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Control of Limit Cycle Amplitude Through Nonsmooth Bifurcations

Fabiola Angulo, Mario di Bernardo and Gerard Olivar

Abstract— **In this paper we present two methods for controlling limit cycles in smooth planar systems making use of the theory of nonsmooth bifurcations. They correspond to two novel switching control strategies for reducing the amplitude of undesired limit cycles in autonomous smooth systems. The switching controller is designed by using the theory of nonsmooth bifurcations. By designing an appropriate switching controller, the occurrence of a corner-sliding bifurcation or a grazing-sliding bifurcation is induced on the system and the amplitude of the target limit cycle is controlled. The switching control action is only started when the system trajectories enter an appropriately defined region in phase space. This technique is illustrated through a representative example.**

I. INTRODUCTION

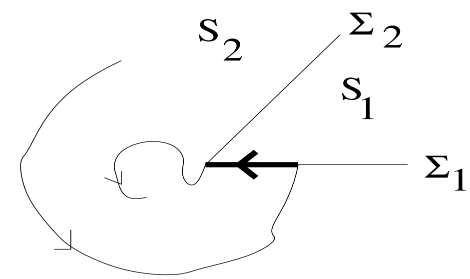
Recently a new technique for controlling (or suppressing) limit cycles has been proposed in the literature [1], [2] which is based on the exploitation of recent results in the theory of non-smooth bifurcations [3]. The idea for the controller is to induce a nearby corner collision event, which does not change the global properties but affects the local phase space structure, which is desirable in most applications.

By using the theory of corner collisions presented in [4], it is possible to assess analytically the effects of the control action on the structure of the local Poincaré map describing the system of interest. In so doing, it is possible to select the control law as to render unstable the fixed point on the map associated to the limit cycle of interest. It has been shown that the technique proposed has good performance and it successfully changes the local properties of the fixed point and hence of the associated limit cycle. However, with this technique the limit cycle can be suppressed but no other local attractor is introduced with the system trajectory moving towards the nearest attracting limit point in phase space. In [5] an alternative technique based on a corner collision event was presented. In this case the control signal was introduced in a small zone of the state space with two objectives. The first was to introduce a desired equilibrium point, and the second being that of stabilizing the system at this point. This technique can be easily extended to higher order autonomous systems, introducing a corner region in the corresponding n-dimensional state space.

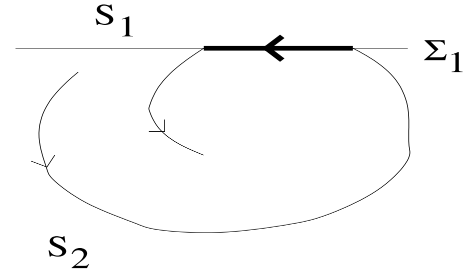
Fabiola Angulo is with Universidad Nacional de Colombia, sede Manizales. Cra 27 No 64-60, Manizales, Colombia. Email: fangulo@nevado.manizales.unal.edu.co

Mario di Bernardo is with the Department of Systems and Computer Science, University of Naples Federico II, Naples 80133, Italy, also with the Department of Engineering Mathematics, University of Bristol, Bristol BS8 1TR, U.K. Email: m.dibernardo@bristol.ac.uk

Gerard Olivar is with Technical University of Catalonia (UPC), EPSEVG and FME, Av. Victor Balaguer, s/n, E-08800, Vilanova i la Geltru, Spain, also with Universidad Nacional de Colombia, sede Manizales. Cra 27 No 64-60, Manizales, Colombia. Email: gerard@ma4.upc.edu



(a) Method I



(b) Method II

Fig. 1. Partial sliding motion is introduced in the trajectories of the system

In this paper we introduce two different techniques to reduce the amplitude of a non-desired limit cycle. Both techniques are supported again by nonsmooth bifurcation theory. In the first one, a corner-sliding bifurcation (a special type of corner collision bifurcation) is introduced. This technique can reduce the amplitude of the limit cycle arbitrarily, and even one can suppress it. The other technique relies on a grazing-sliding bifurcation [6]. This type of nonsmooth bifurcation is introduced in the system, reducing also arbitrarily the amplitude of the (undesired) original limit cycle. Both techniques introduce partial sliding motion along a sliding surface (see Fig.1).

