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Sampled-Data $\mathcal{H}_\infty$ Control Design for a Class of PWM Systems

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Abstract— **Robust control of a class of switched dynamical systems is considered. The switching is controlled by pulse-width modulation (PWM). This idea is used, for example, in power electronics for power conversion. The class of systems considered includes digitally controlled power converters of many different types. The traditional approach to control design for power converters rely on an averaged model that ignores the high frequency behavior and the inherent time delay due to sampling. In contrast, the method presented here is based on a sampled-data model which takes the switched nature of the system into account. The sampled-data model is approximated by a linear quadratic model to which sampled-data $\mathcal{H}_\infty$ theory can be extended. The approach is applied to a bidirectional boost converter which is subjected to a large load disturbance.**

I. Introduction

We consider robust control of a class of switched dynamical systems where control authority is gained by means of fast switching between independent vector fields. This idea is used, for example, in power electronics for energy conversion. Such power converters may be connected to sensitive equipments or be part of large networks which makes proper performance essential. A primary design criteria of a switched network of this type is its ability to sustain large changes in operating point due to load and/or source variations. Here we consider sampled-data $\mathcal{H}_\infty$ control theory as a means to obtain robustness to load disturbances and model uncertainty.

In this paper we consider a class of systems where the state is switched between two affine vector fields in a given order. The switching is controlled by means of pulse-width modulation (PWM). This means that a duty ratio signal is used to determine the fractions of the period in which each of the two dynamics is active. The mathematical model we adopt covers digitally controlled power converters of many standard topologies [8]. In particular, switch-mode fixed-frequency DC-DC converters of buck and boost type can be modeled by the system we consider in this paper.

Our aim is to take into account the nonlinear and switched nature of the system in the control design. This should be contrasted with the traditional approach where the design is based on an averaged model that ignores the high frequency behavior and the inherent time delay due to sampling [1]. Recently, there have been several contributions where hybrid techniques successfully are used for design and analysis of power conversion devises [5], [9], [11]. In [5], [9], dynamic programming and model predictive control techniques are used to explore the intricacies of the nonlinear switching dynamics. Here we consider design of output feedback controllers.

For a precise description of the system we derive a sampled-data model for the error dynamics and a corresponding robust performance measure in terms of lifted variables. This gives us an exact characterization of the induced gain from the disturbance to the output and it can be used to derive analysis tools for power converters. The aim of this paper is, however, not analysis but robust control design for the switching system. The corresponding optimal control problem based on the sampled-data model and the robust performance constraint is highly nonlinear and computationally intractable. By considering the small signal behavior near a desired operating point, we arrive at a sampled-data $\mathcal{H}_\infty$ problem which has reasonable computational complexity, where the switching dynamics is still explicitly taken into account. By extending the loop shifting technique to this situation we derive a corresponding discrete-time $\mathcal{H}_\infty$ control problem that can be solved using standard techniques. We apply the method to design a controller for a boost converter that is subject to a large load disturbance.

Due to the page limitation, all technical results in this paper are stated without proofs.

Notation

For a given continuous-time signal f, a discrete-time signal defined by ideal sampling of f will be denoted $\bar{f}$:

$$\bar{f}_k = f(kT)$$

where $T > 0$ is the sampling period. Also, $\hat{f}$ denotes the lifted signal of f:

$$\hat{f}_k(\theta) = f(kT + \theta), \quad \theta \in [0, T).$$

If G is an STPBC (system with two point boundary conditions) over the interval $[t_0, t_1]$:

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad M_0 x(t_0) + M_1 x(t_1) = 0,$$

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then we write

$$G \overset{\mathrm{STPBC}}{=} \left(\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right], M_0, M_1 \right)$$

where it is assumed that : $M_0 + M_1 \mathrm{e}^{A(t_1 - t_0)}$ is invertible. The notation is simplified to

$$G \overset{\mathrm{STPBC}}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

when $M_1 = 0$. In the derivation of our results we exploit state space formulas for algebraic manipulation of these STPBC operators introduced in [10].

II. Problem Formulation

In this section, we formulate the control synthesis problem considered in this paper.

A. System Description

Consider the switching system governed by

$$\begin{aligned} \dot{x}(t) &= \begin{cases} A_1 x(t) + B_1 + D_1 w(t) & \text{when } t \in T_1 \\ A_2 x(t) + B_2 + D_2 w(t) & \text{when } t \in T_2 \end{cases} \\ v(t) &= Cx(t) \\ y_k &= Mx(kT) \end{aligned} \tag{1}$$

where intervals T_1 and T_2 are $[kT,\, kT + d_k T)$ and $[kT + d_k T,\, (k+1)T)$, respectively. The positive integer T is the sampling period and the signal $w(t)$ is a finite-energy external disturbance which is assumed to have bounded amplitude. Namely, we assume that $w \in \mathbf{L}_2 \cap \mathbf{L}_\infty$. The duty ratio d is a discrete-time signal whose elements d_k belong to $[0,\, 1]$ for any k. Symbol v denotes the continuous-time scalar signal to be regulated, and y represents the discrete-time measurement output based on which the feedback controller is to be designed.

The lifting representation of system (1) is given by

$$\left\{ \begin{array}{rcl} \bar{x}_{k+1} &=& \Phi_{d_k} \bar{x}_k + \Gamma_{d_k} + \Upsilon_{d_k} \hat{w}_k \\ \hat{v}_k &=& \Psi_{d_k} \bar{x}_k + \Theta_{d_k} + \Delta_{d_k} \hat{w}_k \\ y_k &=& M \bar{x}_k \end{array} \right. \tag{2}$$

where $\Phi_{\mathsf{d}}\colon \mathbb{R}^n \to \mathbb{R}^n$, $\Gamma_{\mathsf{d}}\colon \mathbb{R} \to \mathbb{R}^n$, $\Psi_{\mathsf{d}}\colon \mathbb{R}^n \to \mathbf{L}_2[0,\, T]$, and $\Theta_{\mathsf{d}}\colon \mathbb{R} \to \mathbf{L}_2[0,\, T]$ are defined with a parameter $\mathsf{d} \in [0,\, 1]$

$$\begin{aligned} \left[\begin{array}{cc} \Phi_{\mathsf{d}} & \Gamma_{\mathsf{d}} \end{array} \right] &:= \left[\begin{array}{cc} I_n & 0 \end{array} \right] e^{\check{A}_2 (1-\mathsf{d}) T} e^{\check{A}_1 \mathsf{d} T}, \\ \left[\begin{array}{cc} \Psi_{\mathsf{d}} & \Theta_{\mathsf{d}} \end{array} \right] &:= \Omega_{\mathsf{d}}. \end{aligned}$$

