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# Robust Observer Design for a Class of Nonlinear Systems

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*Abstract*— **A Luenberger-based observer is proposed to the state estimation of a class of nonlinear systems subject to parameter uncertainty and bounded disturbance signals. A nonlinear observer gain is designed in order to minimize the effects of the uncertainty, error estimation and exogenous signals in an $\mathcal{H}_\infty$ sense by means of a set of state- and parameter-dependent linear matrix inequalities that are solved using standard software packages. A numerical example illustrates the approach.**

## I. Introduction

Since the seminal works of Kalman in [1] and Luenberger in [2], state estimation of dynamical systems is an active topic of research in a wide diversity of areas varying from control theory to fault detection and information fusion. We can define the observer problem as the task of estimating a function of the states of a dynamical system from its output which may be corrupted by disturbance signals and parameter uncertainty. The most recognized and well-studied state estimators for linear system are probably the Kalman Filter and the Luenberger observer [3], in which it is required a certain level of accuracy on the system model. When the model is uncertain the observer may have a poor performance or even assume an erratic behavior. Moreover, in many practical situations the signal to be observed results from a nonlinear map yielding approximate solutions based on the system linearization such as the extended Kalman filter or EKF [4].

On the other hand, the problem of robustness and disturbance rejection in the control theory has been addressed by means of convex optimization techniques. To this end, the control problem is recast as a set of linear matrix inequalities (LMIs) through the Lyapunov theory and then a solution is obtained using very efficient interior-point method algorithms [5]. In this framework, one can cite several solutions to the stability analysis and performance, and control synthesis [5]. However, the LMI framework cannot be applied in a straightforward way to deal with nonlinear dynamical systems. Nevertheless, some researchers have recently proposed sufficient LMI conditions to handle nonlinear systems such as the works of [6] and [7]. In this setting, the proposed solutions vary from the class of Lyapunov functions and also the way that the nonlinear terms are transformed into convex conditions.

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From the above discussion, a nonlinear state estimator based on the ideas of the Luenberger's observer for linear systems is developed by considering the uncertainty and the disturbance signal as an exogenous input, and then the observer gain is designed such that an upper-bound on the $\mathcal{L}_2$-gain of the input-to-output operator from the error system is minimized. To make this point clear, for the following class of nonlinear systems

$$\begin{aligned}&\dot{x}(t) = (A_y(y) + A_x(x,\delta))x(t) + B_v(x,\delta)v(t),\\ &y(t) = C_y x(t),\ z(t) = C_z x(t),\end{aligned}$$

where $\delta$ is the uncertainty, $v(t)$ the disturbance, $y(t)$ the measurement, and $z(t)$ the signal to be estimated, we rewrite the error system as follows

$$\begin{aligned}&\dot{e}(t) = (A_y(y) + L(y)C_y)e(t) + B_p(x,\delta)p(t),\\ &z_e(t) = C_z e(t).\end{aligned}$$

with $p(t)$ being a fictitious exogenous signal that represents both uncertainty and disturbance signal. Thus, a relaxed version of the Bounded Real Lemma is applied to design the nonlinear observer-gain $L(y)$ such that the $\mathcal{L}_2$-gain from $p(t)$ to $z_e(t)$ is minimized.

To present the proposed methodology, the rest of this paper is as follows. Section II introduces the problem to be addressed, the general ideas of the proposed solution are given in Section III, and a convex characterization of the general solution is presented in Section IV. An illustrative example is given in Section V, and Section VI ends the paper.

**Notation**. $\mathbb{R}^n$ denotes the set of $n$-dimensional real vectors, $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices, $I_n$ is the $n\times n$ identity matrix, $0_{n\times m}$ is the $n\times m$ matrix of zeros, $0_n$ is the $n\times n$ matrix of zeros and $\mathrm{diag}\{\ldots\}$ represents a block-diagonal matrix. For a real matrix $S$, $S'$ denotes its transpose and $S>0$ means that $S$ is symmetric and positive-definite. For a symmetric block matrix, the symbol $\star$ denotes the transpose of the symmetric block outside the main diagonal block. The time derivative of a function $r(t)$ will be denoted by $\dot{r}(t)$ and the argument $(t)$ is often omitted. For polytopes $X_1\subset\mathbb{R}^n$ and $X_2\subset\mathbb{R}^m$, the notation $X_1\times X_2$ represents that $(X_1\times X_2)\subset\mathbb{R}^{(n+m)}$ is a meta-polytope obtained by the cartesian product, and $\mathcal{V}(X_i)$ refers to the set of all vertices of $X_i$. Matrix and vector dimensions are omitted whenever they can be inferred from the context.