The rest of the paper is outlined as follows. In section II the general situations which are relevant for the systems that we are interested in, are developed. In sections III and IV the control strategies proposed are presented. In section V the technique for controlling limit cycles is applied to the standard normal form of a Hopf bifurcation. Finally the conclusions are presented in section VI.

II. BACKGROUND

Let us consider a general autonomous system defined by:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \quad (1)$$

where $\mathbf{F} := (F_1, F_2, \ldots, F_n) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a smooth vector field. If certain conditions are met, we can consider also

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piecewise smooth vector fields. We assume that the system exhibits a stable limit cycle of period T. Our objective is to design a feedback controller to reduce in amplitude (or even suppress) such periodic oscillations, based on a nonsmooth bifurcation event [1],[6].

A. Corner–Sliding: a brief description

In many control systems and electronic switching devices, switching conditions may be governed by several overlapping inequalities. A generic feature of such examples is that the discontinuity boundary has a corner-type singularity formed by the intersection between smooth codimension one surfaces $\Sigma_1 := \{\mathbf{x} \in \mathbb{R}^n : H_1(\mathbf{x}) = 0\}$ and $\Sigma_2 := \{\mathbf{x} \in \mathbb{R}^n : H_2(\mathbf{x}) = 0\}$ at a non-zero angle.

The locus of corners $\mathcal{C}$ will in general be a $(n-2)$-dimensional subset of the phase space $\mathbb{R}^n$. The passage of a trajectory through a point in $\mathbf{c} \in \mathcal{C}$ is a non-smooth bifurcation event because, in a neighborhood of the corner, there are distinct trajectories that do not behave similarly with respect to regions S_1 and S_2 on either side of $\Sigma_1 \cup \Sigma_2$. If such a corner-colliding trajectory is part of an isolated periodic orbit $p(t)$, we shall refer to this as a *corner-collision grazing bifurcation*, or 'corner collision' for short (this is the case, for example, of DC/DC buck converters [7]).

Here, there are two different regions, namely S_1 and S_2. In each zone the system presents a different dynamical behavior described by different vector fields. Whenever the boundary between the two regions is crossed (i.e. whenever the trajectory reaches the corner region), the system vector field looses continuity.

As some parameter is varied, a corner-collision can occur where a limit cycle hits the tip of the corner region (see Fig.2). Further parameter variations can lead to several different scenarios. To classify the possible scenarios following a corner collision the key issue is to be able to construct the Poincaré normal form map of the cycle undergoing the bifurcation. Recently it was shown that a local map describing the dynamics of the system close to a corner-collision point can be derived by using the concept of *discontinuity map* [8].

If, after the corner collision, the orbit slides on the border of the corner region, we will refer to that as a corner-sliding. Thus, this is a special case of a corner collision bifurcation.

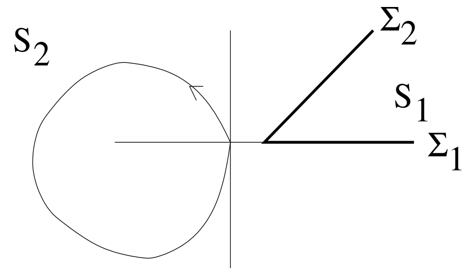
In the case presented in this paper we assume that the system can be described as:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \mathbf{u}(\mathbf{x}) \qquad (2)$$

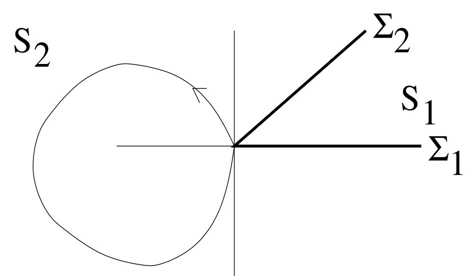
where $\mathbf{u}(\mathbf{x})$ is the switching control signal, which will vanish on S_2. $\mathbf{u}(\mathbf{x})$ is responsible for changing the characteristics of the vector field in the corner region.