In the above expression, $\Omega_{\mathsf{d}}\colon \mathbb{R}^{n+1} \to \mathbf{L}_2[0,\, T]$ is an operator defined as

$$(\Omega_{\mathsf{d}} \underline{x})(\theta) := \left\{ \begin{array}{ll} (\Omega_{1,\mathsf{d}} \underline{x})(\theta), & \text{when } \theta \in [0,\, \mathsf{d}T) \\ (\Omega_{2,\mathsf{d}} \mathrm{e}^{\check{A}_1 \mathsf{d}T} \underline{x})(\theta), & \text{when } \theta \in [\mathsf{d}T,\, T) \end{array} \right.$$

$$\Omega_{1,\mathsf{d}} : \mathbb{R}^{n+1} \to \mathbf{L}_2[0, \mathsf{d}T], \quad \Omega_{1,\mathsf{d}} \underline{x} := \check{C} \mathrm{e}^{\check{A}_1 \theta} \underline{x},$$

$$\Omega_{2,\mathsf{d}} : \mathbb{R}^{n+1} \to \mathbf{L}_2[\mathsf{d}T, T], \quad \Omega_{2,\mathsf{d}} \underline{x} := \check{C} \mathrm{e}^{\check{A}_2 (\theta - \mathsf{d}T)} \underline{x}.$$

where matrices $\check{A}_1$, $\check{A}_2$, and $\check{C}$ are of the following forms

$$\check{A}_1 := \begin{bmatrix} A_1 & B_1 \\ 0 & 0 \end{bmatrix}, \quad \check{A}_2 := \begin{bmatrix} A_2 & B_2 \\ 0 & 0 \end{bmatrix}, \quad \check{C} := \left[\begin{array}{cc} C & 0 \end{array} \right].$$

The other two d-dependent operators in (2), $\Upsilon_{\mathsf{d}}\colon \mathbf{L}_2[0,\, T] \to \mathbb{R}^n$ and $\Delta_{\mathsf{d}}\colon \mathbf{L}_2[0,\, T] \to \mathbf{L}_2[0,\, T]$, have the forms described below:

$$\Upsilon_{\mathsf{d}} = \mathrm{e}^{A_2 (1-\mathsf{d}) T} \Upsilon_{1,\mathsf{d}} \mathcal{P}_{1,\mathsf{d}} + \Upsilon_{2,\mathsf{d}} \mathcal{P}_{2,\mathsf{d}},$$

$$(\Delta_{\mathsf{d}} \hat{w}_k)(\theta) = \left\{ \begin{array}{l} (\Delta_{11,\mathsf{d}} \mathcal{P}_{1,\mathsf{d}} \hat{w}_k)(\theta), \text{when } \theta \in [0,\, \mathsf{d}T) \\ ((\Delta_{21,\mathsf{d}} \mathcal{P}_{1,\mathsf{d}} + \Delta_{22,\mathsf{d}} \mathcal{P}_{2,\mathsf{d}}) \hat{w}_k)(\theta), \\ \qquad\qquad\qquad\qquad \text{when } \theta \in [\mathsf{d}T,\, T) \end{array} \right.$$

where $\mathcal{P}_{1,\mathsf{d}} : \mathbf{L}_2[0,\, T] \to \mathbf{L}_2[0,\, \mathsf{d}T]$ and $\mathcal{P}_{2,\mathsf{d}} : \mathbf{L}_2[0,\, T] \to \mathbf{L}_2[\mathsf{d}T,\, T]$ are truncation operators which work as follows

$$\begin{aligned} f_1 &= \mathcal{P}_{1,\mathsf{d}} f = f(t), \quad t \in [0, \mathsf{d}T], \\ f_2 &= \mathcal{P}_{2,\mathsf{d}} f = f(t), \quad t \in [\mathsf{d}T, T]. \end{aligned}$$

Namely, signal f_1 is signal f restricted in time interval $[0,\, \mathsf{d}T]$, while signal f_2 takes the value of f in the interval $[\mathsf{d}T,\, T]$. Finally, characterizations of operators $\Upsilon_{1,\mathsf{d}}\colon \mathbf{L}_2[0,\, \mathsf{d}T] \to \mathbb{R}^n$, $\Upsilon_{2,\mathsf{d}}\colon \mathbf{L}_2[\mathsf{d}T,\, T] \to \mathbb{R}^n$, $\Delta_{11,\mathsf{d}}\colon \mathbf{L}_2[0,\, \mathsf{d}T] \to \mathbf{L}_2[0,\, \mathsf{d}T]$, $\Delta_{21,\mathsf{d}}\colon \mathbf{L}_2[0,\, \mathsf{d}T] \to \mathbf{L}_2[\mathsf{d}T,\, T]$, and $\Delta_{22,\mathsf{d}}\colon \mathbf{L}_2[\mathsf{d}T,\, T] \to \mathbf{L}_2[\mathsf{d}T,\, T]$ are given in terms of integral forms and systems with two point boundary conditions:

$$\Upsilon_{1,\mathsf{d}} w_1 := \int_0^{\mathsf{d}T} \mathrm{e}^{A_1 (\mathsf{d}T - \tau)} D_1 w_1(\tau)\, \mathrm{d}\tau,$$

$$\Upsilon_{2,\mathsf{d}} w_2 := \int_{\mathsf{d}T}^{T} \mathrm{e}^{A_2 (T - \tau)} D_2 w_2(\tau)\, \mathrm{d}\tau,$$

$$\Delta_{11,\mathsf{d}} \overset{\mathrm{STPBC}}{=} \left[\begin{array}{c|c} A_1 & D_1 \\ \hline C & 0 \end{array} \right], \quad \Delta_{22,\mathsf{d}} \overset{\mathrm{STPBC}}{=} \left[\begin{array}{c|c} A_2 & D_2 \\ \hline C & 0 \end{array} \right],$$

$$\Delta_{21,\mathsf{d}} := \Omega_{21,\mathsf{d}} \Upsilon_{1,\mathsf{d}},$$

where

$$\Omega_{21,\mathsf{d}} := \Omega_{2,\mathsf{d}} \left[\begin{array}{c} I_n \\ 0 \end{array} \right] = C \mathrm{e}^{A_2 (\theta - \mathsf{d}T)}. \tag{3}$$

