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Robust Receding-Horizon Estimation for Discrete-time Linear Systems in the Presence of Bounded Uncertainties

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Abstract— **Receding-horizon state estimation is addressed for a class of uncertain discrete-time linear systems with disturbances acting on the dynamic and measurement equations. The estimates are obtained by minimizing a least-squares cost function in the worst case, i.e., by solving a min-max problem. With respect to previous results (see [1]), the proposed solution is not conservative and, if the computation is too demanding, the problem may be solved approximately with a reduced computational burden. The stability of the estimation errors is guaranteed under suitable conditions. Simulation results are quite satisfying in performance if compared with other methods.**

I. INTRODUCTION

Receding-horizon estimation and control has become a hot topic in the last decade. A survey on receding-horizon control il given in [2]. As to receding horizon estimation, more recent results are those reported in [3], [4], [5]. Recent researches on this subject have aimed at developing filter and controllers with guaranteed robustness properties with respect to system uncertainties. In the framework of robust control, special attention has been paid to various receding-horizon techniques (see, e.g., [6], [7], [8], [9]), where the controller is synthesized by using a worst-case criterion. In particular, a min-max approach is usually followed, which means that the design is made by minimizing a control cost function in the case corresponding to the most pessimistic conditions given, for example, by the disturbances and/or the uncertainties.

A similar approach has been proposed in [1] in the framework of robust receding-horizon estimation, that consists in the minimization of an upper bound on a worst-case quadratic cost defined over a sliding window. Though conservative, this method enables to find solutions with a low computational effort. Such a goal has been pursued by using recent results on the solution of min-max least-squares problems (see [10]). In general, least-squares problems often arise in different research areas and it may occur the need of finding solution robust with respect to uncertain data. The approach to robust least squares in [10] is based on regularization. Such an approach has also been applied with success to robust Kalman-filtering [11]. As to other min-max filtering techniques for linear systems, the most important results in the past literature are summarized in [12] and in the references therein.

In this paper, a new approach with some advance with respect to [1] is proposed, in that the introduction of a modified quadratic cost function allows one to solve exactly the considered min-max receding-horizon estimation problem. This entails the on-line minimization of a unimodal cost function. If the required effort of computation is too high, an approximate solution can be found with a small computational burden as in [1]. Simulation results show a considerable improvement in the performance of the proposed filters if compared with that of [1]. The proofs are omitted for the sake of brevity.

We conclude this section by defining some notations used throughout this paper. Given a generic, symmetric, positive definite matrix P, let us denote by $\underline{\sigma}(P)$ and $\bar{\sigma}(P)$ the minimum and maximum eigenvalues of P, respectively; moreover, $P^{1/2}$ is the unique positive definite square root of the matrix P. Given a generic matrix M, M' and $M^{\dagger}$ indicate the matrix transpose and the pseudoinverse of M, respectively. Furthermore, $\|M\| = [\bar{\sigma}(M'M)]^{1/2}$. Given a generic vector v, $\|v\|$ denotes the Euclidean norm of v and, given a positive definite matrix P, $\|v\|_P$ denotes the weighted norm of v, $\|v\|_P \triangleq (v'Pv)^{1/2}$. For a generic time-varying vector v_t, let us define $v_{t-N}^t \triangleq \text{col}\,(v_{t-N}, v_{t-N+1}, \ldots, v_t)$; similarly, given a generic time-varying matrix M_t, let us define $M_{t-N}^t \triangleq \text{col}\,(M_{t-N}, M_{t-N+1}, \ldots, M_t)$. Finally, let us denote the ordered product of a sequence of matrices $\{M_1, M_2, \ldots M_n\}$ as $\prod_{i=1}^{n} M_i \triangleq M_1 M_2 \cdots M_n$.

II. STATEMENT OF THE PROBLEM

Let us consider an uncertain linear dynamic system described by the following discrete-time equations

$$x_{t+1} = (A + \delta A_t)\, x_t + (B + \delta B_t)\, u_t + w_t \quad \text{(1a)}$$
$$y_t = (C + \delta C_t)\, x_t + v_t \quad \text{(1b)}$$

where $t = 0, 1, \ldots$ is the time instant, $x_t \in \mathbb{R}^n$ is the state vector (the initial state x_0 is unknown), $u_t \in \mathbb{R}^p$ is the control vector, $w_t \in \mathbb{R}^n$ is the system noise vector, $y_t \in \mathbb{R}^m$ is the vector of the measures, and $v_t \in \mathbb{R}^m$ is the measurement noise vector. The matrices δA_t, δB_t, and δC_t represent time-varying uncertainties in the knowledge of the system, and are supposed to belong to the known compact sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively. More specifically, we shall consider unknown but bounded uncertainties of the

