Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Control oriented uncertainty modelling using μ sensitivities and skewed μ analysis tools

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Abstract— **In this paper, an approach for the computation of bounds on parametric model uncertainty for robust control design is proposed. The approach uses μ sensitivities to identify which uncertain parameters in the model are most critical in terms of closed-loop robustness objectives. Using skewed μ analysis tools, the maximum possible uncertainty bounds for model parameters which are most difficult or expensive to identify exactly are then computed. The application of the proposed approach is illustrated via a flight control law design example.**

I. Introduction

Mathematical models which are developed for control design now generally come with associated uncertainty models. If the model is developed using system identification approaches, control-oriented robust identification techniques [16], [17] are available with which to obtain appropriate nominal models with suitably defined uncertainty representations for robust control design. The standard approach is to fit experimental data to obtain a nominal model and perform error analyses to calculate tolerances on the parameters, i.e. uncertainty bounds. The associated robust model validation problem is then based on an in-validation test of the robust model using a different series of input/output data measurements. In the case of models which have been developed based on physical principles as opposed to system identification, bounds on the maximum expected uncertainty/variation in the values of various parameters are also routinely supplied - see for example [8] for the state-of-the-art in flight mechanics modelling.

A key issue in formulating suitable uncertainty models for robust control design is to understand which uncertain parameters are of most importance, i.e. which uncertain parameters most compromise closed-loop stability and performance properties. Such understanding is traditionally gained via sensitivity analysis of the eigenvalues of the open-loop system. However, as will be shown, the "important" parameters identified by such analyses can be completely different from the parameters which actually limit stability and performance properties once a controller has been designed. One implication of the present study is that, for systems which are to be eventually implemented in closed loop with a controller, *closed-loop* μ sensitivity analysis using a sensible (but perhaps quite preliminary) control design is likely to provide more accurate and useful results.

Since certain model parameters are typically much more difficult (i.e. expensive) to estimate accurately than others, it is also of significant interest to calculate relative uncertainty bounds which allow the maximum possible uncertainty in such parameters.

While the above issues are of interest in practically all areas of control applications, they are of particular significance in aerospace control, where the problems of high performance requirements and significant levels of model uncertainty combine to make control law design and analysis a highly challenging and expensive task. For aerospace systems, look-up tables of aircraft aerodynamic coefficients and stability derivatives are obtained using least-square fitting and mean values from large databases obtained from a mix of tests and analyses such as:

* Simplified linear analysis techniques applied to dense grids of the parameters [10].
* Wind tunnel and computer testing [15], [10].
* In-flight data acquisition and analysis.

The cost of performing these tests and reconciling the data for this type of system can be enormous due to their scale, the intrinsic safety problem associated with performing flight-tests at critical points, and the difficulty of measuring some variables at certain operating points [15], [18]. For all these reasons there is currently a drive to change the modelling process for aerospace systems, i.e. to rely less on very accurate but costly empirical data acquisition (wind-tunnels and in-flight) and more on advanced computational tools [12] and techniques which allow larger tolerances on model parameters [11]. This trend is even more pronounced in UAV applications, where typically, the smaller, lighter airframes are more difficult to model accurately via wind-tunnel tests, and the time available for such tests is only a fraction of what would traditionally be available for a manned aircraft development programme. In this paper, a method is proposed based on μ sensitivities and the skewed μ analysis tool which can be used to maximize the allowable level of uncertainty in certain "difficult" parameters, thus making the complex trade-off between model fidelity and closed-loop performance more transparent and easier to manage.

II. Theoretical Preliminaries

In this section, the system theoretic tools which form the basis of the proposed modelling approach, namely the structured singular value μ, skewed-mu ν, and μ sensitivity are briefly reviewed.

A concept widely used in modern robust control is the structured singular value μ, which allows efficient evaluation of the robustness of complex uncertain systems [14]:

Definition II.1
The structured singular value, $\mu_\Delta(M)$, of a matrix $M \in \mathcal{C}^{n\times n}$

This research was supported under EPSRC research grant $GR/S61874/01$.

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with respect to the uncertain matrix Δ *is:*

$$\mu_\Delta(M) = \frac{1}{min_\Delta\{\bar{\sigma}(\Delta) : det(I - \Delta M) = 0\}} \quad (1)$$

where $\mu_\Delta(M) = 0$ *if there is no* Δ *that satisfies the determinant condition.*

Note that this definition is given in terms of a '$M - \Delta$' model where the M component represents the nominal system at a given frequency point and the Δ-block is a diagonal or block-diagonal matrix containing the uncertainty (or parametric variations) which is scaled to have unit norm, see Fig. 1.