Note that $\Delta_{22,\mathsf{d}}$ operates over the time horizon $[\mathsf{d}T, T)$. The derivation of (2) is straightforward by noting that one has the following expressions for x:

$$\hat{x}_k(\theta) = \left\{ \begin{array}{l} \mathrm{e}^{A_1 \theta} \hat{x}_k(0) + \displaystyle\int_0^{\theta} \mathrm{e}^{A_1 (\theta - \tau)} B_1 \mathrm{d}\tau \\ \quad + \displaystyle\int_0^{\theta} \mathrm{e}^{A_1 (\theta - \tau)} D_1 \hat{w}_k(\theta)\, \mathrm{d}\tau, \quad \theta \in [0, d_k T) \\ \mathrm{e}^{A_2 (\theta - d_k T)} \hat{x}_k(d_k T) + \displaystyle\int_{d_k T}^{\theta} \mathrm{e}^{A_2 (\theta - \tau)} B_2 \mathrm{d}\tau \\ \quad + \displaystyle\int_{d_k T}^{\theta} \mathrm{e}^{A_2 (\theta - \tau)} D_2 \hat{w}_k(\theta)\, \mathrm{d}\tau, \quad \theta \in [d_k T, T) \end{array} \right.$$

and the standard formula for matrix exponentials:

$$\mathrm{e}^{\check{A}_i \theta} = \begin{bmatrix} \mathrm{e}^{A_i \theta} & \displaystyle\int_0^{\theta} \mathrm{e}^{A_i (\theta - \tau)} B_i\, \mathrm{d}\tau \\ 0 & 1 \end{bmatrix}.$$

We assume that the *disturbance free* plant (1) attains a periodic solution x^0 of period T if the duty ratio d is set to d^0 and the system is initialized by $x(0) = \mathsf{x}^0$ where $\mathsf{x}^0 := x^0(0)$. The periodicity of x^0 implies that

$$\mathsf{x}^0 = \Phi \mathsf{x}^0 + \Gamma$$

or equivalently

$$(I - \mathrm{e}^{\check{A}_2(1-\mathsf{d}^0)T}\mathrm{e}^{\check{A}_1\mathsf{d}^0T})\check{\mathsf{x}}^0 = 0, \quad \text{where } \check{\mathsf{x}}^0 := \begin{bmatrix} \mathsf{x}^0 \\ 1 \end{bmatrix}. \quad (4)$$

and $\Phi := \Phi_{\mathsf{d}^0}$, $\Gamma := \Gamma_{\mathsf{d}^0}$. The continuous-time output corresponding to $x^0(t)$, denoted by $v^0(t)$, is also periodic. The signal satisfies

$$\hat{v}_k^0 = \Psi \mathsf{x}^0 + \Theta \quad (5)$$

for any k, where $\Psi := \Psi_{\mathsf{d}^0}$, $\Theta := \Theta_{\mathsf{d}^0}$.

B. Problem Formulation

The objective of our control design is to ensure asymptotic convergence of the solution of (1) to a T-periodic solution x^0 in such a way that the energy amplification γ is small; i.e., ultimately we would like to minimize γ subject to

$$\int_0^\infty \left(|v(t) - v^0(t)|^2 \; - \gamma^2 |w(t)|^2\right) \, \mathrm{d}t \le 0 \quad (6)$$

for all $w \in \mathbf{L}_2$. For the purpose, we derive a sampled-data model for the error dynamics in the next section.

III. Sampled-Data Model for Error Dynamics

Let the error between x and x^0 be $z := x - x^0$ and let $e := v - v^0$. Using (2), (4) and (5), one can derive a lifted system for the error dynamics:

$$\begin{aligned} \bar{z}_{k+1} &= \Phi_{d_k}\bar{z}_k + \tilde{\Gamma}_{d_k} + \Upsilon_{d_k}\hat{w}_k \\ \hat{e}_k &= \Psi_{d_k}\bar{z}_k + \tilde{\Theta}_{d_k} + \Delta_{d_k}\hat{w}_k \end{aligned} \quad (7)$$

where $\tilde{\Gamma} : \mathbb{R} \to \mathbb{R}^n$ and $\tilde{\Theta} : \mathbb{R} \to \mathbf{L}_2[0, T]$ are defined as

$$\begin{aligned} \tilde{\Gamma}_{\mathsf{d}} &:= \begin{bmatrix} I_n & 0 \end{bmatrix} (\mathrm{e}^{\check{A}_2(1-\mathsf{d})T}\mathrm{e}^{\check{A}_1\mathsf{d}T} - \mathrm{e}^{\check{A}_2(1-\mathsf{d}^0)T}\mathrm{e}^{\check{A}_1\mathsf{d}^0T})\check{\mathsf{x}}^0, \\ \tilde{\Theta}_{\mathsf{d}} &:= (\Omega_{\mathsf{d}} - \Omega)\check{\mathsf{x}}^0, \quad \Omega := \Omega_{\mathsf{d}^0}. \end{aligned}$$