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## II. Problem Statement

Consider the following class of systems

$$\begin{array}{rcl}\dot{x}(t) &=& a(x(t),\delta,u(t),v(t)),\ x(0)\in\mathcal{R}_0 \\ y(t) &=& C_y x(t),\ z(t) \ = \ C_z x(t),\end{array} \quad (1)$$

where $x(t)\in\mathbb{R}^n$ is the state, $\delta\in\mathbb{R}^{n_\delta}$ is a vector of bounded time invariant parameters, $u(t)\in\mathbb{R}^{n_u}$ is the control input, $v(t)\in\mathbb{R}^{n_v}$ is the disturbance signal, $y(t)\in\mathbb{R}^{n_y}$ is the measurement, $z(t)\in\mathbb{R}^{n_z}$ represents the vector of signals to be estimated, $a(\cdot)$ is a nonlinear vector function, and $C_y, C_z$ are constant matrices having appropriate dimensions. In addition, consider the following assumptions over the above system:

- **A1** The right-hand side of the differential equation is continuous and bounded on its arguments.
- **A2** The vector of (constant) uncertain parameters $\delta$ lies in a polytope $\Delta$, i.e. $\delta\in\Delta$.
- **A3** The state-space domain in which the observer design will be performed is bounded by a known polytopic region $\mathcal{X}\subset\mathbb{R}^n$.
- **A4** The initial condition $x_0 = x(0)$ is unknown but belonging to the following level surface

$$\mathcal{R}_0 = \{x(t) : x(t)'Rx(t)\le 1,\ R>0\}\subset\mathcal{X}.$$

- **A5** The disturbance signal $v(t)$ belongs to the set of integrable vector functions on $[0,T]$, i.e. $v(t)\in\mathcal{L}_{2,[0,T]}$.

When the equilibrium point of system (1) is not GAS (globally asymptotically stable), the input signals ($u(t)$ and $v(t)$) may drive the system to instability. To simplify the analysis, one shall consider the following additional assumption

- **A6** System (1) is exponentially stable in $\mathcal{X}$ for all $x_0\in\mathcal{R}_0$, $v(t)\in\mathcal{L}_{2,[0,T]}$ and $u(t)$, for the interval of time $T$.

Also, for simplicity, we suppose whenever $x\in\mathcal{X}$ the output $y$ is also contained in a given polytope, which is guaranteed by the following assumption:

- **A7** The matrix $C_y$ satisfies $C_y'C_y\le I_n$.

The problem to be addressed in this paper is to synthesize an estimation of $z(t)$ in (1) while requiring some performance on the estimation error in a numerical and tractable way. To this end, let us recast the differential equation in (1) as follows

$$\dot{x} = (A_y(y)+A_x(x,\delta))x + b(y,u) + B_v(x,\delta)v \quad (2)$$

where $A_y(\cdot), A_x(\cdot), b(\cdot), B_v(\cdot)$ are nonlinear bounded functions on their arguments.

From Luenberger's observer theory, the following estimator is proposed

$$\left\{\begin{array}{rcl}\dot{\hat{x}}(t) &=& A(y)\hat{x}(t) + b(y,u) + L(y)(y-\hat{y}), \\ \hat{z}(t) &=& C_z\hat{x}(t),\ \hat{y} = C_y\hat{x},\ \hat{x}(0)=0,\end{array}\right. \quad (3)$$

where $\hat{x}\in\mathbb{R}^n$ is the observer state; $\hat{z}(t),\hat{y}$ are the estimation of $z(t)$ and $y(t)$, respectively; and $L(y)$ is a nonlinear matrix function of $y(t)$ to be determined.

In practical applications, the estimated signal $\hat{z}(t)$ has to track $z(t)$ as close as possible for all $\delta\in\Delta$ regardless the presence of disturbance signals, i.e., one has to guarantee some performance on the estimation error $z_e(t)\triangleq z(t)-\hat{z}(t)$ for all possible circumstances.