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form

$$[\delta A_t \quad \delta B_t] = D\Delta_t [E \quad F] , \tag{2a}$$
$$\delta C_t = G\bar{\Delta}_t H , \tag{2b}$$

for $t = 0, 1, \ldots,$ where $D, E, F, G,$ and H are known matrices, and Δ_t and $\bar{\Delta}_t$ are arbitrary contractions, i.e.,

$$\|\Delta_t\| \le 1 , \quad \|\bar{\Delta}_t\| \le 1 .$$

We assume the statistics of the random variables x_0, w_t, and v_t to be unknown, and consider them as deterministic variables of an unknown kind. Moreover, we assume our estimates to be based on data obtained in the recent past according to a receding-horizon strategy [3], [13], [1]. Then we define the information vector as

$$I_t^N \stackrel{\triangle}{=} \mathrm{col}\,(y_{t-N}, \ldots, y_t, u_{t-N}, \ldots, u_{t-1}) ,$$

for $t = N, N+1, \ldots.$ $N+1$ is the number of measurements made at sliding-window stages from $t-N$ to t.

At any time $t = N, N+1, \ldots$, the objective is to find estimates of the state vectors $x_{t-N}, \ldots, x_t$ on the basis of the information vector I_t^N and of some "prediction" $\bar{x}_{t-N}$ of the state x_{t-N} at the beginning of the sliding window. Let us denote by $\hat{x}_{t-N,t}, \ldots, \hat{x}_{t,t}$ the estimates (to be made at time t) of $x_{t-N}, \ldots, x_t$, respectively.

As we have assumed the statistics of the disturbances and of the initial state to be unknown, a natural criterion to derive the estimator consists in resorting to a least-squares approach. Towards this end, we shall address the minimization of the following loss function

$$\begin{aligned} J_t = {} & \| \hat{x}_{t-N,t} - \bar{x}_{t-N} \|_M^2 \\ & + \sum_{j=t-N}^{t-1} \| \hat{x}_{j+1,t} - (A + \delta A_j)\hat{x}_{j,t} - (B + \delta B_j)u_j \|_Q^2 \\ & + \sum_{j=t-N}^{t} \| y_j - (C + \delta C_j)\hat{x}_{j,t} \|_R^2 \end{aligned} \tag{3}$$

where the matrices M, Q, and R are assumed to be positive definite and can be regarded as design parameters. The first term, weighted by the matrix M, penalizes the distance of the state estimate at the beginning of the sliding window from the prediction $\bar{x}_{t-N}$. The second contribution, weighted by the matrix Q, takes into account the evolution of the state in terms of the state equation (1a). Finally, the third term, weighted by the matrix R, penalizes the distances of the state estimates from the measures. As to the prediction $\bar{x}_{t-N}$, different choices are possible. For example, following [1], it can be determined via the state equation of the nominal system by the estimate $\hat{x}_{t-N-1,t-1}$, that is, $\bar{x}_{t-N} = A\hat{x}_{t-N-1,t-1} + Bu_{t-N-1}$. A second possibility consists in assigning to $\bar{x}_{t-N}$ the value of the estimate of x_{t-N} made at the previous time instant $t-1$, that is, $\bar{x}_{t-N} = \hat{x}_{t-N,t-1}$. In both cases, the vector $\bar{x}_0$ denotes an a-priori prediction of x_0. In the following, we shall consider the former definition, since it will make easier the derivation of the convergence results (see Section IV).

It is worth noting that cost (3) is a function, not only of the estimates $\hat{x}_{t-N,t}^t$, but also of the uncertain matrices δA_{t-N}^{t-1}, δB_{t-N}^{t-1}, and δC_{t-N}^t, and consequently, in the light of equations (2), of the arbitrary contraction matrices Δ_{t-N}^{t-1} and $\bar{\Delta}_{t-N}^t$, that is,

$$J_t = J_t\left(\hat{x}_{t-N,t}^t, \Delta_{t-N}^{t-1}, \bar{\Delta}_{t-N}^t\right) .$$

As to the uncertainties in the system matrices, we shall follow a min-max approach; then, at any time $t = N, N+1, \ldots$, the following problem has to be solved.