The energy of e can be expressed in terms of $\bar{z}_k$ and $\hat{w}_k$

$$\begin{aligned} \int_0^\infty |e(t)|^2 \mathrm{d}t &= \sum_{k=0}^\infty \int_0^T |\hat{e}_k(\theta)|^2 \mathrm{d}\theta \\ &:= \sum_{k=0}^\infty \langle \hat{e}_k, \hat{e}_k \rangle = \sum_{k=0}^\infty \left\langle \begin{bmatrix} \bar{z}_k \\ 1 \\ \hat{w}_k \end{bmatrix}, \mathcal{Q}(d_k) \begin{bmatrix} \bar{z}_k \\ 1 \\ \hat{w}_k \end{bmatrix} \right\rangle \end{aligned} \quad (8)$$

where

$$\begin{aligned} \mathcal{Q}(d_k) &:= \begin{bmatrix} \mathcal{Q}_{11}(d_k) & \mathcal{Q}_{12}(d_k) & \mathcal{Q}_{13}(d_k) \\ \mathcal{Q}_{12}^*(d_k) & \mathcal{Q}_{22}(d_k) & \mathcal{Q}_{23}(d_k) \\ \mathcal{Q}_{13}^*(d_k) & \mathcal{Q}_{23}^*(d_k) & \mathcal{Q}_{33}(d_k) \end{bmatrix} \\ &= \begin{bmatrix} \begin{pmatrix} I_n \\ 0 \end{pmatrix} & \check{\mathsf{x}}^0 & 0 \\ 0 & -\check{\mathsf{x}}^0 & 0 \\ 0 & 0 & I \end{bmatrix}^* \Pi(d_k) \begin{bmatrix} \begin{pmatrix} I_n \\ 0 \end{pmatrix} & \check{\mathsf{x}}^0 & 0 \\ 0 & -\check{\mathsf{x}}^0 & 0 \\ 0 & 0 & I \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \Pi(d_k) &= \begin{bmatrix} \Pi_{11}(d_k) & \Pi_{12}(d_k) & \Pi_{13}(d_k) \\ \Pi_{12}^*(d_k) & \Pi_{22} & \Pi_{23}(d_k) \\ \Pi_{13}^*(d_k) & \Pi_{23}^*(d_k) & \Pi_{33}(d_k) \end{bmatrix} \\ &:= \begin{bmatrix} \Omega_{d_k}^* \\ \Omega^* \\ \Delta_{d_k}^* \end{bmatrix} \begin{bmatrix} \Omega_{d_k} & \Omega & \Delta_{d_k} \end{bmatrix}. \end{aligned}$$

We observe that

$$\begin{aligned} \mathcal{Q}_{11}(d_k) &= \begin{pmatrix} I_n & 0 \end{pmatrix} \Pi_{11}(d_k) \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \\ \mathcal{Q}_{12}(d_k) &= \begin{pmatrix} I_n & 0 \end{pmatrix} (\Pi_{11}(d_k) - \Pi_{12}(d_k))\check{\mathsf{x}}^0, \\ \mathcal{Q}_{22}(d_k) &= (\check{\mathsf{x}}^0)'(\Pi_{11}(d_k) - \Pi_{12}(d_k) - \Pi_{12}^*(d_k) + \Pi_{22})\check{\mathsf{x}}^0, \\ \mathcal{Q}_{13}(d_k) &= \begin{pmatrix} I_n & 0 \end{pmatrix} \Pi_{13}(d_k), \\ \mathcal{Q}_{23}(d_k) &= (\check{\mathsf{x}}^0)'(\Pi_{13}(d_k) - \Pi_{23}(d_k)), \;\; \mathcal{Q}_{33}(d_k) = \Pi_{33}(d_k). \end{aligned}$$

IV. Design of Feedback Control

In our control problem we have access to the measurement output $\mathcal{Y} = \{y_1, y_2, \ldots, y_k, \ldots\}$, the reference signal $\mathsf{v}_{\mathrm{ref}}$, and the stationary duty ratio d^0. Ideally we would like to optimize the control policy by solving the optimal control problem

$$\begin{aligned} &\min_{d(\mathcal{Y})} \gamma \;\; \text{subject to} \\ &\sum_{k=0}^\infty \int_0^T |\hat{e}_k(\theta)|^2 - \gamma^2 |\hat{w}_k(\theta)|^2 d\theta \le 0 \\ = &\min_{d(\mathcal{Y})} \gamma \;\; \text{subject to} \\ &\sum_{k=0}^\infty L(d_k, \bar{z}_k, \hat{w}_k) \le \sum_{k=0}^\infty \int_0^T \gamma^2 |\hat{w}_k(\theta)|^2 d\theta \end{aligned} \quad (9)$$

where the error dynamics model is defined as in (7) and

$$L(\mathsf{d}, \mathsf{z}, \hat{w}) := \left\langle \begin{bmatrix} \mathsf{z} \\ 1 \\ \hat{w} \end{bmatrix}, \mathcal{Q}(\mathsf{d}) \begin{bmatrix} \mathsf{z} \\ 1 \\ \hat{w} \end{bmatrix} \right\rangle \quad (10)$$

and $\mathcal{Q}(\mathsf{d})$ is defined in (8). Note that in (9) the variable to be optimized is the duty ratio signal $d(\mathcal{Y})$, where the notation emphasizes the dependency of the signal d on $\mathcal{Y}$. Despite the nice structure of this optimization problem it is highly nonlinear and γ is generally finite only when the disturbance input w is small and the state is restricted within some neighborhood of the origin where the error model (7) is stabilizable. Therefore, in the rest of the paper we will consider a quadratic approximation of $L(\mathsf{d}, \mathsf{z}, \hat{w})$ together with a linearized error dynamics for (7). This leads to a sampled-data $\mathcal{H}_\infty$ control design problem. By solving such a problem, a controller which locally stabilizes system (7) against small amplitude disturbance w can be obtained.