Defining the error signal as $e(t)\triangleq x(t)-\hat{x}(t)$, the following error dynamics is obtained

$$\dot{e} = (A_y(y)+L(y)C_y)e + A_x(x,\delta)x + B_v(x,\delta)v, e(0)=x_0.$$

From assumption **A6**, the state trajectory belongs to the $\mathcal{L}_{2,[0,T]}$ space. Thus, the term $A_x(x,\delta)x$ that appears on the error dynamics can be viewed as a disturbance signal to the error dynamics driven by the uncertain part of (1). As a result, we can define a fictitious input signal namely $p(t)$ as follows

$$p(t) = \left[\begin{array}{cc} x'(t) & v'(t)\end{array}\right]',$$

and rewrite the error dynamics as follows

$$\begin{array}{l}\dot{e}(t) = A_e(y)e(t) + B_p(x,\delta)p(t), \\ z_e(t) = C_z e(t),\ e(0)=x_0,\end{array} \quad (4)$$

where $p(t)\in\mathbb{R}^{n\times n_p}$, $n_p = n+n_v$, $A_e(y) = A_y(y) + L(y)C_y$, and $B_p(x,\delta) = [\begin{array}{cc} A_x'(x,\delta) & B_v'(x,\delta)\end{array}]'$.

One can measure the effects of the signal $p(t)$ on $z_e(t)$ by means of the (finite-horizon) $\mathcal{L}_2$-gain of the input-to-output $\mathcal{G}_{pz_e}$ operator, i.e.,

$$\|\mathcal{G}_{pz_e}\|_{\infty,[0,T]} = \sup_{0\neq p\in\mathcal{L}_{2,[0,T]}} \frac{\|z_e(t)\|_{2,[0,T]}}{\|p(t)\|_{2,[0,T]}}, \quad (5)$$

for all $x\in\mathcal{X}$ and $\delta\in\Delta$. Now, we are ready to state the problem of concern in this paper:

*Problem 1:* Design the observer gain in (3) such that an upper bound $\gamma$ on $\|\mathcal{G}_{pz_e}\|_{\infty,[0,T]}$ is minimized for all $(x,\delta)\in\mathcal{X}\times\Delta$. □

*Remark 1:* (Regional Stability of Disturbed Systems) The assumption **A6** is crucial to the correctness of the approach. In practical applications, the regional stability of system (1) should be checked in advance regarding the set of initial conditions and the class of admissible disturbances. To the same class of systems considered in this paper, the convex technique proposed in [8] can be applied to this purpose. □

## III. Preliminaries

One can characterize the $\mathcal{L}_2$-gain of a system by means of the Lyapunov theory [9]. The following result is a straightforward application of the input-to-output stability analysis of nonlinear systems.

*Lemma 1:* ($\mathcal{L}_2$-gain Analysis) Consider a nonlinear system $\dot{x}(t) = f(x,u)$, $y(t) = g(x,u)$, where $f:\mathcal{X}\times\mathcal{U}\mapsto\mathbb{R}^n$, and $g:\mathcal{X}\times\mathcal{U}\mapsto\mathbb{R}^{n_y}$ (with $\mathcal{X}\in\mathbb{R}^n$, $\mathcal{U}\in\mathcal{L}_{2,[0,T]}^{n_u}$) are continuous and bounded on their arguments. Let $V:\mathcal{X}\mapsto\mathbb{R}$ be a continuously differentiable function and $\epsilon_1,\epsilon_2,c,\gamma$ be positive scalars such that

$$\epsilon_1 x'x \le V(x) \le \epsilon_2 x'x \quad (6)$$

$$\dot{V}(x) + \frac{1}{\gamma}y'y - \gamma u'u < 0, \quad (7)$$

$$\mathcal{R}_c = \{x : V(x)\le c\}\subset\mathcal{X} \quad (8)$$

for all $x \in \mathcal{X}$, then the unforced system is exponentially stable in $\mathcal{R}_c$, and the $\mathcal{L}_2$-gain from $u$ to $y$ satisfies

$$\|\mathcal{G}_{uy}\|_{\infty,[0,T]} \leq \gamma,\ \forall\ x \in \mathcal{X},\ u \in \mathcal{U}.$$

□

In order to determine a Lyapunov function that proves the stability and also provides an upper bound on the $\mathcal{L}_2$-gain, we normally constraint the problem to a given class of functions and then compute some parameterized solution. In this paper, we consider the following parameter-dependent Lyapunov function candidate:

$$V(e,\delta) = e'P(\delta)e,\ P(\delta) > 0,\ \forall\ \delta \in \Delta, \tag{9}$$

where $P(\delta)$ is a matrix function of $\delta$ to be determined.