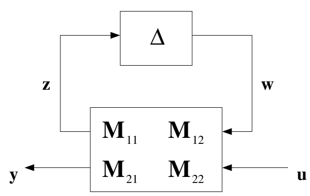

Fig. 1. Upper linear fractional transformation.

From the figure it is seen by inspection that:

$$y = [M_{22} + M_{21}(I - \Delta M_{11})^{-1}\Delta M_{12}]u \quad (2)$$

The linear feedback formulation of an uncertain system given by (2) is known as an upper linear fractional transformation (LFT), and it represents the standard modelling paradigm for robustness analysis. The exact calculation of μ is in general NP-hard [3], and therefore in practice upper and lower bounds are generally calculated for realistic problems: the upper bound provides the maximum allowable size of uncertainty that satisfies the robustness requirements while the lower bound obtains a worst-case (i.e. smallest) uncertainty which violates them [19]. Their combined use provides an indication of the conservatism associated with the upper bound calculation. Numerous algorithms exist for computing bounds on μ for different types of uncertainty - see [1], [6] for a comprehensive overview.

The skewed-mu ν concept [13] is a particular case of μ, where the norm of a subset of the parameters in the uncertain set Δ is not allowed to vary freely:

Definition II.2
The skewed-mu, $\nu_{\Delta_s}(M)$*, of a matrix* $M \in \mathcal{C}^{n\times n}$ *with respect to the uncertain matrix* $\Delta_s = diag(\Delta_v, \Delta_f)$ *is defined as:*

$$\nu_{\Delta_s}(M) = \frac{1}{min_{\Delta_s}\{\bar{\sigma}(\Delta_v) : \ det(I - \Delta_s M) = 0\}} \quad (3)$$

where $|\Delta_f| \leq 1$*. If there is no* Δ_s *that satisfies* $det(I - \Delta_s M) = 0$ *then* $\nu_{\Delta_s}(M) = 0$.

The main task of sensitivity analysis is to provide a measure of the change in the system behaviour due to parameter variations. If the system behaviour is characterized from a frequency-domain perspective by the structured singular value μ, the μ sensitivities allow the identification of the uncertain system parameters most responsible for the level of (or lack of) robustness of the system [2], [5]:

$$Sen^{\mu}_{p_j} = \frac{\partial\mu(M)}{\partial p_j} \approx \frac{|\mu(M) - \mu(M_\epsilon)|}{\Delta p_j} \quad (4)$$

The perturbed system M_ϵ is defined by assuming each uncertainty in the diagonal block Δ is multiplied by a real scalar α_i nominally of value one except for the j^{th} perturbed scalar $\alpha_j = 1 + \epsilon$. This yields a diagonal matrix $\alpha = diag(\alpha_1 = 1, \ldots, \alpha_j = 1 + \epsilon, \ldots, \alpha_n = 1)$ which multiplies the original uncertain block Δ and which can be absorbed into M to form M_ϵ:

$$M_\epsilon = \begin{bmatrix} \alpha M_{11} & \alpha^{1/2}M_{12} \\ \alpha^{1/2}M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} \alpha M_{11} & \alpha M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad (5)$$

The uncertain block Δ can be constructed to include real parametric variations $p = [p_1 \ p_2 \ldots p_n]^\top$, which can be perturbed by $\epsilon = \Delta p_j$. Hence, calculating $\mu(M_\epsilon)$ implies calculating the robustness properties of the perturbed system $M(p + \Delta p_j)$. Note that the versatility of the Δ construction [14] allows the definition of diagonal blocks of repeated parameters which can be perturbed simultaneously, i.e. $\Delta = diag(p_1 I_1, \ p_2 I_2, \ldots p_n I_n)$ where I_i represents an identity matrix of dimension equal to the number of repetitions of the i^{th} parameter. Note however, that μ sensitivities must generally be computed using (upper) μ bounds rather than the actual value of μ as the exact calculation of μ is NP-hard.

III. Control Oriented Uncertainty Modelling

In this section, a method for control oriented uncertainty modelling is proposed based on the use of the analytical tools described in the previous section. The main idea of the approach is to calculate the increase in uncertainty size for a specific set of parameters based on the possible decrease in size for the other parameters while keeping an established level of robust stability or performance. Before describing the method in detail, several observations regarding the skewed μ approach are made.