Problem $\mathcal{E}_t$ *For a given pair $(\bar{x}_{t-N}, I_t^N)$, find the optimal estimates $\hat{x}_{t-N,t}^\circ, \ldots, \hat{x}_{t,t}^\circ$ that minimize the maximum of cost (3) over all the possible uncertainties, i.e., find the solutions of the min-max optimization problem*

$$\min_{\hat{x}_{t-N,t}^t} \max_{\Delta_{t-N}^{t-1};\, \bar{\Delta}_{t-N}^t} J_t\left(\hat{x}_{t-N,t}^t, \Delta_{t-N}^{t-1}, \bar{\Delta}_{t-N}^t\right) \tag{4}$$

with $\|\Delta_i\| \le 1$ for $i = t-N, \ldots, t-1$ and $\|\bar{\Delta}_i\| \le 1$ for $i = t-N, \ldots, t$. □

Obviously, concerning the propagation of the estimation procedure from Problem $\mathcal{E}_t$ to Problem $\mathcal{E}_{t+1}$, only the estimate $\hat{x}_{t-N,t}^\circ$ has to be retained. This estimate becomes the optimal prediction for Problem $\mathcal{E}_{t+1}$ via the nominal state equation, i.e., $\bar{x}_{t-N+1,t+1} = A\hat{x}_{t-N,t}^\circ + Bu_{t-N}$. Such a recursion is initialized at stage N with some a-priori prediction $\bar{x}_0$ of the initial state vector.

The following proposition ensures the well-posedness of Problem $\mathcal{E}_t$.

Proposition 1: Suppose that $M > 0$ and $Q > 0$, then

$$\max_{\Delta_{t-N}^{t-1};\, \bar{\Delta}_{t-N}^t} J_t\left(\hat{x}_{t-N,t}^t, \Delta_{t-N}^{t-1}, \bar{\Delta}_{t-N}^t\right)$$

is a strictly convex radially unbounded function[1] of $\hat{x}_{t-N,t}^t$ and, consequently, Problem $\mathcal{E}_t$ has a unique finite solution. □

Before proceeding to the derivation of the unique solution of Problem $\mathcal{E}_t$, it is important to remark the main differences between the approach described above and the one proposed in [1]. With this respect, it is worth noting that in [1] a similar min-max approach for the receding-horizon estimation of uncertain linear systems was proposed, in which a slightly different quadratic cost was considered of the form

$$\begin{aligned} J_t' = {} & \| \hat{x}_{t-N,t} - \bar{x}_{t-N} \|_M^2 \\ & + \sum_{j=t-N}^{t} \| y_j - (C + \delta C_j)\hat{x}_{j,t} \|^2 . \end{aligned}$$

Furthermore, the estimates $\hat{x}_{t-N+1,t}, \ldots, \hat{x}_{t,t}$ were generated recursively by the estimate at the beginning of the sliding window $\hat{x}_{t-N,t}$ through the state equation (1a), that is,

$$\hat{x}_{i+1,t} = (A + \delta A_i)\,\hat{x}_{i,t} + (B + \delta B_i)u_i , \tag{5}$$

[1]Recall that a function $f(z)$ is said to be radially unbounded if $f(z) \to +\infty$ as $\|z\| \to +\infty$.

for $i = t-N, \ldots, t-1$. By applying (5), the cost J'_t turns out to be a function of the estimate of the state $\hat{x}_{t-N,t}$ and of the uncertain matrices Δ_{t-N}^{t-1} and $\bar{\Delta}_{t-N}^{t}$.

Note that the summation in cost J_t weighted by the matrix Q can be seen as a "soft" version of the hard constraints (5). Hence Problem $\mathcal{E}_t$ turns out to be a relaxation of the min-max problem addressed in [1] (in the following, we shall call it Problem $\mathcal{E}_t^h$).

While the introduction of the hard constraints (5) leads to an apparent simplification in the estimation scheme (in that only the estimate at the beginning of the sliding-window has to be computed), it turns out that the dependence of cost J'_t on the contraction matrices Δ_{t-N}^{t-1} and $\bar{\Delta}_{t-N}^{t}$ is quite complex. More specifically, in [1] it was shown that Problem $\mathcal{E}_t^h$ can be written as a linear-fractional Structured Robust Least Squares (SRLS) problem (see [14]). As it is well known, such a problem is in general very difficult to solve. In [1], an alternative conservative reformulation of Problem $\mathcal{E}_t^h$ was proposed that consisted in the minimization of an upper bound on the worst-case cost and lead to a less computationally demanding solution.

On the contrary, it is immediate to see that, for fixed values of the estimates $\hat{x}_{t-N,t}^{t}$, cost J_t depends quadratically on the contraction matrices Δ_{t-N}^{t-1} and $\bar{\Delta}_{t-N}^{t}$, thus making it possible to derive a convenient form for the solution of Problem $\mathcal{E}_t$.

III. Form of the Solution

In order to solve Problem $\mathcal{E}_t$, we shall refer to the following technical lemma that summarizes some of the results presented in [10].