Taking into account the error system dynamics as defined in (4), the time-derivative of $V(e)$ is as follows

$$\dot{V}(e,\delta) = \begin{bmatrix} e \\ p \end{bmatrix}' \begin{bmatrix} \Phi(y,\delta) & P(\delta)B_p(x,\delta) \\ \star & 0 \end{bmatrix} \begin{bmatrix} e \\ p \end{bmatrix} \tag{10}$$

where $\Phi = \Phi(y,\delta)$ is given by

$$\Phi = A_y'(y)P(\delta)+P(\delta)A_y(y)+C_y'L'(y)P(\delta)+P(\delta)L(y)C_y.$$

Notice for linear systems that Lemma 1 is the well-known Bounded Real Lemma (BRL) [5]. Recently, several authors have proposed improved stability conditions for systems of the type $\dot{x} = A(\delta)x$ to avoid the product term $P(\delta)A(\delta)$ that appears in $\dot{V}$, and so making possible the use of parameter-dependent Lyapunov functions [10], [11].

From the above analysis, one can propose the following nonlinear version of the improved BRL initially proposed in [12] applied to system (1).

*Lemma 2:* Consider the system (1), with assumptions **A1-A7**, and the observer system as defined in (3). Let $\mathcal{X}$ and $\Delta$ be given polytopes. Suppose there exist matrices $W(\delta), G, M(y)$, and a positive scalar $\gamma$ that solves the following optimization problem for all $(x,\delta) \in \mathcal{X} \times \Delta$.

$$\min_{W(\delta),G,M(y)} \gamma \quad \text{subject to: } \begin{bmatrix} R & G' \\ G & W(\delta) \end{bmatrix} > 0, \tag{11}$$

$$\begin{bmatrix} \Phi_a & \star & \star & C_z' \\ B_p'G' & -2\gamma I_{n_p} & \star & 0 \\ \Phi_b & GB_p & -W(\delta) & 0 \\ C_z & 0 & 0 & -\frac{\gamma I_{n_z}}{2} \end{bmatrix} < 0 \tag{12}$$

where $B_p = B_p(x,\delta)$, $\hat{A} = A(y) + I_n$, $\tilde{A} = A(y) - I_n$, $\Phi_b = G\hat{A} + M(y)C_y$, and

$$\Phi_a = W(\delta) + G\tilde{A} + \tilde{A}'G + M(y)C_y + C_y'M'(y)$$

Then, the unforced error system (4) with

$$L(y) = G^{-1}M(y),$$

is exponentially stable in $\mathcal{R}_0$ for all $\delta \in \Delta$. Moreover, the finite horizon $\mathcal{L}_2$-gain satisfies

$$\|\mathcal{G}_{pz_e}\|_{\infty,[0,T]} \leq \gamma,\ \forall\ (x,\delta) \in (\mathcal{X} \times \Delta).$$

□

**Proof.** For convenience, define the following notation:

$$W_a = \begin{bmatrix} W(\delta) & 0 \\ 0 & 2\gamma I_{n_p} \end{bmatrix},\ G_a = \begin{bmatrix} G & 0 \\ 0 & \gamma I_{n_p} \end{bmatrix},$$
$$\hat{A}_a = \begin{bmatrix} \hat{A} + L(y)C_y & B_p \\ 0 & 0_{n_p} \end{bmatrix},$$
$$\tilde{A}_a = \begin{bmatrix} \tilde{A} + L(y)C_y & B_p \\ 0 & -2\gamma I_{n_p} \end{bmatrix}, \tag{13}$$

$$A_a = \begin{bmatrix} A_e(y) & B_p \\ 0 & -I_{n_p} \end{bmatrix},\ C_a = \begin{bmatrix} C_z & 0 \end{bmatrix},$$
$$P_a = G_a'W_a^{-1}G_a = \begin{bmatrix} P(\delta) & 0 \\ 0 & -\frac{\gamma I_{n_p}}{2} \end{bmatrix}$$

From the above notation, the expression in (12) can be rewritten as follows

$$\begin{bmatrix} W_a + G_a\tilde{A}_a + \tilde{A}_a'G_a' & \star & C_a' \\ G_a\hat{A}_a & -W_a & 0 \\ C_a & 0 & -\frac{\gamma I_{n_p}}{2} \end{bmatrix} < 0.$$

where $M(y) = GL(y)$.