Firstly, if the norm bounds for the uncertainty sets Δ_f and Δ_v are both left free to vary, the standard μ calculation from Definition II.1 is recovered (i.e. skewed μ is a special case of μ). Indeed, for robustness tests the μ value is an upper/lower bound (depending on whether the test is satisfied or not) on the skewed μ value. Therefore, an algebraic interpretation of skewed μ can be given by a 2-norm distance problem:

$$\min_{s.t. \ |\Delta_f| \leq 1} |\mu_\Delta(M) - \nu_{\Delta_s}(M)| \quad (6)$$

This means that for a fixed subset of parameters in the uncertain set Δ_s and a given $\mu_\Delta(M)$, the goal of the skewed μ calculation is to find the minimum k_ν of Δ_v which minimizes (6) subject to the conditions from Definition II.2.

Secondly, an alternative interpretation in terms of sensitivity analysis for the skewed-mu calculation is provided in the following lemma:

Lemma III.1 (Skewed $\mu \equiv \mu$ Sensitivity Analysis)
The skewed μ *calculation of a matrix* $M \in \mathcal{C}^{n\times n}$ *with respect to the uncertain set* $\Delta_s = diag(\Delta_v, \Delta_f)$ *is equivalent to a* μ

sensitivity analysis of the nominal system M with respect to perturbations of the set Δ_v.

Proof: First, if a skewed μ calculation on M is performed with $\Delta_s = diag(\Delta_v, \Delta_f)$, we get from Definition II.2 that $|\Delta_v| = k_\nu = \frac{1}{\nu}$ and $|\Delta_f| \leq 1$. Furthermore, the resulting uncertain matrix Δ_s can be transformed as:

$$\begin{aligned}\Delta_s &= \begin{bmatrix}\Delta_v & 0\\ 0 & \Delta_f\end{bmatrix} = \begin{bmatrix}k_\nu \bar{\Delta}_v & 0\\ 0 & \Delta_f\end{bmatrix}\\ &= \begin{bmatrix}k_\nu & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\bar{\Delta}_v & 0\\ 0 & \Delta_f\end{bmatrix} = \begin{bmatrix}1+\epsilon_\nu & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\bar{\Delta}_v & 0\\ 0 & \Delta_f\end{bmatrix}\end{aligned} \quad (7)$$

with $|\bar{\Delta}_v| = 1$.

Absorbing the matrix $\alpha_\nu = diag(k_\nu, I)$ into M following equation (5), the matrix M_{ϵ_ν} is obtained:

$$M_{\epsilon_\nu} = \begin{bmatrix}M_{11}^\epsilon & M_{12}^\epsilon\\ M_{21}^\epsilon & M_{22}^\epsilon\end{bmatrix} = \begin{bmatrix}k_\nu M_{11} & k_\nu M_{12}\\ M_{21} & M_{22}\end{bmatrix} \quad (8)$$

Note that both the μ and ν algorithms, look to satisfy $det(I - \Delta M) = 0$. Therefore, for the above skewed-mu calculation on M and Δ_s:

$$det(I - \Delta_s M) = det\begin{pmatrix}I - \Delta_v M_{11} & -\Delta_v M_{12}\\ -\Delta_f M_{21} & I - \Delta_f M_{22}\end{pmatrix} \quad (9)$$

Similarly for $\mu(M_{\epsilon_\nu})$ with $\bar{\Delta}_s = diag(\bar{\Delta}_v, \Delta_f)$:

$$\begin{aligned}det(I - \bar{\Delta}_s M_{\epsilon_\nu}) &= det\begin{pmatrix}I - \bar{\Delta}_v M_{11}^\epsilon & -\bar{\Delta}_v M_{12}^\epsilon\\ -\Delta_f M_{21}^\epsilon & I - \Delta_f M_{22}^\epsilon\end{pmatrix}\\ &= det\begin{pmatrix}I - (\Delta_v \frac{1}{k_\nu})(k_\nu M_{11}) & -(\Delta_v \frac{1}{k_\nu})(k_\nu M_{12})\\ -\Delta_f M_{21} & I - \Delta_f M_{22}\end{pmatrix}\end{aligned} \quad (10)$$

which is equal to the determinant in equation (9). Therefore, both calculations are equivalent (in reality, $\nu_{\Delta_s}(M) = \frac{1}{k_\nu}\mu_{\bar{\Delta}_s}(M_\epsilon)$).