Lemma 1: Let $\phi : \mathbb{R}^q \to \mathbb{R}$ be a strictly positive and convex function and consider the constrained maximization problem over the vector η

$$\max_{\|\eta\| \le \phi(\zeta)} \|\Psi\zeta - \omega + \Pi\eta\|_{\Omega}^2 \tag{6}$$

where Ψ, Π, and $\Omega > 0$ are known matrices, and ω and ζ are known vectors. Then problem (6) is equivalent to the constrained minimization problem over the scalar Lagrange multiplier λ

$$\min_{\lambda \ge \|\Pi'\Omega\Pi\|} \|\Psi\zeta - \omega\|_{\Omega(\lambda)}^2 + \lambda\phi^2(\zeta) \tag{7}$$

where

$$\Omega(\lambda) \triangleq \Omega + \Omega\Pi'(\lambda I - \Pi'\Omega\Pi)^{\dagger}\Pi'\Omega\,.$$

Furthermore, the optimal Lagrange multiplier λ° that solve problem (7) depends continuously on the vector ζ. □

By equivalence of problems (6) and (7), we mean that the maximum of the quadratic cost in (6) is equal to the minimum of the cost in (7).

Let us now address the problem of maximizing cost J_t over all the possible uncertain matrices Δ_{t-N}^{t-1} and $\bar{\Delta}_{t-N}^{t}$. Towards this end, it is important to note that the generic term

$$\|\,\hat{x}_{j+1,t} - (A + \delta A_j)\hat{x}_{j,t} - (B + \delta B_j)u_j\,\|_Q^2\,, \tag{8}$$

for $j = t-N, \ldots t-1$, depends only on the contraction Δ_j. In a similar way, the generic term

$$\|\,y_j - (C + \delta C_j)\hat{x}_{j,t}\,\|_R^2\,, \tag{9}$$

for $j = t-N, \ldots, t$, depends only on the contraction $\bar{\Delta}_j$. As a consequence, since the contraction matrices Δ_{t-N}^{t-1} and $\bar{\Delta}_{t-N}^{t}$ are supposed to be arbitrary and hence mutually independent, one can consider each maximization independently.

Let us first consider the generic term (8). By applying Lemma 1, one can replace the maximization of (8) over the contraction Δ_j with a minimization over a scalar parameter. More specifically, we can state the following proposition.

Proposition 2: The maximum of (8) over Δ_j under the constraint $\|\Delta_j\| \le 1$ is equal to

$$\min_{\lambda_{j,t} \ge \|D'QD\|} \Big\{ \|\hat{x}_{j+1,t} - A\hat{x}_{j,t} - Bu_j\|_{Q(\lambda_{j,t})}^2 + \lambda_{j,t}\,\|E\hat{x}_{j,t} + Fu_i\|^2 \Big\} \tag{10}$$

where

$$Q(\lambda_{j,t}) \triangleq Q + QD(\lambda_{j,t}I - D'QD)^{\dagger}D'Q\,. \tag{11}$$

□

As to the generic term (9), in a similar way one can replace the maximization of (9) over the contraction $\bar{\Delta}_j$ with a minimization over a scalar parameter.

Proposition 3: The maximum of (9) over $\bar{\Delta}_j$ under the constraint $\|\bar{\Delta}_j\| \le 1$ is equal to

$$\min_{\mu_{j,t} \ge \|G'RG\|} \Big\{ \|y_j - C\hat{x}_{j,t}\|_{R(\mu_{j,t})}^2 + \mu_{j,t}\,\|H\hat{x}_{j,t}\|^2 \Big\} \tag{12}$$

where

$$R(\mu_{j,t}) \triangleq R + RG(\mu_{j,t}I - G'RG)^{\dagger}G'R\,. \tag{13}$$

□

For the sake of brevity, let us define the following cost:

$$\begin{aligned} L_t\left(\hat{x}_{t-N,t}^{t}\,,\,\lambda_{t-N,t}^{t-1}\,,\,\mu_{t-N,t}^{t}\right) &\triangleq \|\,\hat{x}_{t-N,t} - \bar{x}_{t-N}\,\|_M^2 \\ &+ \sum_{j=t-N}^{t-1} \Big\{ \|\hat{x}_{j+1,t} - A\hat{x}_{j,t} - Bu_j\|_{Q(\lambda_{j,t})}^2 \\ &\qquad + \lambda_{j,t}\,\|E\hat{x}_{j,t} + Fu_i\|^2 \Big\} \\ &+ \sum_{j=t-N}^{t} \Big\{ \|y_j - C\hat{x}_{j,t}\|_{R(\mu_{j,t})}^2 + \mu_{j,t}\,\|H\hat{x}_{j,t}\|^2 \Big\}. \end{aligned}$$

It is important to remark that cost L_t is similar to a standard least-squares cost for the nominal system, in which the weight matrices Q and R are suitably corrected, according to (11) and (13), and some regularization terms are added.