Applying the Schur complement to the above matrix inequality yields:

$$\begin{bmatrix} \begin{pmatrix} \hat{A}_a'G_a'W_a^{-1}G_a\hat{A}_a + \\ W_a + G_a\tilde{A}_a + \tilde{A}_a'G_a' \end{pmatrix} & \star \\ C_a & -\frac{\gamma I_{n_p}}{2} \end{bmatrix} < 0 \tag{14}$$

From the fact that[1]

$$W_a + G_a\tilde{A}_a + \tilde{A}_a'G_a' \geq -\tilde{A}_a'G_a'W_a^{-1}G_a\tilde{A}_a,$$

the matrix inequality in (14) implies the following

$$\begin{bmatrix} \hat{A}_a'P_a\hat{A}_a - \tilde{A}_a'P_a\tilde{A}_a & \star \\ C_a & -\frac{\gamma I_{n_p}}{2} \end{bmatrix} < 0 \tag{15}$$

for all $(x,\delta) \in \mathcal{X} \times \Delta$.

Noticing that $\hat{A}_a'P_a\hat{A}_a - \tilde{A}_a'P_a\tilde{A}_a = 2(A_a'P_a + P_aA_a)$, we get for (15) the following

$$A_a'P_a + P_aA_a + \frac{C_a'C_a}{\gamma} < 0,\ \forall\ (x,\delta) \in \mathcal{X} \times \Delta.$$

Pre- and post-multiplying the above by $[\ e' \quad p'\ ]$ and its transpose, respectively, leads to

$$\dot{V}(e,\delta) + \frac{z_e'z_e}{\gamma} - \gamma p'p < 0. \tag{16}$$

The LMI in (11) implies the following from Schur complement

$$R - G'W(\delta)^{-1}G > 0,\ W(\delta) - GR^{-1}G' > 0.$$

As $R > 0$ by assumption, one can infer that $W(\delta) > 0$ and so $P(\delta) = G'W(\delta)^{-1}G > 0$ and $R - P(\delta) > 0$. Thus,

$$V(e,\delta) = e'P(\delta)e > 0, \text{ and}$$
$$\{e : e'Re \leq 1\} \subset \{e : V(e,\delta) \leq 1\}.$$

[1] See [13] for instance.

As the elements of $P(\delta)$ are bounded for all $\delta \in \Delta$, there exists scalars $\epsilon_1, \epsilon_2$ such that

$$\epsilon_1 e'e \leq V(e,\delta) \leq \epsilon_2 e'e,$$

and the proof is completed from Lemma 1. ■

*Remark 2:* To guarantee the feasibility of Lemma 2, we have to check a priori if the pair $(A_y(y), C_y)$ is observable for all $y \in \mathcal{X}$. □

## IV. CONVEX CHARACTERIZATION

The contribution of Lemma 2 is mainly one of existence of solutions, since the conditions are characterized by a set of Nonlinear Matrix Inequalities (NLMIs) [14] and so they are hard to solve [15]. However, using some previous results from the nonlinear control literature we can obtain a convex characterization of Lemma 2.

In fact, there exist two basic approaches to deal with this problem. The first one considers a different model representation for the system under analysis such as the linear fractional representation (LFR) of El Ghaoui and co-authors in [6], [16] and the differential-algebraic one (DAR) of Trofino and collaborators in [7], [17]. The second one is the sums of squares (SOS) initially proposed in [18] where multivariable polynomial conditions resulting from Lyapunov-like inequalities are rewritten as a sum of squares in a computational and tractable way. In spite of the promising results of this approach, the method cannot handle parameterizations like $M = PL$, where $P$ represents the Lyapunov matrix and $L$ some control gain to be determined. Hence, the SOS methodology is more indicated to stability analysis rather than control design.

It should be noted that the design conditions on Lemma 2 are different from the ones usually found in stability analysis and control design of nonlinear systems. More precisely, in general the conditions are in the form:

$$\sigma'\mathcal{T}(\sigma)\sigma > 0, \ \forall\ \sigma \in \Sigma,$$

where $\sigma \in \mathbb{R}^{n_\sigma}$ is a generic parameter, $\Sigma \subset \mathbb{R}^{n_\sigma}$ is a given polytopic region, and $\mathcal{T}(\sigma)$ is a nonlinear function of $\sigma$. It turns out that there is coupling between the vector $\sigma$ and the arguments of $\mathcal{T}(\sigma)$. So, some algebraic manipulations can be performed to obtain a convex characterization of the original problem as shown, e.g., in [19]. However, in Lemma 2 one get conditions like

$$\rho'\mathcal{T}(\sigma)\rho > 0, \ \forall\ \rho \in \mathbb{R}^{n_\rho}, \ \sigma \in \Sigma, \tag{17}$$