Now, note that performing a μ calculation of $(M_{\epsilon_\nu}, \bar{\Delta}_s)$ is indeed the same as calculating μ for a nominal system $(M, \bar{\Delta}_s)$ which is perturbed by $\epsilon_\nu = k_\nu - 1$ along the input-output uncertain channels of $\bar{\Delta}_v \in \bar{\Delta}_s$.

Finally, using the algebraic interpretation of skewed μ from (6), together with the equivalence between $\mu_{\bar{\Delta}_s}(M_{\epsilon_\nu})$ and $\nu_{\Delta_s}(M)$, and the definition of μ-sensitivity (4) yields:

$$|\mu_\Delta(M) - \nu_{\Delta_s}(M)| \equiv |\mu_\Delta(M) - \mu_{\bar{\Delta}_s}(M_{\epsilon_\nu})| \approx \epsilon_\nu \, Sen^\mu_{\Delta_v} \quad (11)$$

from which the equivalence of ν and μ-sensitivity can be clearly seen. ■

Thirdly, it is noted that the quality of the skewed μ bound calculation is due to both the number of uncertain parameters that are fixed, i.e. contained in Δ_f, and to the choice of parameters whose norm is to be optimized, i.e. contained in Δ_v. Furthermore, the uncertain parameters that when fixed maximize the allowable uncertainty for the rest of the parameters are those with the largest μ sensitivities. These observations provide guidelines for the placement of the uncertainties in Δ_f or Δ_v, which can be summarised in the following result:

Lemma III.2 (skewed μ Uncertainty Classification) *Given an uncertain set Δ and a complex matrix $M \in \mathcal{C}^{n\times n}$, the decomposition of Δ into $\Delta_s = diag(\Delta_v, \Delta_f)$ where $\Delta_f = \{\delta_i \in \Delta : |\delta_i| \leq 1\}$ and $\Delta_v = \{\delta_i \in \Delta/\Delta_f : |\delta_i| \leq k_v\}$ such that the resulting skew μ value, $\nu = \frac{1}{k_v}$, maximizes $|\mu_\Delta(M) - \nu_{\Delta_s}(M)|$ requires:*

1 *The dimension of Δ_f must be maximized with respect to that Δ_v.*

2 *The uncertain parameters with large μ sensitivities must be placed in Δ_f.*

Proof: The first property is proven by considering the geometric interpretation of ν [6], shown in Fig. 2, where for simplicity of presentation a three dimensional parameter space is assumed:

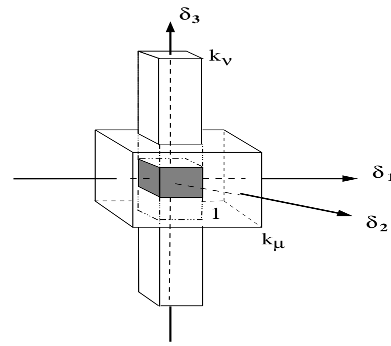

Fig. 2. Geometric interpretation of real $\nu(M)$ and $\mu(M)$.

Assuming the system satisfies the closed-loop robustness objectives, i.e. $\mu \leq 1$, the calculation of μ finds a hypercube in the parameter space of size $k_\mu \geq 1$. The skewed μ computation works by initially using a hypercube of size one inside the μ hypercube (i.e. the shaded cube in Fig. 2 with $|\Delta_f| = 1$ and $|\Delta_v| = 1$). It then starts increasing the norm k_ν for those parameters that are free to vary, $\delta_i \in \Delta_v$, until a solution to $det(I - \Delta_s M) = 0$ is obtained (iterating until the minimum value of k_ν that satisfies the determinant is calculated). For the case of three parameters with two δ's fixed, this search for the minimum k_ν yields a co-axial hyper-rectangle along the axis of the free parameter, e.g. δ_3 in Fig. 2, with a norm $k_\nu > k_\mu$.

The above shows that when uncertain parameters are fixed, the skewed μ yields a larger uncertainty size for the rest of the parameters (i.e. those in Δ_v). Furthermore, it is well-known that as the uncertainty is reduced in a system, the robustness measure μ decreases for a given controller (i.e. the norm k_μ increases). Equivalently, as more parameters are fixed (i.e. their uncertainty size is assumed to be smaller than k_μ), the parameters in Δ_v must compensate for the more stringent restriction in size and a larger value of k_ν is obtained for them. Note that this results in relative bounds on the uncertain parameters, with those corresponding to "free" parameters (i.e. in Δ_v) being maximized.