In the light of Propositions 2 and 3, the original Problem $\mathcal{E}_t$ turns out to be equivalent to

$$\min_{\hat{x}_{t-N,t}^{t}} \; \min_{\lambda_{t-N}^{t-1};\,\mu_{t-N}^{t}} L_t\left(\hat{x}_{t-N,t}^{t}, \lambda_{t-N}^{t-1}, \mu_{t-N}^{t}\right) \tag{14}$$

with $\lambda_i \ge \|D'QD\|$ for $i = t-N, \ldots, t-1$ and $\mu_i \ge \|G'RG\|$ for $i = t-N, \ldots, t$. By inverting the positions of the two minimizations in (14), one can consider the equivalent problem

$$\min_{\lambda_{t-N}^{t-1};\mu_{t-N}^{t}} \min_{\hat{x}_{t-N,t}^{t}} L_t\left(\hat{x}_{t-N,t}^{t}, \lambda_{t-N}^{t-1}, \mu_{t-N}^{t}\right). \tag{15}$$

Note that, for fixed values of the Lagrange multipliers $\lambda_{t-N,t}^{t-1}$ and $\mu_{t-N,t}^{t}$, cost L_t is a quadratic function of $\hat{x}_{t-N,t}^{t}$, hence it is possible to derive a closed-form expression for the solution of the innermost minimization in (15). Let $\tilde{x}_{t-N,t}^{t}\left(\lambda_{t-N,t}^{t-1}, \mu_{t-N,t}^{t}\right)$ be the estimate vectors that yield such a minimum (for the reader's convenience, the exact expression of $\tilde{x}_{t-N}^{t}$ is derived in the appendix). Then the following theorem holds.

Theorem 1: The optimal Lagrange multipliers $\tilde{\lambda}_{t-N,t}^{t-1}$ and $\tilde{\mu}_{t-N,t}^{t}$ can be obtained as solution of the minimization problem

$$\min_{\lambda_{t-N,t}^{t-1};\mu_{t-N,t}^{t}} \tilde{L}_t\left(\lambda_{t-N,t}^{t-1}, \mu_{t-N,t}^{t}\right) \tag{16}$$

where

$$\begin{aligned}\tilde{L}_t\left(\lambda_{t-N,t}^{t-1}, \mu_{t-N,t}^{t}\right) &\triangleq \min_{\hat{x}_{t-N,t}^{t}} L_t\left(\hat{x}_{t-N,t}^{t}, \lambda_{t-N}^{t-1}, \mu_{t-N}^{t}\right)\\ &= L_t\left[\tilde{x}_{t-N,t}^{t}\left(\lambda_{t-N,t}^{t-1}, \mu_{t-N,t}^{t}\right), \lambda_{t-N,t}^{t-1}, \mu_{t-N,t}^{t}\right].\end{aligned}$$

If all the optimal Lagrange multipliers $\tilde{\lambda}_{t-N,t}^{t-1}$ and $\tilde{\mu}_{t-N,t}^{t}$ are finite, then the unique solution of Problem $\mathcal{E}_t$ can be obtained as

$$\hat{x}_{i,t}^{\circ} = \tilde{x}_{i,t}\left(\tilde{\lambda}_{t-N,t}^{t-1}, \tilde{\mu}_{t-N,t}^{t}\right)$$

for $i = t-N, \ldots, t$. □

We would like to point out once more that the form of the solution, i.e., the function $\tilde{x}_{t-N}^{t}\left(\lambda_{t-N,t}^{t-1}, \mu_{t-N,t}^{t}\right)$, is known and can be easily computed (see the appendix). Such a solution depends on $2N+1$ parameters, i.e., the Lagrange multipliers $\lambda_{t-N,t}^{t-1}$ and $\mu_{t-N,t}^{t}$, that have to be determined by means of the minimization (16) in order to find the optimal estimates $\hat{x}_{t-N,t}^{\circ}, \ldots, \hat{x}_{t,t}^{\circ}$. Unfortunately, in general it is not possible to derive an analytical expression for the optimal Lagrange multipliers $\tilde{\lambda}_{t-N,t}^{t-1}$ and $\tilde{\mu}_{t-N,t}^{t}$. As a consequence they have to be determined on line by means of some nonlinear programming routine. Note that the minimization problem (16) is always well posed, in that one can search for the global minimum of $\tilde{L}_t$ without worrying about local minima. More specifically, the following proposition holds.

Proposition 4: Suppose that $M > 0$ and $Q > 0$, then the function $\tilde{L}_t$ is unimodal. □

If the computation of the optimal Lagrange multipliers via (16) is not feasible (e.g., lack of computation time in the sampling period), by following [1], [11], one can obtain a reasonable approximation of the optimal solution by assigning to each of the $2N+1$ parameters $\lambda_{t-N,t}^{t-1}$ and $\mu_{t-N,t}^{t}$ a fixed value, which can be suitably tuned off line by means of numerical simulations. As will be shown in Section V, a suitable selection of such fixed values can make the performance of the approximate filter very close to those of the optimal one.