where there is no (a straightforward) coupling between $\rho$ and $\sigma$. To overcome the above problem, one can apply the technique proposed in [20] for nonlinear discrete-time systems. Note that condition (17) may be tested by means of $\mathcal{T}(\sigma) > 0, \ \forall\ \sigma \in \Sigma$. If it is assumed that $\mathcal{T}(\sigma)$ can be written as follows

$$\mathcal{T}(\sigma) = \begin{bmatrix} I \\ \mathcal{M}(\sigma) \end{bmatrix}' \mathbf{T} \begin{bmatrix} I \\ \mathcal{M}(\sigma) \end{bmatrix},$$

where $\mathbf{T}$ is a constant matrix and $\mathcal{M}(\sigma)$ is a nonlinear function of $\sigma$ satisfying the following constraint[2]

$$\Xi_1(\sigma) + \Xi_2(\sigma)\mathcal{M}(\sigma) = 0,$$

for some $\Xi_1, \Xi_2$ which are affine matrix functions of $\sigma$ with $\Xi_2$ having column full-rank for all $\sigma \in \Sigma$. Then, condition (17) is satisfied (at the cost of some conservativeness) if the following holds for all $\sigma \in \mathcal{V}(\Sigma)$:

$$\mathbf{T} + N\Xi(\sigma) + \Xi'(\sigma)N' > 0,$$

where $\Xi(\sigma) = [\ \Xi_1(\sigma) \quad \Xi_2(\sigma)\ ]$ and $N$ is a free multiplier to be determined.

To allow the application of the above technique, we suppose that the error dynamics matrices can be decomposed as follows

$$A_y(y) = A_0 + A_1\Pi_1(y), B_p(x,\delta) = B_0 + B_1\Psi_1(x,\delta), \tag{18}$$

where $A_0, A_1, B_0, B_1$ are constant matrices with appropriate dimensions and $\Pi_1(y) \in \mathbb{R}^{m_\pi \times n}$, $\Psi_1(x,\delta) \in \mathbb{R}^{m_\psi \times n_p}$ are nonlinear functions of $(x,\delta)$ satisfying the following constraints

$$\Omega_0(y) + \Omega_1(y)\Pi_1(y) = 0, \ \Upsilon_0(x,\delta) + \Upsilon_1(x,\delta)\Psi_1(x,\delta) = 0,$$

with $\Omega_0(y) \in \mathbb{R}^{q\times n}$, $\Omega_1(y) \in \mathbb{R}^{q\times m_\pi}$, $\Upsilon_0(x,\delta) \in \mathbb{R}^{r\times n_p}$, and $\Upsilon_1(x,\delta) \in \mathbb{R}^{r\times m_\psi}$ being affine functions on their arguments.

To guarantee that the matrix decomposition defined in (18) is well-posed, we further assume

**A8** The matrices $\Omega_1(y)$ and $\Upsilon_1(x,\delta)$ are column full-rank for all $y, x$ and $\delta$ of interest.

For simplicity of notation, we may represent the matrix decomposition above defined as follows

$$\begin{aligned} A_y(y) &= \mathbf{A}\Pi, \ \Omega\Pi = 0, \\ B_p(x,\delta) &= \mathbf{B}\Psi, \ \Upsilon\Psi = 0, \end{aligned} \tag{19}$$

where $\mathbf{A} = [\ A_0 \quad A_1\ ]$, $\mathbf{B} = [\ B_0 \quad B_1\ ]$, $\Omega = [\ \Omega_0(y) \quad \Omega_1(y)\ ]$, $\Upsilon = [\ \Upsilon_0(x,\delta) \quad \Upsilon_1(x,\delta)\ ]$, and

$$\Pi = \begin{bmatrix} I_n \\ \Pi_1(y) \end{bmatrix}, \ \Psi = \begin{bmatrix} I_{n_p} \\ \Psi_1(x,\delta) \end{bmatrix}.$$

Following the same steps of [20], we propose the following convex characterization of Lemma 2.

*Theorem 1:* Consider system (1), with **A1-A7**, the representation defined in (2), the observer in (3), the error dynamics as defined in (4) and the matrix decomposition defined in (19) with **A8**. Let $\mathcal{X}$ and $\Delta$ be given polytopes. Further define the following notation:

$$N_\pi = [\ I_n \quad 0_{n\times m_\pi}\ ], \quad N_\psi = [\ I_{n_p} \quad 0_{n_p\times m_\psi}\ ], \quad \Lambda = \begin{bmatrix} \Omega & 0 & 0 & 0 \\ 0 & \Upsilon & 0 & 0 \\ 0 & 0 & \Omega & 0 \\ 0 & 0 & 0 & 0_{n_z} \end{bmatrix}.$$

[2]See Lemma 3 of [20] for further details.