The second condition requires the use of the equivalence result from Lemma III.1. From the definition of μ sensitivity (4), it is observed that parameters with large μ-sensitivities result in large differences in $|\mu(M) - \mu(M_\epsilon)|$ for small ϵ variations in their corresponding input-output channels. Therefore, assuming these parameters are perturbed toward the inside of the hypercube by ϵ (i.e. their uncertainty is reduced or equivalently they are fixed to a lower stability margin $k < k_\mu$), then by similar arguments to the above discussion based on the geometric interpretation of Fig. 2, the other parameters will achieve a larger uncertain norm $k_{\nu_{Shigh}} > k_\mu$. Similarly, if the parameters with low sensitivity are perturbed by the same ϵ in the same manner (i.e. fixed within the hypercube k_μ), the rest of the parameters will achieve a norm $k_{\nu_{Slow}} > k_\mu$ but smaller than $k_{\nu_{Shigh}}$ as the net change is smaller.

Using (11) it is then concluded that the higher the μ sensitivity of the parameters in Δ_f, the larger the difference $|\mu(M)-\mu(M_\epsilon)|$ and hence, smaller the value of $\nu(M) = \frac{1}{k_\nu}$ (larger size for uncertain parameters in Δ_v). ■

In the above discussion, it has been assumed the calculation of μ was exact. Recall that in practice this is NP-hard and upper and lower bounds are calculated. Since the interest of this research is to find the worst-case combination of the uncertain parameters, the μ calculation will be always performed using lower bound algorithms.

Using these results, it can be concluded that by selecting as components of the fixed uncertainty set Δ_f: a) all those parameters which are easy or inexpensive to estimate with a high level of accuracy, and b) those parameters with largest μ-sensitivities, a skewed μ lower bound calculation for the resulting system will provide the largest possible allowable uncertainty bounds for the remaining parameters contained in the uncertain set Δ_v.

Finally, in order to represent the relative change in uncertainty size, the uncertain parameters are defined based on a nominal value c_o, a specified percentage σ_c, and a normalized uncertain parameter $|\delta| \leq 1$, i.e. $c = c^o(1+\sigma_c\delta)$. Then, if the skewed μ computation finds that the normalized δ can be allowed to achieve a larger norm $\bar{\delta} = \bar{\sigma}_c\delta$ then the actual percentage level of uncertainty that is allowable for the parameter c is given by $\sigma_c\bar{\sigma}_c$.

IV. A Flight Control Application

A. Sensitivity Analysis of an Aircraft Model

In this section, a sensitivity analysis of the aircraft model used to illustrate the proposed approach to uncertainty modelling is presented. The equations of motion and aerodynamic data are taken from reference [6] and represent the lateral/directional motion of a conventional rigid transport aircraft.

There are four states: sideslip angle β (rad), roll rate p (rad/sec), yaw rate r (rad/sec), roll angle ϕ (rad), and four output signals: lateral acceleration n_y (g), roll and yaw rates, and roll angle. Control is performed through the aileron δ_r and rudder δ_p deflections (the same nomenclature as in [6] is followed). The rigid aircraft equations of motion, with no wind effects or flexible modes, obtained after linearizing the nonlinear model at the equilibrium point given by the initial angles of attack and pitch (α_o, θ_o) are as follows:

$$\begin{aligned}\dot{\beta} =& Y_\beta\beta + (Y_p + sin\alpha_o)p + (Y_r - cos\alpha_o)r \\ &+ \frac{g}{V}\phi + Y_{\delta_p}\delta_p + Y_{\delta_r}\delta_r\end{aligned} \tag{12}$$

$$\dot{p} = L_\beta\beta + L_p p + L_r r + L_{\delta_p}\delta_p + L_{\delta_r}\delta_r \tag{13}$$

$$\dot{r} = N_\beta\beta + N_p p + N_r r + N_{\delta_r}\delta_r \tag{14}$$

$$\dot{\phi} = p + tan\theta_o r \tag{15}$$

$$n_y = -\frac{V}{g}(Y_\beta\beta + Y_p p + Y_r r + Y_{\delta_p}\delta_p + Y_{\delta_r}\delta_r) \tag{16}$$

The stability derivatives are defined depending on the force or moment being affected by angle/rate/control-surface: the sideforce derivatives are Y_β, Y_p, Y_r, Y_{δ_p}, and Y_{δ_r}; the rolling moment derivatives are L_β, L_p, L_r, L_{δ_p}, and L_{δ_r}; and the yawing moment derivatives are N_β, N_p, N_r, N_{δ_r}. The derivatives are normalized assuming a 10 percent uncertainty with respect to their nominal values, e.g. $Y_\beta = Y_\beta^o(1 + \sigma_c\delta_1)$ where $\sigma_c = 0.1$. Therefore, the model has fourteen normalized uncertainties δ_i corresponding to the 14 stability derivatives (numbered in the order mentioned above).