A. Small Perturbation Model for J

We proceed with computing the dominating term of L in a small neighborhood around the origin. Suppose that $\bar{z}_k \approx 0$, and w is small, then

$$L(d_k, \bar{z}_k, \hat{w}_k) = \tilde{L}(u_k, \bar{z}_k, \hat{w}_k) + O((\bar{z}_k, u_k, \hat{w}_k)^3) \quad (11)$$

where u_k is equal to $d_k - \mathsf{d}^0$. The quadratic approximation $\tilde{L}(u_k, \bar{z}_k, \hat{w}_k)$ has the explicit form given in the next lemma.

Lemma 1: $\tilde{L}$ is of the form

$$\left\langle \begin{bmatrix} \bar{z}_k \\ u_k \\ \hat{w}_k \end{bmatrix}, \begin{bmatrix} \Psi^*\Psi & \Psi^*\bar{\Theta} & \Psi^*\Delta \\ \bar{\Theta}^*\Psi & 2\bar{\Theta}^*\bar{\Theta} & \bar{\Theta}^*\Delta \\ \Delta^*\Psi & \Delta^*\bar{\Theta} & \Delta^*\Delta \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ u_k \\ \hat{w}_k \end{bmatrix} \right\rangle \quad (12)$$

where $\Delta := \Delta_{\mathsf{d}^0}$ and $\bar{\Theta}$: $\mathbb{R} \to \mathbf{L}_2[0,\, T]$ is defined as

$$(\bar{\Theta} \cdot c)(\theta) = \begin{cases} 0 & \theta \in [0, \mathsf{d}^0 T) \\ (\Omega_{21}\eta \cdot c)(\theta) & \theta \in [\mathsf{d}^0 T, T] \end{cases}.$$

In the above expression, Ω_{21} is equal to Ω_{21,d^0} which in turn is equal to $C\mathrm{e}^{\mathrm{A}_2(\theta-\mathsf{d}^0\mathrm{T})}$ according to (3), and

$$\eta := T \begin{bmatrix} A_1 - A_2 & B_1 - B_2 \end{bmatrix} \mathrm{e}^{\check{A}_1 \mathsf{d}^0 T} \check{\mathrm{x}}^0.$$

From Lemma 1, we obtain an equivalent expression for $\tilde{L}$: $\tilde{L}(u_k,\, \bar{z}_k,\, \hat{w}_k) = \langle \hat{\varepsilon}_k, \hat{\varepsilon}_k \rangle$, where

$$\hat{\varepsilon}_k := \begin{bmatrix} \Psi & \bar{\Theta} & \Delta \\ 0 & \bar{\Theta} & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ u_k \\ \hat{w}_k \end{bmatrix}. \tag{13}$$

B. Sampled-Data $\mathcal{H}_\infty$ Control

Now, consider the linearization of the error dynamics (7) around the origin

$$\bar{z}_{k+1} = \Phi \bar{z}_k + \bar{\Gamma} u_k + \Upsilon \hat{\omega}_k \tag{14}$$

where

$$\bar{\Gamma} := \frac{\partial \tilde{\Gamma}}{\partial d_k}(\mathsf{d}^0) = \mathrm{e}^{A_2(1-\mathsf{d}^0)T}\eta, \quad \Upsilon := \Upsilon_{\mathsf{d}^0}.$$

If we restrict our attention to the quadratic approximation $\tilde{L}(u_k,\, \bar{z}_k,\, \hat{w}_k)$ and consider only the linearized error dynamics (14), the the optimal control problem (9) is reduced to the $\mathcal{H}_\infty$ control synthesis problem for the generalized plant:

$$\begin{bmatrix} \bar{z}_{k+1} \\ \hat{\varepsilon}_k \\ \psi_k \end{bmatrix} = \left[\begin{array}{ccc} \Phi & \Upsilon & \bar{\Gamma} \\ \hline \Psi & \Delta & \bar{\Theta} \\ 0 & 0 & \bar{\Theta} \\ \hline M & 0 & 0 \end{array}\right] \begin{bmatrix} \bar{z}_k \\ \hat{w}_k \\ u_k \end{bmatrix} \tag{15}$$

with the measurement output $\psi_k = M\bar{z}_k = y_k - M\mathrm{x}^0$. We will search for a controller of the form

$$\begin{bmatrix} z^K_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} z^K_k \\ \psi_k \end{bmatrix} \tag{16}$$

such that the closed-loop system composed of (15) and (16) is internally stable; i.e.,

$$\rho(A_{c\ell}) < 1, \quad A_{c\ell} := \begin{bmatrix} \Phi + \bar{\Gamma} D_K M & \bar{\Gamma} C_K \\ B_K M & A_K \end{bmatrix}$$

and the $\mathcal{H}_\infty$ norm of the closed-loop system is less than γ; i.e.,

$$\sup_{z \in \mathbb{C},\, |z|>1} \left\| C_{c\ell}(zI - A_{c\ell})^{-1} B_{c\ell} + D_{c\ell} \right\| < \gamma$$

where the norm in the above expression is the $\mathbf{L}_2[0,\, T]$-induced norm and

$$B_{c\ell} := \begin{bmatrix} \Upsilon \\ 0 \end{bmatrix}, \quad C_{c\ell} := \begin{bmatrix} \Psi + \bar{\Theta} D_K M & \bar{\Theta} C_K \\ \bar{\Theta} D_K M & \bar{\Theta} C_K \end{bmatrix},$$

$$D_{c\ell} := \begin{bmatrix} \Delta \\ 0 \end{bmatrix}.$$

The generalized plant (15) has operator entries acting on a space of functions. Applying the same procedure for the standard sampled-data $\mathcal{H}_\infty$ control problem, we derive an equivalent discrete-time problem as below:

Theorem 1: Suppose that $\gamma > \|\Delta\|$. The following two conditions are equivalent:

(i) The closed-loop system composed of (15) and (16) is internally stable and the $\mathcal{H}_\infty$ norm is less than γ.

