neq 0$, then any equilibrium with $M \neq N$ is unstable. Now suppose $M = N$, i.e., consider equilibria of the form $\phi_i = \phi_0$, $i = 1,\ldots,N$, where $\phi_0 = \bar{\phi}_0 + 2n\pi$, $\bar{\phi}_0 \in [-\pi/2, 3\pi/2)$, n some integer and $\bar{\phi}_0$ satisfies (14). Then, for $K > 0$, the corresponding S^1 set of equilibria is exponentially stable if $\bar{\phi}_0 \in (\pi/2, 3\pi/2)$ and unstable if $\bar{\phi}_0 \in (-\pi/2, \pi/2)$. For $K < 0$, the corresponding S^1 set of equilibria is exponentially stable if $\bar{\phi}_0 \in (-\pi/2, \pi/2)$ and unstable if $\bar{\phi}_0 \in (\pi/2, 3\pi/2)$. If $K \neq 0$ and $\phi_0 = \pi/2 + m\pi$ for some integer m, then all eigenvalues of the linearization are zero.

Proof: First we find the Jacobian J for the system (8). Using $\phi = B^T\theta$, we compute

$$J = \frac{K}{N} B \mathrm{diag}(\cos\phi) B^T. \qquad (19)$$

The matrix BB^T is called the Laplacian for the graph defined by the incidence matrix B [10]. It is a symmetric, positive semi-definite, $N \times N$ matrix with rank $N-1$. Thus, all its eigenvalues are real and positive except for one zero eigenvalue. The eigenvector corresponding to the zero eigenvalue is $\mathbf{1}_e$. Thus, the zero eigenvalue corresponds to the rotational symmetry, i.e., for every equilibrium value of ϕ there is a circle of equilibria corresponding to the arbitrariness in the choice of $\theta_1 \in S^1$.

Consider the case that $M = N$ and $\phi_i = \phi_0$, $i = 1,\ldots,N$, where $\phi_0 = \bar{\phi}_0 + 2n\pi$, $\bar{\phi}_0 \in [-\pi/2, 3\pi/2)$, n some integer and $\bar{\phi}_0$ satisfies (14). Since $\mathrm{diag}(\cos\phi) = I_N \cos\phi_0$, where I_N is the $N \times N$ identity matrix, the Jacobian is

$$J = \frac{K}{N} \cos\phi_0 BB^T.$$

Suppose that $\bar{\phi}_0 \in (\pi/2, 3\pi/2)$. Then $\cos\phi_0 < 0$. If $K > 0$ then all eigenvalues of J are negative, except for the zero eigenvalue corresponding to the symmetry and so the S^1 set of equilibria is exponentially stable. If $K < 0$ then all $N-1$ nonzero eigenvalues are positive and so the equilibrium set is unstable. Now suppose that $\bar{\phi}_0 \in (-\pi/2, \pi/2)$ so that $\cos\phi > 0$. In this case, if $K > 0$ the equilibrium set is unstable since there are $N-1$ positive eigenvalues and if $K < 0$ the equilibrium set is exponentially stable since there are $N-1$ negative eigenvalues and one zero eigenvalue corresponding to the symmetry direction. In the case that $\phi_0 = \pi/2 + m\pi$, m some integer, $\cos\phi_0 = 0$ and the Jacobian is the zero matrix.

If $M \neq N$, then $\phi_i = \phi_0 \in S^1$ and $\phi_j = \pi - \phi_0$ for some $i \neq j$. Therefore, $\cos\phi_i = -\cos\phi_j$. Suppose that $\phi_i \neq \pi/2 + m\pi$ for any integer m. Let $z = \mathrm{sgn}(\cos\phi))$. Then, the Jacobian becomes

$$J = \frac{K}{N} \cos\phi_0 B \mathrm{diag}(z) B^T. \qquad (20)$$

Note that $z_i = -z_j$ which implies that at least one eigenvalue of J will be positive for any $K \neq 0$. To see this, assume without loss of generality that $z_1 = -1$ and $z_N = 1$. Let $Y = B\mathrm{diag}(z)B^T$. Let $w = (1,0,\ldots,0,2) \in \mathbb{R}^N$ and $x = (2,0,\ldots,0,1) \in \mathbb{R}^N$. Then, $w^TYw = 4 > 0$ and $x^TYx = -2 < 0$ and so Y is indefinite. Thus, for any choice $K \neq 0$ there will be at least one positive eigenvalue of the Jacobian J and the equilibrium set will be unstable. ∎

V. GEOMETRY OF STABLE EQUILIBRIA

In this section we examine the geometry of the equilibrium values of $\theta \in T^N$ (equivalently $\phi \in T^N$) that were proved to be exponentially stable in Theorem 2. To draw the graph of θ, we plot a point for each particle $k = 1,\ldots,N$ at the equilibrium heading angle θ_k on the unit circle and draw a line between every pair of points that are coupled. The graph represents the equilibrium solution in T^N. It also illustrates the equilibrium formation in $SE(2)^N$ (headings and positions in the plane for each particle) that corresponds to the solution $\phi \in T^N$ when the particles are also circling the fixed center of mass of the group. See Figure 2 for examples.