In classical flight mechanics, see reference [18], the stability derivatives which have the largest effect on the aircraft dynamics for a conventional aircraft in this type of motion are known to be Y_β, L_β, L_p, L_r, N_β, N_p and N_r (i.e. $\{\delta_1, \delta_6, \delta_7, \delta_8, \delta_{11}, \delta_{12}, \delta_{13}\}$).

This is confirmed, see Fig. 3, by applying a standard open-loop eigenvalue sensitivity analysis [9]. Fig. 3 shows the average of the maximum eigenvalues sensitivities $Sen_p^\lambda = \frac{1}{4}\sum_{i=1}^4 max(Sen_p^{\lambda_i})$ for each parameter, where the sensitivity of the i^{th} eigenvalue λ_i $(i = 1, 2, 3, 4)$ to variations in the j^{th} parameter p_j $(j = 1, 2, \ldots, 14)$ is given by the finite-difference equation:

$$Sen_{p_j}^{\lambda_i} = \frac{\partial\lambda_i(p)}{\partial p_j} \approx \frac{|\lambda_i(p + \Delta p_j) - \lambda_i(p)|}{\Delta p_j} \tag{17}$$

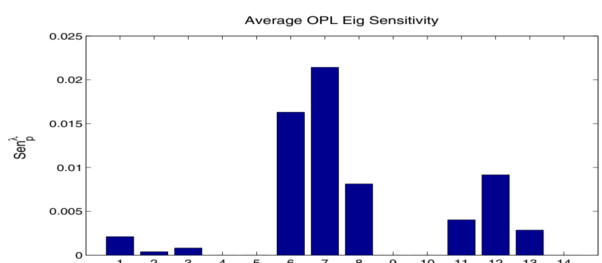

Fig. 3. Eigenvalue Sensitivity Analysis (open-loop).

This agreement is expected since the physical/experimental insight used in flight mechanics is typically obtained by estimating the effect of variations with respect to flight condition of the stability derivatives for conventional aircraft dynamics, which is basically an open-loop sensitivity analysis.

In order to perform a closed-loop sensitivity analysis, a simple static output feedback controller K_1 (taken from [6]) designed using classical methods is connected to the aircraft model:

$$K_1 = \begin{bmatrix} -629.8858 & 11.5254 & 3.3110 & 9.4278 \\ 285.9496 & 0.3693 & -2.6301 & -0.5489 \end{bmatrix} \tag{18}$$

The calculation of the closed-loop μ sensitivities is performed by using a mixed upper-bound μ algorithm [7] and

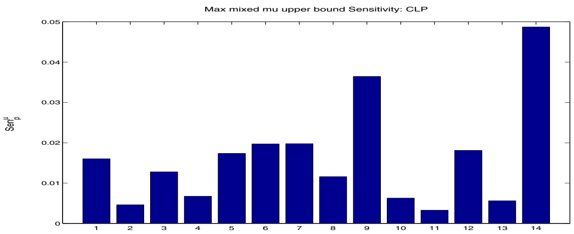

Fig. 4. μ Sensitivity Analysis (closed-loop K_1).

perturbing each normalized parameter by 0.1 (i.e. corresponding to a deviation of $0.1\sigma_c = 0.1{\cdot}0.1 = 1$ percent with respect to the nominal value).

Note that the results of this analysis, see Fig. 4, are completely different to those of the standard open-loop eigenvalue sensitivity analysis. Specifically, two control derivatives: the rudder effect on roll moment L_{δ_p} (δ_9) and the aileron contribution to yaw moment N_{δ_r} (δ_{14}) now emerge as by far the most sensitive parameters.