B. Grazing–Sliding: a brief description

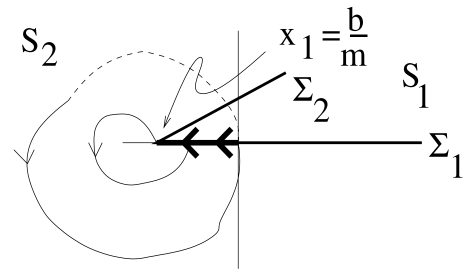
In nonsmooth Filippov systems sliding motions are possible due to vector fields on adjacent regions of the state space can be oriented towards the switching surface, one opposite to the other. Thus limit cycles can have part of the orbit in the sliding region of the switching surface. When a non-sliding



(a) Before the bifurcation



(b) Corner-Sliding bifurcation



(c) After the bifurcation

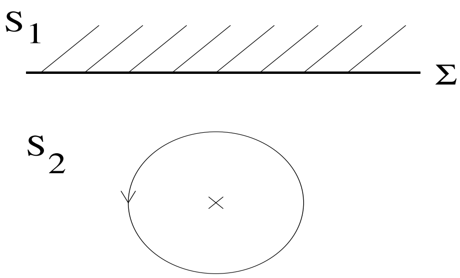
Fig. 2. Method I: A corner-sliding bifurcation is introduced in the system.

limit cycle grazes the sliding region as a parameter is varied, a nonsmooth transition named grazing-sliding occurs [4]. After this nonsmooth transition, as the bifurcation parameter is further varied, the resulting limit cycle has part of it in the sliding surface (see Fig.3). The sliding flow moves locally towards the boundary of the sliding region, when perturbed from the bifurcation point.

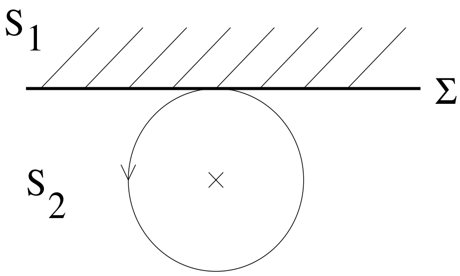
Recently it has been shown that the so-called discontinuity map is very useful in determining the structure of this bifurcation, namely the normal form [4]. In [4] several types of sliding bifurcations and their normal forms are classified and computed. It was shown that the Poincaré map of a grazing-sliding bifurcation is piecewise-linear. We make use of this theoretical result in section V to efficiently compute the corresponding map.

III. Method 1: Corner-Sliding

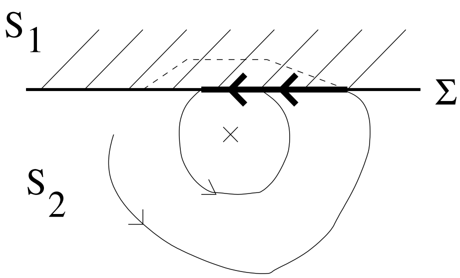
According to the theory of corner-collisions, the Poincaré map of a limit cycle undergoing such a bifurcation is piecewise-linear and dependent on the vector fields inside and outside the corner. The corner is also supposed such that sliding on one of its boundaries is forced to occur. Without loss of generality, we select $\mathbf{u}$ as the switching controller