(ii) The closed-loop system composed of G_γ and (16) is internally stable and the $\mathcal{H}_\infty$ norm is less than γ, where G_γ is a discrete-time generalized plant given by

$$\begin{bmatrix} \bar{z}_{k+1} \\ \epsilon_k \\ \psi_k \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & 0 & \mathcal{D}_{12} \\ M & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ \omega_k \\ u_k \end{bmatrix}, \tag{17}$$

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_2 \end{bmatrix} := \begin{bmatrix} \Phi & \bar{\Gamma} \end{bmatrix} + \Lambda_{03}, \quad \mathcal{B}_1 \mathcal{B}_1^* := \gamma^2 \Lambda_{01},$$

$$\begin{bmatrix} \mathcal{C}_1^* \\ \mathcal{D}_{12}^* \end{bmatrix} \begin{bmatrix} \mathcal{C}_1 & \mathcal{D}_{12} \end{bmatrix} := \Lambda_{02} + \bar{\Theta}^* \bar{\Theta},$$

and

$$\Lambda_0 = \begin{bmatrix} \Lambda_{01} & \Lambda_{03} \\ \Lambda_{03}' & \Lambda_{02} \end{bmatrix}$$
$$:= \begin{bmatrix} \Upsilon & 0 \\ 0 & \begin{bmatrix} \Psi^* \\ \bar{\Theta}^* \end{bmatrix} \end{bmatrix} \begin{bmatrix} \gamma^2 I & -\Delta^* \\ -\Delta & I \end{bmatrix}^{-1} \begin{bmatrix} \Upsilon^* & 0 \\ 0 & \begin{bmatrix} \Psi & \bar{\Theta} \end{bmatrix} \end{bmatrix}.$$

Standard algorithms for discrete-time $\mathcal{H}_\infty$ control can now be applied to (17) to find a controller of the form (16). The remaining tasks are to

(a) verify that $\|\Delta\| < \gamma$, and

(b) compute Λ_0 and $\bar{\Theta}^* \bar{\Theta}$.

In the rest of this session, computational algorithms for (a) and (b) will be developed. First, we note that $\bar{\Theta}^* \bar{\Theta}$ can be easily expressed as

$$\bar{\Theta}^* \bar{\Theta} = \eta' \left(\int_0^{(1-\mathsf{d}^0)T} \mathrm{e}^{A_2'\tau} C' C \mathrm{e}^{A_2 \tau}\, \mathrm{d}\tau. \right) \eta,$$

and hence is easily computable using the standard formula for the integral with matrix exponentials. On the other hand, checking $\|\Delta\| < \gamma$ and computing Λ_0 are more involved. To begin with, we make the following observation. Given a $\mathbf{L}_2[0,\, T]$ signal f, one can express it in a lifted form

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{1,\mathsf{d}^0} f \\ \mathcal{P}_{2,\mathsf{d}^0} f \end{bmatrix}$$

where $f_1 \in \mathbf{L}_2[0,\, \mathsf{d}^0 T]$ and $f_2 \in \mathbf{L}_2[\mathsf{d}^0 T,\, T]$. The lifted form of signals in $\mathbf{L}_2[0,\, T]$ leads to the following equivalent expressions for operators Δ, Υ, and $\begin{bmatrix} \Psi & \bar{\Theta} \end{bmatrix}$

$$\Upsilon = \begin{bmatrix} \mathrm{e}^{A_2(1-\mathsf{d}^0)T}\Upsilon_1 & \Upsilon_2 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_{11} & 0 \\ \Delta_{21} & \Delta_{22} \end{bmatrix},$$
$$\begin{bmatrix} \Psi & \bar{\Theta} \end{bmatrix} = \begin{bmatrix} \Psi_1 & 0 \\ \Omega_{21} \mathrm{e}^{A_1 \mathsf{d}^0 T} & \Omega_{21}\eta \end{bmatrix} \tag{18}$$

where $\Upsilon_1 := \Upsilon_{1,\mathsf{d}^0}$, $\Upsilon_2 := \Upsilon_{2,\mathsf{d}^0}$, $\Omega_{21} := \Omega_{21,\mathsf{d}^0}$, $\Delta_{11} := \Delta_{11,\mathsf{d}^0}$, $\Delta_{21} := \Omega_{21,\mathsf{d}^0}\Upsilon_{1,\mathsf{d}^0}$, $\Delta_{22} := \Delta_{22,\mathsf{d}^0}$, and

$$\Psi_1 := \Omega_{1,\mathsf{d}^0} \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

Using expression (18), one can verify that

$$
\Lambda_0 = \left[\begin{array}{cc|cc} e^{A_2(1-\mathsf{d}^0)T}\Upsilon_1 & 0 & \Upsilon_2 & 0 \\ \hline 0 & \Psi_1^* & 0 & e^{A_1'\mathsf{d}^0T}\Omega_{21}^* \\ 0 & 0 & 0 & \eta'\Omega_{21}^* \end{array}\right]
\times \left[\begin{array}{cc|cc} \gamma^2 I & -\Delta_{11}^* & 0 & -\Delta_{21}^* \\ -\Delta_{11} & I & 0 & 0 \\ \hline 0 & 0 & \gamma^2 I & -\Delta_{22}^* \\ -\Delta_{21} & 0 & \Delta_{22} & I \end{array}\right]^{-1}
\times \left[\begin{array}{c|cc} \Upsilon_1^* e^{A_2'(1-\mathsf{d}^0)T} & 0 & 0 \\ 0 & \Psi_1 & 0 \\ \hline \Upsilon_2^* & 0 & 0 \\ 0 & \Omega_{21}e^{A_1\mathsf{d}^0T} & \Omega_{21}\eta \end{array}\right]. \tag{19}
$$