It has been mentioned before, that in as much as the controller used for the closed-loop μ sensitivities is sensibly designed, it is not important that it might represent only a preliminary design. To illustrate this, a second controller K_2 is designed which, compared to the previous K_1 controller, has reduced robust stability:

$$K_2 = \begin{bmatrix} -629.8858 & 5.1525 & 7.3110 & 5.4278 \\ 285.9496 & 0.03693 & 1.8301 & -0.2489 \end{bmatrix} \quad (19)$$

The closed-loop μ sensitivity result for this second controller is given in Figure 5. Although, it might seem that it is quite different to that from Figure 4, taking the six parameters with the highest Sen_p^μ –in descending order– from both figures yields $K = \{\delta_{14}, \delta_9, \delta_7, \delta_6, \delta_5, \delta_1\}$ and $K_2 = \{\delta_{14}, \delta_1, \delta_6, \delta_9, \delta_5, \delta_4\}$. Comparing the parameters, it is observed that they differ only by one parameter, δ_7 and δ_4 respectively, and that the the ordering is also quite similar.

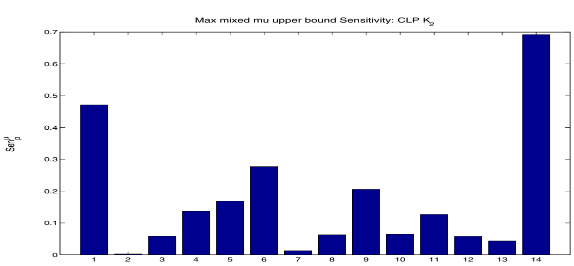

Fig. 5. μ Sensitivity Analysis (closed-loop K_2).

These results clearly reveal the limitations of standard open-loop sensitivity analysis for systems which are to be implemented in closed loop. In particular, decisions about what level of uncertainty should be allowed for each parameter, or which parameters to neglect based on such analysis are likely to be completely erroneous. The results of the closed-loop μ sensitivity analysis, on the other hand, can now be used to simplify the uncertainty set and/or to systematically produce maximum uncertainty bounds for any chosen combination of parameters, as shown in the following section.

B. Application of Control Oriented Uncertainty Modelling

Among all the uncertain aircraft parameters considered in the study, the damping cross-stability derivatives L_r and N_p are typically the most difficult to estimate accurately via wind-tunnel testing - additionally these parameters often have a large variation in magnitude with respect to changes in Mach number [18]. In the following, we therefore investigate the maximum allowable uncertainty which can be specified for these parameters (δ_8, δ_{12}), and the resulting trade-off with respect to the bounds on the other parameters in the model.

As mentioned before, initial uncertainty bounds of $\sigma_c = \pm 10\%$ are placed on all normalized uncertain parameters in the model. Based on the closed-loop μ sensitivity, the three parameters with the lowest μ sensitivities are discarded ($\delta_2, \delta_{11}, \delta_{13}$). All real skewed μ lower bound calculations were performed using the algorithm of [4] as implemented in [7].

Table I shows six models obtained by placing several different combinations of parameters in the two skewed μ uncertain sets, Δ_f and Δ_v. The first model places the nine uncertainties in Δ_v and is equivalent to a standard μ_{LB} calculation. The three subsequent models correspond to placing different combinations of two parameters in Δ_f: model 2 is for medium-to-low μ sensitivity parameters, model 3 to medium-to-high and model 4 to the two highest μ sensitivity parameters. The comparison of these three models enables us to check the influence of the choice of parameters to be fixed on the quality of the skewed μ lower bound. The last two models, 5 and 6, together with any of the previous models, allow us to assess the effect of increasing the number of parameters declared fixed: model 5 fixes five parameters, while model 6 fixes seven. Note that model 6 is guaranteed to compute the largest possible allowable uncertainty for the two parameters which are most difficult to measure accurately, i.e. (δ_8, δ_{12}).

TABLE I

MODELS FOR SKEWED μ ANALYSIS.

	Δ_f	Δ_v
1	$\{-\}$	$\{\delta_1, \delta_3, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{12}, \delta_{14}\}$
2	$\{\delta_8, \delta_{12}\}$	$\{\delta_1, \delta_3, \delta_5, \delta_6, \delta_7, \delta_9, \delta_{14}\}$
3	$\{\delta_6, \delta_9\}$	$\{\delta_1, \delta_3, \delta_5, \delta_7, \delta_8, \delta_{12}, \delta_{14}\}$
4	$\{\delta_9, \delta_{14}\}$	$\{\delta_1, \delta_3, \delta_5, \delta_6, \delta_7, \delta_8, \delta_{12}\}$
5	$\{\delta_3, \delta_5, \delta_6, \delta_9, \delta_{14}\}$	$\{\delta_1, \delta_7, \delta_8, \delta_{12}\}$
6	$\{\delta_1, \delta_3, \delta_5, \delta_7, \delta_9, \delta_{12}, \delta_{14}\}$	$\{\delta_8, \delta_{12}\}$