(a) Before the bifurcation



(b) Grazing-Sliding bifurcation



(c) After the bifurcation

Fig. 3. Method II: A grazing-sliding bifurcation is introduced in the system

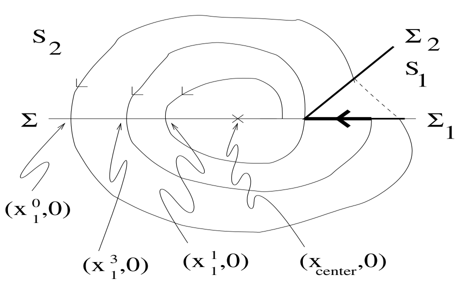


Fig. 4. A Poincaré section is computed.

defined by:

$$\mathbf{u} = \begin{cases} \mathbf{0} & \text{if} \quad \mathbf{x} \in S_2 \\ \phi(\mathbf{x}, t) & \text{if} \quad \mathbf{x} \in S_1 \end{cases} \tag{3}$$

with $S_1 \subseteq R^2$ being the region (corner) limited by the manifolds defined as $\Sigma_1 = \{H_1(\mathbf{x}) = 0\}$ and $\Sigma_2 = \{H_2(\mathbf{x}) = 0\}$ as depicted in Fig. 2. With this choice of $\mathbf{u}$ the controlled system becomes:

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{F}(\mathbf{x}) & \text{if} \quad \mathbf{x} \in S_2 \\ \mathbf{F}(\mathbf{x}) + \phi(\mathbf{x}, t) := \mathbf{G}(\mathbf{x}) & \text{otherwise} \end{cases} \tag{4}$$

In order for the control to be effective we need to select the boundaries of regions S_1 and S_2, i.e. define the corner in phase space. According to the theory of corner-sliding, the corner must be such that (i) sliding or Filippov solutions are forced to occur on one of its boundaries; (ii) it penetrates the cycle to be controlled as one of its defining parameters is changed so that at some critical parameter value the target limit cycle undergoes a corner-collision bifurcation.

In order to force sliding mode [8] we choose $H_1(x)$ and $H_2(x)$ so that $\langle \nabla H_1(\mathbf{x}), \mathbf{F}(\mathbf{x}) \rangle < 0$, $\langle \nabla H_1(\mathbf{x}), \mathbf{G}(\mathbf{x}) \rangle > 0$.

The point at which corner collision occurs will be identified as $\mathbf{x}^0$ and corresponds to the intersection of Σ_1 and Σ_2. For simplicity, we suppose a counterclockwise direction of the vector field in a neighborhood of the corner collision point. We select $H_1(\mathbf{x}) := -x_2$, and $H_2(\mathbf{x}) := x_2 - mx_1 + b$, with m and b being real constants. It is easy to see that it is possible to rescale the system coordinates so that a corner-collision occurs when $b = 0$ at the point $\mathbf{x}^0 = (0, 0)$. Therefore varying b we can move the tip of the corner and hence yield a corner-collision bifurcation.

We then have that the local interaction of the cycle with the corner can be described by using the local mapping Π defined in a convenient Poincaré section Σ (for x_1 in the neighborhood of $\frac{b}{m}$) (see Fig.4):

$$\mathbf{x} = (x_1, x_2) \mapsto \begin{cases} \Pi(\mathbf{x}) := (P \cdot x_1 + x_c \cdot (1 - P), 0) & \text{if} \quad x_1 \in (x_1^1, x_c) \\ \Pi(\mathbf{x}) := (x_1^3, 0) & \text{if} \quad x_1 < x_1^1 \end{cases} \tag{5}$$

where P corresponds to the attraction of the original stable limit cycle and can be approximately computed, as

$$x_{image} - x_c = P(x_{inicial} - x_c)$$

and x_c stands for x_{center}.

Note that we only need to know $c_1 \geq 0$ and $c_2 \geq 0$ such that $|F_i(x)| \leq c_i$, $i = 1, 2$ for $x \in [x_{min}, x_{max}]$, being $x \in [x_{min}, x_{max}]$ a neighborhood of the intersection of the cycle with the corner.