IV. Stability of the Estimator

For the sake of simplicity and without loss of generality, suppose that the weight matrices M, Q, and R are diagonal, i.e.,

$$M = mI, \quad Q = qI, \quad R = rI.$$

Furthermore, let us consider the matrices

$$\mathcal{F}_t^{(N)} \triangleq \begin{bmatrix} (C+\delta C_{t-N}) \\ (C+\delta C_{t-N+1})\,(A+\delta A_{t-N}) \\ \vdots \\ (C+\delta C_t)\prod_{i=1}^{N}(A+\delta A_{t-i}) \end{bmatrix},$$

$$\mathcal{H}_t^{(N)} \triangleq \begin{bmatrix} 0 & \cdots & 0 \\ (C+\delta C_{t-N+1}) & \cdots & 0 \\ (C+\delta C_{t-N+2})(A+\delta A_{t-N+1}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ (C+\delta C_t)\prod_{i=1}^{N-1}(A+\delta A_{t-i}) & \cdots & (C+\delta C_t) \end{bmatrix},$$

and define the quantities

$$\begin{aligned} f^{(N)} &\triangleq \min_{\Delta_{t-N}^{t-1};\bar{\Delta}_{t-N}^{t}} \underline{\sigma}\left\{\mathcal{F}_t^{(N)\prime}\mathcal{F}_t^{(N)}\right\},\\ h^{(N)} &\triangleq \max_{\Delta_{t-N}^{t-1};\bar{\Delta}_{t-N}^{t}} \|\mathcal{H}_t^{(N)}\|^2, \end{aligned}$$

with $\|\Delta_i\| \le 1$ for $i = t-N, \ldots, t-1$ and $\|\bar{\Delta}_i\| \le 1$ for $i = t-N, \ldots, t$. Note that, with the exception of the contractions Δ_t and $\bar{\Delta}_t$, all the other matrices that describe the uncertain system (1) are time invariant. This ensures the time-invariance of the constants $f^{(N)}$ and $h^{(N)}$. Clearly, $f^{(N)}$ represents a measure of the worst-case degree of observability of the system.

In order to show the convergence properties of the proposed estimator, the following assumptions are needed.

A1. At any time stage $t = 0, 1, \ldots$, the system noise vector w_t, the control vector u_t, and the measurement noise vector v_t belong to the bounded sets $\mathcal{W}$, $\mathcal{U}$, and $\mathcal{V}$, respectively.

A2. There exists a bounded set $\mathcal{X}$ such that, for any admissible initial condition x_0, any system noise sequence $\{w_t\}$, and any uncertain sequence $\{\Delta_t\}$, the control sequence $\{u_t\}$ ensures that $x_t \in \mathcal{X}$ for $t = 0, 1, \ldots$.

A3. System (1) is uniformly completely observable in N steps, i.e., for any $t = N, N+1, \ldots$, any Δ_{t-N}^{t} and any $\bar{\Delta}_{t-N}^{t}$ the observability matrix $\mathcal{F}_t^{(N)}$ is of full rank.

Note that Assumptions A1 and A2 are quite reasonable from a practical point of view when considering the state estimation problem for a physical system: it is very typical that the state variables and the disturbances are bounded in some way. Furthermore, if Assumption A1 is satisfied, then Assumption A2 is automatically verified, provided that the considered system is asymptotically stable. Again this is a quite common assumption when addressing the stability of some estimation scheme in an uncertain framework. As to Assumption A3, it ensures that $f^{(N)} > 0$.

We are now in the position to state the following convergence result.

Theorem 2: Suppose that Assumptions A1, A2, and A3 are satisfied. Then the norm of the quadratic estimation error $e_{t-N} \stackrel{\triangle}{=} \|x_{t-N} - \hat{x}^{\circ}_{t-N,t}\|^2$ is bounded from above as

$$e_{t-N} \leq \zeta_{t-N} \quad , \quad t = N, N+1, \ldots \quad .$$

The sequence $\{\zeta_t\}$ is defined recursively as

$$\begin{aligned} \zeta_0 &= \beta_0 \,, \\ \zeta_t &= \alpha\, \zeta_{t-1} + \beta \,, \quad t = 1, 2, \ldots \,. \end{aligned} \tag{17}$$

where

$$\alpha = \frac{2\, m\, a \left(2\, q + 3\, r h^{(N)}\right)}{q \left(r f^{(N)} + m\right)}, \quad a = \|A\|^2 \,,$$

and β_0, β are suitable positive constants.