Fig. 6 shows the real skewed μ calculations for the six models. As expected, the lower-bound real μ of model 1 gives an upper bound for the other models. Looking at the effect of the choice of parameters to be fixed (i.e. models 2, 3 and 4), it is observed that the higher the μ-sensitivity of the fixed parameters the smaller the value of ν, which implies the size of the worst-case perturbation increases, i.e. $k_\nu = \frac{1}{\nu}$. The bottom plots in the figure correspond to models 5 and 6, and show that as the number of fixed parameters increases, the skewed μ lower bound algorithm is forced to search

TABLE II

WORST-CASE PARAMETER COMBINATIONS FOR SKEWED μ ANALYSIS MODELS.

Model	δ_1	δ_3	δ_5	δ_6	δ_7	δ_8	δ_9	δ_{12}	δ_{14}
1	4.9864	-2.9144	6.3477	-6.3477	-6.3477	**-6.3477**	6.3477	**6.3477**	-6.3477
2	-6.4910	1.0582	-7.3242	7.3242	-7.3242	**-1.0000**	-7.3242	**1.0000**	-7.3242
3	5.5291	7.5684	7.5684	-1.0000	-7.5684	**-7.5684**	1.0000	**7.5684**	-7.3389
4	10.0295	-10.7422	10.7422	-10.7422	-6.9167	**-10.7422**	1.0000	**-10.7422**	-1.0000
5	11.7752	-1.0000	-1.0000	-1.0000	-14.5878	**-14.6484**	-1.0000	**14.6484**	-1.0000
6	-0.5003	-0.7863	1.0000	-1.0000	-1.0000	**-27.3438**	1.0000	**27.3438**	-1.0000

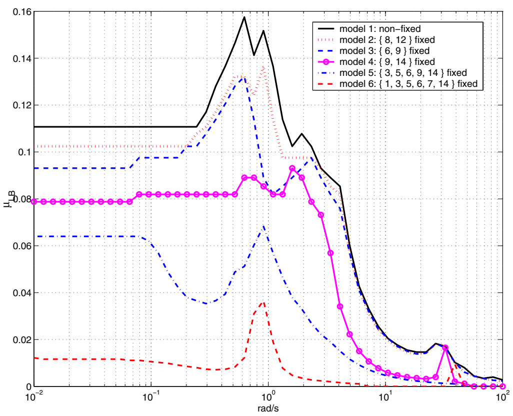

Fig. 6. Lower-bound real skewed μ for six models

for larger norms for the remaining "free" parameters (and therefore, smaller values of ν are obtained).

Table II provides the worst-case parameter combinations calculated by the skewed μ lower bound algorithms for the six models. This table can also be interpreted as providing the relative uncertainty bounds between the uncertain parameters for each different model considered.

It is observed that all the models yield a norm of one for the system parameters in the uncertain set Δ_f, thus $\bar{\sigma}_c = 1$. This implies that the uncertainty bound associated with these parameters is the initially chosen ten percent of their nominal values, i.e. $\bar{\delta} = \bar{\sigma}_c \sigma_c = 1 \times 0.1 = 10\%$. Furthermore, the table provides a quantitative analysis of the effects that maximizing the uncertainty bounds for some of the parameters have on the bounds for the rest of the parameters. For example, in model 6 the two most-difficult parameters to identify (δ_8, δ_{12}) are allowed to have up to $\bar{\delta} = \bar{\sigma}_c \sigma_c = 27 \times 0.1 = \pm 270\%$ uncertainty if the rest of the parameters are kept within a ten percent bound - note, however, that this represents a reduction in the allowable uncertainty for parameters δ_1 and δ_3 (which might not be acceptable). These results show how the proposed approach can be used to systematically manage the trade-offs between allowable uncertainty levels among different combinations and numbers of parameters.

V. CONCLUSIONS

In this paper, an approach for the computation of bounds on model parametric uncertainty for robust control design was proposed. The approach uses μ sensitivities to identify which uncertain parameters in the model are most critical, and hence, need to be identified with the greatest accuracy. Using skewed μ analysis tools and the uncertainty classification given by the μ sensitivity analysis, the maximum possible uncertainty bounds for those parameters which are most difficult or expensive to identify exactly can then be computed. The application of the proposed approach was illustrated via an aircraft example.

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