Suppose the matrices $W(\delta) = W(\delta)', \mathbf{M}(y), G, K$ and a scalar $\gamma$ are a solution to the following optimization problem where the LMIs are constructed on $\mathcal{V}(\mathcal{X} \times \Delta)$.

$$\min \quad \gamma \quad \text{subject to:} \begin{bmatrix} R & G' \\ G & W(\delta) \end{bmatrix} > 0, \tag{20}$$

$$\begin{bmatrix} Q_a & \star & \star & \star \\ \mathbf{B}'G'N_\pi & -2\gamma N_\psi' N_\psi & \star & 0 \\ Q_b & N_\pi' G\mathbf{B} & -N_\pi' W(\delta) N_\pi & 0 \\ C_z N_\pi & 0 & 0 & -\frac{\gamma I_{n_z}}{2} \end{bmatrix} + $$
$$+K\Lambda + \Lambda' K' < 0 \tag{21}$$

where $W(\delta) = W_0 + \sum_{i=1}^{n_\delta} \delta_i W_i$, $\mathbf{M}(y) = M_0 + \sum_{i=1}^{n_y} y_i M_i$, $Q_a = N_\pi' W(\delta) N_\pi + N_\pi' G \tilde{\mathbf{A}} + \tilde{\mathbf{A}}' G' N_\pi + \mathbf{M}(y) C_y N_\pi + N_\pi' C_y' \mathbf{M}(y)'$, $Q_b = N_\pi' G \hat{\mathbf{A}} + \mathbf{M}(y) C_y N_\pi$, $\tilde{\mathbf{A}} = [\ (A_0 - I_n) \quad A_1\ ]$, and $\hat{\mathbf{A}} = [\ (A_0 + I_n) \quad A_1\ ]$.

Then, the unforced error system (4) with

$$L(y) = G^{-1} \Pi' \mathbf{M}(y)$$

is exponentially stable in $\mathcal{R}_0$ for all $\delta \in \Delta$. Moreover, the finite horizon $\mathcal{L}_2$-gain satisfies

$$\|\mathcal{G}_{pz_e}\|_{\infty,[0,T]} \leq \gamma, \ \forall \ (x, \delta) \in (\mathcal{X} \times \Delta).$$

□

**Proof.** Suppose that the LMIs in Theorem 1 hold for all $(x, \delta) \in \mathcal{V}(\mathcal{X} \times \Delta)$, thus for convexity they are also satisfied for all $x \in \mathcal{X}$ and $\delta \in \Delta$. Notice from **A7** and $y = C_y x$ that $y \in \mathcal{X}$.

Consider the LMI in (20). From the Schur complement, we get the following

$$R - G' W(\delta)^{-1} G > 0 \text{ and } W(\delta) - G R^{-1} G' > 0, \ \forall \ \delta \in \Delta.$$

Taking into account that $R > 0$ by **A4** and the above, it follows that

$$W(\delta) > 0, \ P(\delta) = G' W(\delta)^{-1} G > 0, \ R > P(\delta).$$

In view of Lemma 2, the following is satisfied for all $\delta \in \Delta$

$$\begin{array}{c} \epsilon_1 e'e \leq V(e, \delta) = e' P(\delta) e \leq \epsilon_2 e'e, \\ \{e : e' R e \leq 1\} \subset \{e : e' P(\delta) e \leq 1\}. \end{array} \tag{22}$$

Now, consider the LMI (21) and rename it as $\Gamma < 0$ for simplicity. Pre- and post-multiplying $\Gamma < 0$ by

$$\Theta = \begin{bmatrix} \Pi' & 0 & 0 & 0 \\ 0 & \Psi' & 0 & 0 \\ 0 & 0 & \Pi' & 0 \\ 0 & 0 & 0 & I_{n_z} \end{bmatrix}$$

and its transpose, respectively, leads to (12) with $M(y) = \Pi' \mathbf{M}(y)$. Notice that the following holds

$$\Lambda \Theta' = 0,$$

by construction. So, the proof is completed from Lemma 2. ■

*Remark 3:* For the class of Bilinear Systems [21], Theorem 1 is oversimplified. For instance, considering a system $\dot{x} = A_y(y)x + A(x, \delta)x + b(y)u + B_v(x, \delta)v$, where $A_y(\cdot), A_x(\cdot), b(\cdot), B_v(\cdot)$ are affine matrix functions on their arguments, the LMI (21) can be replaced by (12) in straightforward way.