The matrix inversion lemma gives us that

$$
\left[\begin{array}{cc|cc} \gamma^2 I & -\Delta_{11}^* & 0 & -\Delta_{21}^* \\ -\Delta_{11} & I & 0 & 0 \\ \hline 0 & 0 & \gamma^2 I & -\Delta_{22}^* \\ -\Delta_{21} & 0 & \Delta_{22} & I \end{array}\right]^{-1}
= \begin{bmatrix} \mathcal{T}_1^{-1} & \mathcal{T}_1^{-1}\begin{bmatrix} 0 & \Delta_{21}^* \\ 0 & 0 \end{bmatrix}\mho_2^{-1} \\ \mho_2^{-1}\begin{bmatrix} 0 & 0 \\ \Delta_{21} & 0 \end{bmatrix}\mathcal{T}_1^{-1} & \mathcal{T}_2 \end{bmatrix}
$$

where

$$
\mathcal{T}_1 := \mho_1 - \begin{bmatrix} \gamma^2\Delta_{21}^*(\gamma^2 I - \Delta_{22}\Delta_{22}^*)^{-1}\Delta_{21} & 0 \\ 0 & 0 \end{bmatrix},
$$

$$
\mathcal{T}_2 := \mho_2^{-1} + \mho_2^{-1}\begin{bmatrix} 0 & 0 \\ \Delta_{21} & 0 \end{bmatrix}\mathcal{T}_1^{-1}\begin{bmatrix} 0 & \Delta_{21}^* \\ 0 & 0 \end{bmatrix}\mho_2^{-1},
$$

$$
\mho_1 := \begin{bmatrix} \gamma^2 I & -\Delta_{11}^* \\ -\Delta_{11} & I \end{bmatrix}, \quad \mho_2 := \begin{bmatrix} \gamma^2 I & -\Delta_{22}^* \\ -\Delta_{22} & I \end{bmatrix}.
$$

As an application of the Schur's lemma, the following theorem provides a characterization for $\|\Delta\| < \gamma$:

Theorem 2: The following two statements are equivalent:

1) $\|\Delta\| < \gamma$.
2) $\|\Delta_{22}\| < \gamma$ and $\mathcal{T}_1$ is positive-definite[1].

Hence, the task of checking whether $\|\Delta\| < \gamma$ is split into two steps: checking whether $\|\Delta_{22}\| < \gamma$ and then whether $\mathcal{T}_1$ is positive-definite. Whether $\|\Delta_{22}\| < \gamma$ can be checked by algorithms provided in the literature[2], see for example [2]. Checking whether $\mathcal{T}_1$ is positive definite can be done using the algorithm in [3]. The important step for applying the algorithm in [3] is to obtain an STPBC representation for $\mathcal{T}_1$. To show this, let matrix Λ_2 be

$$
\Lambda_2 = \begin{bmatrix} \Lambda_{21} & \Lambda_{23} \\ \Lambda_{23}' & \Lambda_{22} \end{bmatrix} := \begin{bmatrix} \Upsilon_2 & 0 \\ 0 & \Omega_{21}^* \end{bmatrix}\mho_2^{-1}\begin{bmatrix} \Upsilon_2^* & 0 \\ 0 & \Omega_{21} \end{bmatrix}.
$$

[1]We say that an operator $\mathcal{O}$ on $\mathbf{L}_2$ is positive-definite if there exists a scalar $\varepsilon > 0$ such that

$$
u^*\mathcal{O}u \geq \varepsilon\|u\|_2^2
$$

for all $u \in \mathbf{L}_2$.

[2]That Δ_{22} operates on $\mathbf{L}_2[\mathsf{d}^0T, T]$ creates a slight complication for applying the standard techniques in the literature to obtain a computational algorithm for calculating its norm. However, the difficulty can be easily resolved by applying an appropriate time shift.

Lemma 2: Suppose that $\|\Delta_{22}\| < \gamma$. Then operator $\mathcal{T}_1$ has an STPBC representation

$$
\mathcal{T}_1 \overset{\text{STPBC}}{=} \left(\left[\begin{array}{cc|cc} -A_1' & 0 & 0 & C' \\ 0 & A_1 & D_1 & 0 \\ \hline D_1' & 0 & \gamma^2 I & 0 \\ 0 & -C & 0 & I \end{array}\right], \right.
\left. \begin{bmatrix} 0 & -e^{A_1'\mathsf{d}^0T}\Lambda_{22}e^{A_1\mathsf{d}^0T} \\ 0 & I \end{bmatrix}, \begin{bmatrix} e^{A_1'\mathsf{d}^0T} & e^{A_1'\mathsf{d}^0T}\Lambda_{22} \\ 0 & 0 \end{bmatrix} \right).
$$

Given that $\|\Delta_{22}\| < \gamma$, computational algorithms for Λ_2 can be found in the sampled-data $\mathcal{H}_\infty$ control literature[3], see [1] and references therein for details.

Finally, we end this section by the following theorem which leads to a tractable computational algorithm for calculating Λ_0:

Theorem 3: Suppose that $\|\Delta\| < \gamma$. Then we have

$$
\Lambda_0 = \begin{bmatrix} e^{A_2(1-\mathsf{d}^0)T} + \Lambda_{23} & 0 \\ e^{A_1'\mathsf{d}^0T}\Lambda_{22} & I \\ \eta'\Lambda_{22} & 0 \end{bmatrix} \mathcal{L} \begin{bmatrix} e^{A_2(1-\mathsf{d}^0)T} + \Lambda_{23} & 0 \\ e^{A_1'\mathsf{d}^0T}\Lambda_{22} & I \\ \eta'\Lambda_{22} & 0 \end{bmatrix}',
+ \begin{bmatrix} I & 0 \\ 0 & e^{A_1'\mathsf{d}^0T} \\ 0 & \eta' \end{bmatrix} \Lambda_2 \begin{bmatrix} I & 0 & 0 \\ 0 & e^{A_1\mathsf{d}^0T} & \eta \end{bmatrix}.
$$

The matrix $\mathcal{L}$ is defined as

$$
\mathcal{L} := \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_3 \\ \mathcal{L}_3' & \mathcal{L}_2 \end{bmatrix},
$$

where

$$
\mathcal{L}_1 := -H_{21}\mathcal{V}_1^{-1}e^{A_2'(1-\mathsf{d}^0)T},
$$

$$
\mathcal{L}_2 := -(e^{A_1'\mathsf{d}^0T}H_{11} - I)\mathcal{V}_1^{-1}\mathcal{V}_2 + e^{A_1'\mathsf{d}^0T}H_{12},
$$

$$
\mathcal{L}_3 := -H_{21}\mathcal{V}_1^{-1}\mathcal{V}_2 + H_{22} - e^{A_1\mathsf{d}^0T},
$$

$$
\mathcal{V}_1 := H_{11} + \Lambda_{22}H_{21}, \quad \mathcal{V}_2 := H_{12} + \Lambda_{22}H_{22} - \Lambda_{22}e^{A_1\mathsf{d}^0T},
$$

and

$$
\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} := \exp\left(\begin{bmatrix} -A_1' & C'C \\ -\gamma^{-2}D_1D_1' & A_1 \end{bmatrix} \mathsf{d}^0T \right).
$$