Moreover, if m, q, and r have been selected such that $\alpha < 1$, the sequence $\{\zeta_t\}$ converges exponentially to the asymptotic value $e_\infty \stackrel{\triangle}{=} \beta/(1-\alpha)$. □

Note that, since under Assumption A3 $f^{(N)} > 0$, condition $\alpha < 1$ can be easily verified for any value of a and $h^{(N)}$ through suitable choices of m, q, and r. As to the constant β, it depends on the system matrices and on the sets $\mathcal{W}$, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ and can be easily computed. More specifically, it turns out that:

1) As expected, if system (2) is noise-free (i.e., $\mathcal{W} = 0$ and $\mathcal{V} = 0$) and without uncertainty (i.e., $\mathcal{A} = 0$, $\mathcal{B} = 0$, and $\mathcal{C} = 0$), then $\beta = 0$, i.e., the proposed filter is an asymptotic observer.
2) The value of β depends continuously on the "amplitudes" of the noises and of the uncertainties (i.e., on the radii of the sets $\mathcal{W}$, $\mathcal{V}$, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$).

Clearly, such properties ensure that there always exist non-null values of the radii of the sets $\mathcal{W}$, $\mathcal{V}$, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ such that the upper bound given in Theorem 2 is significative.

V. A Numerical Example

In this section, a simulation example is given to illustrate the proposed approach to receding-horizon estimation for uncertain systems. Let us consider a test-bed uncertain system described by means of equations (1) with

$$A = \begin{bmatrix} 0.68 & -0.5 \\ 1 & -0.7 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 10 & 1 \end{bmatrix}.$$

TABLE I

Asymptotic values of the RMSEs for the considered filters.

Filter	RRHF	ARRHF	CRRHF	NRHF	NKF
RMSE	0.0169	0.0177	0.0206	0.0262	0.0261

Suppose that the uncertain matrices δA_t, δB_t, and δC_t can be described through equations (2) with

$$D = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0.04 \end{bmatrix}, \quad F = 0,$$
$$G = 1, \quad H = \begin{bmatrix} 3.6 & 1.6 \end{bmatrix}.$$

Furthermore, suppose that, at each time instant, the contractions Δ_t and $\bar{\Delta}_t$ are independent random variables uniformly distributed in the interval $[-1, 1]$. In addition, let us assume x_0, w_t, and v_t to be normally distributed independent random variables with zero-mean and covariance matrices $\sigma_x^2 I$, $\sigma_w^2 I$, and σ_v^2, respectively.

In the following, for the sake of compactness, we shall refer to the proposed estimator as "robust receding-horizon filter" (RRHF). Moreover, we shall refer to the filter obtained by assigning a fixed value to each lagrange multiplier as "approximate robust receding-horizon filter" (ARRHF).

For the sake of comparison, let us now consider the performance index given by the Root Mean Square Error (RMSE). In order to evaluate the improvement in performance achieved with respect to previous works, the proposed estimators will be compared with the "conservative robust receding-horizon filter" (CRRHF) of [1]. Moreover, we shall also consider the "nominal Kalman filter" (NKF) and the "nominal receding-horizon filter" (NRHF), both obtained by considering the nominal system (i.e., with $\delta A = 0$, $\delta B = 0$ and $\delta C = 0$).

Fig. 1 presents the plots of the RMSEs, computed over 100 randomly chosen simulations, for the considered filters with $\sigma_x = 50$, $\sigma_w = 10^{-2}$, and $\sigma_v = 5 \cdot 10^{-2}$ (note that the plot for the NKF is omitted since it is almost coincident to the one for the NRHF). For the reader's convenience, the asymptotic values of the RMSEs are reported in Table I. The weight matrices M, Q and R have been chosen equal to $10^4 I$, $10^2 I$, and 4, respectively. The size N of the sliding window was taken equal to 2. Finally, for the ARRHF, the lagrange multipliers $\lambda_{j,t}$ and $\mu_{j,t}$ have been assigned the fixed values 40 and 10, respectively. As can be seen from Fig. 1 and Table I, the proposed robust filters offer the best performance from the point of views of both the transient and the asymptotic behavior. Furthermore, since the behavior of the ARRHF is very close to that of the RRHF, one may conclude that the approximation of the time-varying parameters $\lambda_{j,t}$ and $\mu_{j,t}$ with fixed values leads to an acceptable decay of the performance of the proposed estimation scheme.

Appendix

Let us now consider the problem of minimizing cost L_t for given values of the Lagrange multipliers $\lambda_{t-N,t}^{t-1}$ and

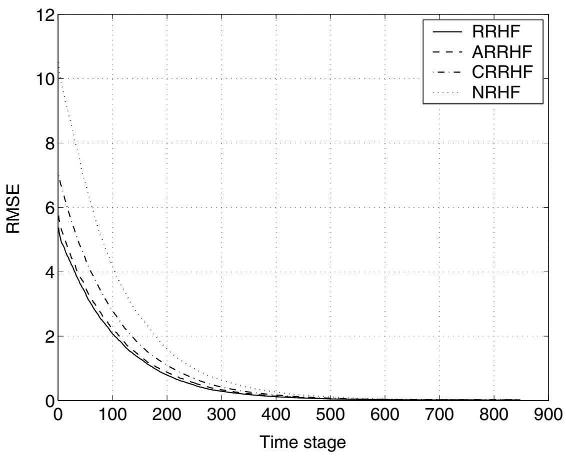

Fig. 1. Plots of the RMSEs for the considered filters.