## V. Illustrative Example

In the following, we give an example to illustrate the approach. The example is based on the following stable Van der Pol's equation:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & -x_1 - (1 - 0.2\delta)(1 - x_1^2) x_2 + u + v, \\ y & = & x_1, \ z \ = \ x_2, \ \delta \in [-1, 1]. \end{array} \tag{23}$$

where $v(t)$ is a disturbance signal satisfying **A5**, and the initial conditions are unknown but bounded by $\mathcal{R}_0$ as defined in **A4** with $R = 4I_2$. Accordingly to **A3** and **A4**, we bound the state space by the following polytope:

$$\mathcal{X} = \{x : |x_i| \leq 0.5, \ i = 1, 2\}.$$

The objective in this example is to estimate the state $x_2$ by means of Theorem 1. To this end, consider the representation (2) with

$$A_y = \begin{bmatrix} 0 & 1 \\ -1 & -(1 - x_1^2) \end{bmatrix}, \ A_x = \begin{bmatrix} 0 & 0 \\ 0 & 0.2\delta(1 - x_1^2) \end{bmatrix},$$

$b = [\ 0 \quad 1\ ]'u$, and $B_v = [\ 0 \quad 1\ ]'$. Applying the parametrization defined in (18) and (19) leads to

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.2 & 0 & -0.2 \end{bmatrix},$$
$$\Omega = \begin{bmatrix} 0 & x_1 & -1 & 0 \\ 0 & 0 & x_1 & -1 \end{bmatrix},$$
$$\Upsilon = \begin{bmatrix} 0 & \delta & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & x_1 & -1 & 0 \\ 0 & 0 & 0 & 0 & x_1 & -1 \end{bmatrix},$$
$$\Pi_1 = \begin{bmatrix} 0 & x_1 \\ 0 & x_1^2 \end{bmatrix}, \ \Psi_1 = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta x_1 & 0 \\ 0 & \delta x_1^2 & 0 \end{bmatrix}.$$

In light of Theorem 1, we get the following

$$\mathbf{M}(y) = \begin{bmatrix} -1371977.1 \\ 4.6666328 \\ -317.24137 \\ 728.82189 \end{bmatrix} + \begin{bmatrix} 1192.0414 \\ 317.29133 \\ -728.70494 \\ -0.2276901 \end{bmatrix} x_1,$$

$$G = \begin{bmatrix} 0.0801244 & 3.3670086 \\ -3.4385201 & 1.4599502 \end{bmatrix}.$$

Figure 1 shows the worst-case time response of $e_2(t) = x_2(t) - \hat{x}_2(t)$ for $\delta = 1$, an initial condition $x(0) = [\ -0.5 \quad 0\ ]'$, and the following input signals

$$u(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < 15sec \\ 0.5 & \phantom{\text{if}} \quad t \geq 15sec \end{cases},$$

$$v(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < 30sec \\ 0.1 & \quad\quad 30 \leq t \leq 50sec \end{cases}.$$

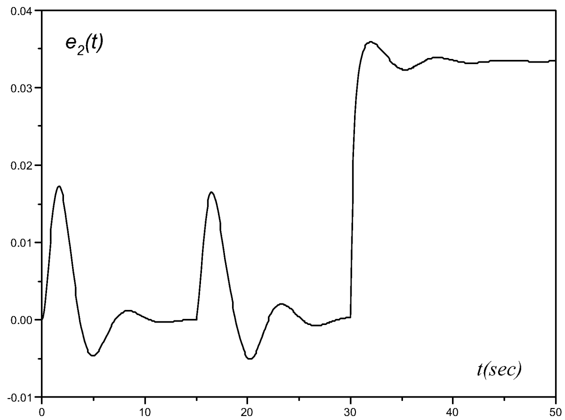


Fig. 1. Estimation error of $z(t)$ for system (23).

## VI. Concluding Remarks

This paper has proposed a convex approach to design robust state observers for a class of nonlinear systems subject to (constant) uncertainties and energy-bounded disturbance signals, with possibly unknown initial conditions. The LMI conditions give sufficient conditions that assure the local stability of the error dynamics while providing a robust domain of stability with guaranteed $\mathcal{H}_\infty$ performance specification on the estimation error. The approach can be extended to handle time-varying uncertainties straightforwardly by requiring that $d\delta/dt$ is bounded by a convex set. Future research is concentrated on devising a discrete-time version of the proposed observer.

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