All of the equilibria that were proved to be exponentially stable in Theorem 2 correspond to *generalized regular polygons*. In [9], a cyclic pursuit strategy was used to control N agents in the plane and equilibrium formations were shown to be generalized regular polygons.

Following [13], [9], let $d \leq N$ be a positive integer and denote p as N/d. Let y_1 be a point on the unit circle. Let R_γ be clockwise rotation by the angle $\gamma = 2\pi/p$. The generalized regular polygon $\{p\}$ is given by the points $y_{i+1} = R_\gamma y_i$, $i = 1,\ldots,N-1$ and edges between points i and $i+1$ for $i = 1,\ldots,N$. The polygon $\{N/1\}$ is called an *ordinary regular polygon* and its edges do not intersect; see, for example, $\{9/1\}$ in Figure 2. If $d > 1$ and N and d are coprime, then the edges intersect and the polygon is a *star* as in the examples $\{9/2\}$ and $\{9/4\}$ of Figure 2. If N and d do have a common factor $l > 1$, then the polygon will consist of l traversals of the same polygon with N/l vertices and edges, see $\{9/3\}$ in the figure. In the case $d = N$ the polygon N/N corresponds to all particles at the same point. In the case that N is even and $d = N/2$, then the polygon is a line with points corresponding to an even index on one end and points corresponding to an odd index on the other end.

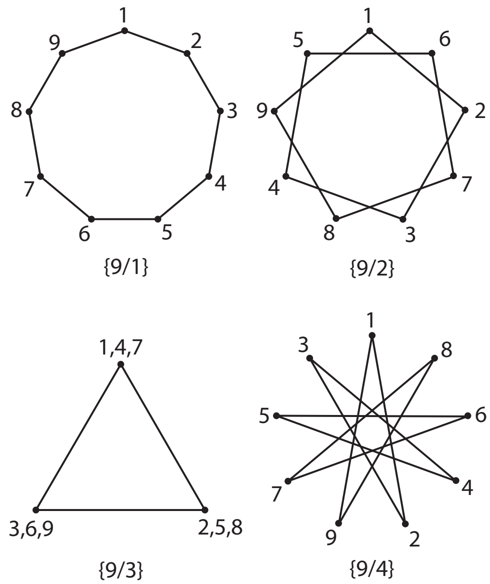

Fig. 2. Regular polygons of type {9/1}, {9/2}, {9/3}, and {9/4}. Types {9/1} and {9/2} are exponentially stable for $K < 0$ and unstable for $K > 0$. Types {9/3} and {9/4} are exponentially stable for $K > 0$ and unstable for $K < 0$. For $K > 0$, $\{9/4\}$ appears to have the largest region of attraction.

In Theorem 2 it was shown that exponentially stable equilibrium are of the form $\phi_i = \phi_0$, $i = 1, \ldots, N$, where $\phi_0 = n(2\pi/N)$, n some integer. These correspond to generalized regular polygons $\{N/d\}$ where $d = n$, i.e., $\gamma = \phi_0$.

For $K > 0$, the exponentially stable equilibria correspond to $\frac{N}{4} < d < \frac{3N}{4}$. For $K < 0$, the exponentially stable equilibria correspond to $0 < d < \frac{N}{4}$ and $\frac{3N}{4} < d \leq N$. For example, in the case $N = 9$, the equilibria corresponding to regular polygons $\{9/3\}$, $\{9/4\}$ $\{9/5\}$, and $\{9/6\}$ are exponentially stable for $K > 0$ and the equilibria corresponding to regular polygons $\{9/1\}$, $\{9/2\}$, $\{9/7\}$, $\{9/8\}$ and $\{9/9\}$ are exponentially stable for $K < 0$. Note that when N is an integer multiple of 4 and $d = N/4$ then $\phi_0 = \pi/2$ and all eigenvalues of the linearization are zero. This is also the case for $d = 3N/4$ and $\phi_0 = 3\pi/2$.

For $K > 0$, numerical simulations suggest that the configurations with the largest regions of attraction are, for N even, those with $n = d = N/2$, i.e., $\phi_0 = \pi$, and for N odd, $n = d = (N \pm 1)/2$, i.e., $\phi_0 = \pi(N \pm 1)/N$. This is consistent with intuition since the case $K > 0$ corresponds to an "anti-synchronizing" control. In the case of $N = 9$ this is the polygon $\{9/4\}$ (and $\{9/5\}$ which looks like $\{9/4\}$ reflected about the vertical axis).

For $K < 0$, numerical simulations suggest that the configurations with the largest regions of attraction are the case $n = d = N$, i.e., $\phi_0 = 0$, corresponding to the synchronized state. This is consistent with intuition since the case $K < 0$ corresponds to a "synchronizing" control.