$\mu^t_{t-N,t}$. For the sake of compactness, we shall use the notations

$$Q_{i,t} \stackrel{\triangle}{=} Q(\lambda_{i,t}), \quad R_{i,t} \stackrel{\triangle}{=} R(\mu_{i,t}).$$

Furthermore, without risk of ambiguity, we shall drop the dependency of all the considered quantities on the Lagrange multipliers $\lambda^{t-1}_{t-N,t}$ and $\mu^t_{t-N,t}$.

The following theorem provides an efficient procedure to compute the estimates $\tilde{x}^t_{t-N}$.

Theorem 3: Suppose that $M > 0$ and $Q > 0$. Then, given the Lagrange multipliers $\lambda^{t-1}_{t-N,t}$ and $\mu^t_{t-N,t}$, the estimates $\tilde{x}^t_{t-N}$ can be computed recursively as

$$\tilde{x}_{i,t} = T_{i,t} Q_{i-1,t} A \tilde{x}_{i-1,t} + z_{i,t}, \quad i = t-N+1, \ldots, t,$$
$$\tilde{x}_{t-N,t} = z_{t-N,t}$$

where the matrices $T_{i,t}$ and the vectors $z_{i,t}$ are computed according to the backward recursions

$$\begin{aligned}
T_{t,t} &= \Big\{ Q_{t-1,t} + C' R_{t,t} C + \mu_{t,t} H' H \Big\}^{-1}, \\
T_{i,t} &= \Big\{ Q_{i-1,t} + A' Q_{i,t} A + \lambda_{i,t} E' E + C' R_{i,t} C \\
&\quad + \mu_{i,t} H' H - A' Q_{i,t} T_{i+1,t} Q_{i,t} A \Big\}^{-1}, \\
&\qquad\qquad i = t-1, \ldots, t-N+1, \\
T_{t-N,t} &= \Big\{ M + A' Q_{t-N,t} A + \lambda_{t-N,t} E' E + C' R_{t-N,t} C \\
&\quad + \mu_{t-N,t} H' H - A' Q_{t-N,t} T_{t-N+1,t} Q_{t-N,t} A \Big\}^{-1},
\end{aligned}$$

and

$$\begin{aligned}
z_{t,t} &= T_{t,t} \left\{ Q_{t-1,t} B u_{t-1} + C' R_{t,t} y_t \right\} \\
z_{i,t} &= T_{i,t} \Big\{ A' Q_{i,t} \left[z_{i+1,t} - B u_i \right] - \lambda_{i,t} E' F u_i \\
&\quad + Q_{i-1,t} B u_{i-1} + C' R_{i,t} y_i \Big\}, \quad i = t-1, \ldots, t-N+1, \\
z_{t-N,t} &= T_{t-N,t} \Big\{ A' Q_{t-N,t} \left[z_{t-N+1,t} - B u_{t-N} \right] \\
&\quad - \lambda_{t-N,t} E' F u_{t-N} + C' R_{t-N,t} y_{t-N} + M \bar{x}_{t-N} \Big\}.
\end{aligned}$$

□

The closed-form formula for the estimates $\tilde{x}^t_{t-N}$ given in Theorem 3 can be obtained with some algebra (omitted for the sake of brevity) by imposing the first-order optimality condition $\nabla_{\hat{x}^t_{t-N,t}} L_t = 0$. Note that the computation of such estimates consists in a quite standard backward-forward algorithm: first the matrices $T_{i,t}$ and the vectors $z_{i,t}$ are computed by means of a suitable backward recursion; then such quantities are used to compute the estimates $\tilde{x}^t_{t-N,t}$ by means of a suitable forward recursion. It is important to remark that, by construction, the matrices $T_{i,t}$ are always positive definite (provided that $M > 0$ and $Q > 0$).

As a final remark, if one considers the approximate estimator obtained by assigning to each of the $2N+1$ parameters $\lambda^{t-1}_{t-N,t}$ and $\mu^t_{t-N,t}$ a fixed value, then the matrices $T_{i,t}$ do not depend on the time index t, i.e., $T_{t-N+i,t} = \bar{T}_i$ for $i = 0, 1, \ldots, N$. Then the $N+1$ matrices $\bar{T}_i$ can be computed once for all off line. On the other hand, even for the approximate estimator, the vectors $z_{i,t}$ depend on the information vector I^N_t and, consequently, have to be computed on line at every time step.

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