Remark 1: It can be shown that the matrix $\mathcal{L}$ has the expression

$$
\mathcal{L} = \begin{bmatrix} \Upsilon_1 & 0 \\ 0 & \Psi_1^* \end{bmatrix} \mathcal{T}_1^{-1} \begin{bmatrix} \Upsilon_1 & 0 \\ 0 & \Psi_1^* \end{bmatrix}^*.
$$

The formulas stated in Theorem 3 are then derived by using the STPBC representation of $\mathcal{T}_1^{-1}$.

[3]Again, that Δ_{22}, Υ_2 and Ω_{21} operate on $\mathbf{L}_2[\mathsf{d}^0T, T]$ creates a slight complication, but the difficulty can be easily resolved by applying an appropriate time shift.

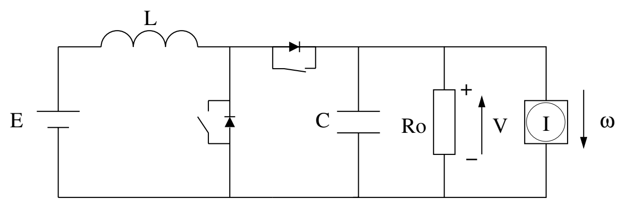

Fig. 1. Bi-directional boost converter with load disturbance modeled as an independent current source.

V. EXAMPLE

The system depicted in Fig. 1 is a bi-directional boost converter with a load disturbance modeled by an independent current source. The system matrices are

$$A_1 = \begin{bmatrix} 0 & -1/l \\ 1/c & -1/(Rc) \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1/(Rc) \end{bmatrix},$$
$$B_1 = B_2 = \begin{bmatrix} E/l \\ 0 \end{bmatrix}, \quad D_1 = D_2 = \begin{bmatrix} 0 \\ -1/c \end{bmatrix},$$
$$C = M = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

with parameter values $E = 1$[V], $R = 30[\Omega]$, $l = 10[\mu\text{H}]$, $c = 50[\mu\text{F}]$, the time period $T = 2 \times 10^{-5}$[s] and reference output voltage $\text{v}_{\text{ref}} = 5$[V]. The example and the parameter values are adopted from the paper [7], where passivity based control is applied to this system.

We have compared the LQ controller in [4] with an $\mathcal{H}_\infty$ controller designed using the ideas presented in this paper. The control parameters for the LQ controller are

$$A_K = \begin{bmatrix} -0.0502 & 0.0249 \\ -0.0405 & 0.0202 \end{bmatrix}, \quad B_K = \begin{bmatrix} 0.5334 \\ 0.4311 \end{bmatrix},$$
$$C_K = \begin{bmatrix} -0.1157 & 0.0575 \end{bmatrix}, \quad D_K = -0.0027.$$

while the control parameters for the $\mathcal{H}_\infty$-controller with $\gamma = 1$ are

$$A_K = \begin{bmatrix} -1.1089 & -0.0162 \\ 0.0879 & 0.0013 \end{bmatrix}, \quad B_K = \begin{bmatrix} -0.0188 \\ 0.0015 \end{bmatrix},$$
$$C_K = \begin{bmatrix} 149.7763 & 2.1787 \end{bmatrix}, \quad D_K = 1.3033.$$

Fig. 2 shows a simulation for the LQ based control. The upper part of the figure shows the output voltage when the system is subjected to a disturbance w in the shape of a rectangular pulse. The lower part shows the pulse which has an amplitude of 0.2[A] and a duration of 0.2[ms]. The (non-saturated) output of the controller is also shown.

Fig. 3 shows the same simulation for the $\mathcal{H}_\infty$ based control. We note that the amplitude of the disturbance input (0.2[A]) is in fact fairly big, compared with the nominal value of the load current which is merely $1/6$[A].

None of our controllers has integral action. Integral action is generally crucial in control of power converters because of variations in the input voltage E. We see that integral action would have been useful for the LQ controller. It is possible to include integral action in the sampled-data $\mathcal{H}_\infty$ design procedure by using methods in, e.g., [6].

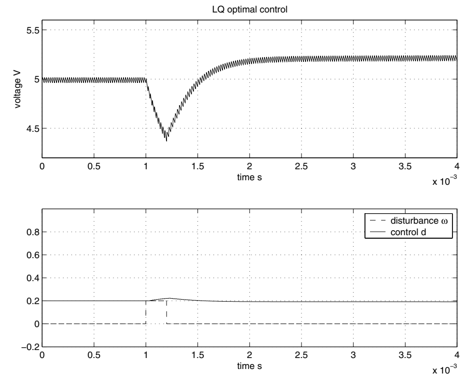

Fig. 2. Above: Output voltage response to current load disturbance ω. Below: Current load disturbance ω and the non-saturated controller output.

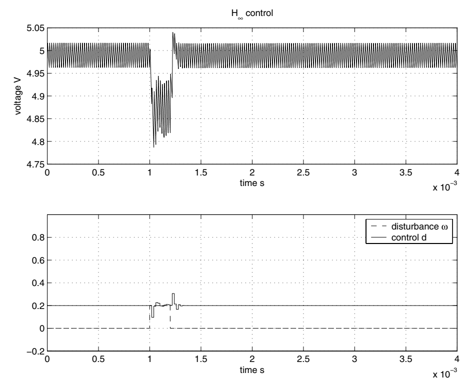

Fig. 3. Above: Output voltage response to current load disturbance ω. Below: Current load disturbance ω and the non-saturated controller output.

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