VI. Ring-Coupled Spacing Control

In this section we consider the case $K > 0$. Recall that the exponentially stable equilibria for the phases (with the likely largest regions of attraction) correspond to the anti-synchronized case in which $\phi_0 = \pi$ for N even and $\phi_0 = \pi(N \pm 1)/N$ for N odd. We add a control term for spacing so that we can drive the particles to a steady circular rotation about the fixed center of mass of the group with radius prescribed by $\rho_0 > 0$ and phases (relative headings) corresponding to the exponentially stable case. Consider the spacing term (4) used in [7], [8]:

$$u_k^{\text{spac}} = -\omega_0(1 + \kappa < \tilde{r}_k, \dot{r}_k >) \tag{21}$$

where $\kappa > 0$, $\omega_0 = 1/\rho_0$. The vector $\tilde{r}_k$ is the position of k^{th} particle with respect to the center of mass of the group, i.e.,

$$\tilde{r}_k = r_k - R = r_k - \frac{1}{N}\sum_{j=1}^{N} r_j = \frac{1}{N}\sum_{j=1}^{N} r_{kj}. \tag{22}$$

This spacing control term requires all-to-all coupling. In the case of ring-coupling, the vectors r_{kj} are only measured for $j = k-1$ and $j = k+1$. So, each particle does not have enough measurements to exactly compute the center of mass of the group.

We propose a spacing control term u_k^{spac} of the form (21) where R is replaced in the control law with an estimate $\hat{R}_k = \hat{R}_k(r_{k-1}, r_k, r_{k+1})$ of the center of mass of the group, i.e.,

$$u_k^{\text{spac}} = -\omega_0(1 + \kappa < r_k - \hat{R}_k, \dot{r}_k >) \tag{23}$$

Each particle makes it own estimate of the center of mass of the group as if the two particles with which it communicates were already anti-synchronized. Thus, the estimate gets better as the phase differences of the communicating particles get closer to the anti-synchronized state. The estimated center of mass for each particle k is

$$\hat{R}_k = \frac{1}{4} r_k + \frac{1}{4}\sum_{j=k-1}^{k+1} r_j.$$

Numerical simulations suggest that a control law using this estimation of the center of mass can effectively stabilize circular motion. Although the particles always converge to circular motion in simulation, certain initial conditions can lead to long transient behavior.

Fig. 3 shows the results of a simulation using control $u = u^{\text{align}} + u^{\text{spac}}$ where u^{align} is given by (6) and u^{spac} is given by (23) with $K = 1$, $\omega_0 = 0.2$, $\kappa = 0.1$, and $N = 9$. Heading control is the same as used in the previous section, except that the gain in this case is given by $K\omega_0^2$. Initial positions and headings were randomized and the system was simulated for 600 time units. The long transient trajectories in the figure are due to the fact that each particle is only communicating with the two particles that are farthest away from it. Thus, spacing adjustments between adjacent particles are only made indirectly.

VII. Final Remarks

In this paper we study collective motion of N constant-speed, planar particles with ring coupling. To control the relative headings of the particles, we use the Kuramoto model of coupled oscillators with identical natural frequencies,

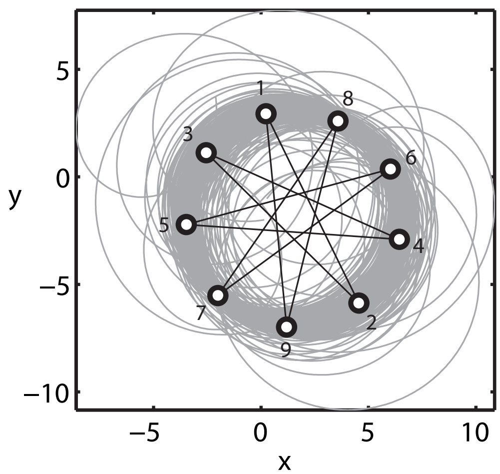

Fig. 3. Numerical simulation results for an $N = 9$ particle system with heading and spacing control given by (6) and (23), respectively. $K = 1$, $\omega_0 = 0.2$, $\kappa = 0.1$. The circles represent the final positions of the particles. The numbers correspond to the indices of the particles, and lines are drawn to represent the interconnections for the final positions of the particles. The trajectories converge to the star pattern $\{9/4\}$.

restricted to ring coupling. We prove convergence of the particle headings (oscillator phases) to the set of equilibria, and we show for $K < 0$ and $K > 0$ which of the equilibria are unstable and which are exponentially stable. We show further that the exponentially stable equilibria correspond to generalized regular polygons. Some of these generalized regular polygons are splay state formations for the particles. In [7], [8] the splay state formations are stabilized with all-to-all coupling and higher harmonics in the sinusoidal coupling terms. Here, instead of adding higher harmonics, we break discrete symmetries by reducing the coupling (i.e., to ring coupling).

We also add a control term to regulate the relative spacing of the particles and drive the particles to a circular motion of prescribed radius about the center of mass of the group. This control term is the spacing control term used in [7], [8] except that each particle uses only an estimate of the center of mass of the particle group since it only measures the relative position of two other particles. A complete generalization of the spacing control law for limited communication is presented in [11]. In this work we analyze stability of the controlled, ring-coupled system with both alignment and spacing control terms.

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