Then we choose $\phi(\mathbf{x}, t) = (-c_1, -c_2)$ in the corner region to make sure that we have sliding, and that the sliding dynamics points towards the center of the cycle. Even, one can obtain an estimate of the amplitude of the reduced cycle to be

$$\delta = P \cdot (\frac{b}{m} - x_{center})$$

Thus one can choose $x_{corner} := \frac{b}{m}$ in order to modify this amplitude δ arbitrarily. Some simulations in a representative example will be shown later.

IV. METHOD 2: GRAZING-SLIDING

With this new method, we introduce a grazing-sliding bifurcation in the system in order to reduce the amplitude of the limit cycle.

First, we consider an appropriate region S_1 of the form

$$S_1 := \{x = (x_1, x_2) : x_2 > m\}$$

for arbitrary m, being

$$\Sigma := \{x = (x_1, x_2) : x_2 = m\}.$$

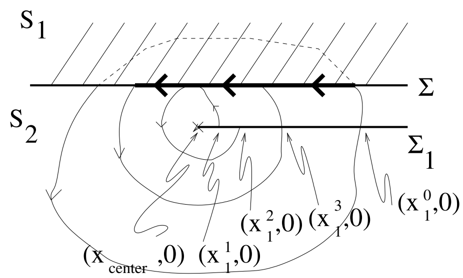


Fig. 5. A Poincaré section is computed.

As parameter m is varied, a grazing-sliding bifurcation occurs and, under the same conditions on F as the previous section, the amplitude of the limit cycle is reduced as desired (see Fig.3).

We choose a Poincaré section Σ_1 as in Fig.5. Let $(x_1^0, 0)$ be the intersection of Σ_1 with the original stable limit cycle. For fixed m, there is a new stable limit cycle with a sliding part on the border of S_1. Let $(x_1^3, 0)$ be the intersection of Σ_1 with the new stable limit cycle which has a sliding part on the border of S_1. For every $m > 0$ there exists a point $(x_1^1, 0) \in \Sigma_1$ such that the orbit starting at $(x_1^1, 0) \in \Sigma_1$ grazes region S_1. The Poincaré map maps $(x_1^1, 0) \in \Sigma_1$ into a point $(x_1^2, 0) \in \Sigma_1$ (see Fig.5).

Then the Poincaré map Π on Σ_1 is defined as

$$\mathbf{x} = (x_1, x_2) \mapsto \begin{cases} \Pi(\mathbf{x}) := (P \cdot x_1 + x_c \cdot (1 - P), 0) & \text{if} \quad x_1 \in (x_c, x_1^1) \\ \Pi(\mathbf{x}) := (x_1^3, 0) & \text{if} \quad x_1 > x_1^1 \end{cases} \tag{6}$$

where x_c stands for x_{center}.

Some simulations on a representative example are shown in the next section.

V. A 2-D REPRESENTATIVE EXAMPLE

We choose the planar normal form of a Hopf bifurcation, described by:

$$\begin{aligned} \dot{x_1} &= \varepsilon(x_1 + 1)\left(a - \sqrt{(x_1+1)^2 + x_2^2}\right) - x_2 \quad &:= F_1 \\ \dot{x_2} &= \varepsilon x_2 \left(a - \sqrt{(x_1+1)^2 + x_2^2}\right) + x_1 + 1 \quad &:= F_2 \end{aligned} \tag{7}$$

This system exhibits a stable limit cycle, which is a perfect circle of radius a centered in $(x_1^*, x_2^*) = (-1, 0)$. The system flow moves in an anti-clockwise direction as time increases and hence crosses the line $x_2 = 0$ upwards. We fix $a = 1$ and $\epsilon = 0.1$

A. Method I

For method 1 we choose the region S_1 as the phase space set (corner) bounded by $H_1(\mathbf{x}) := -x_2 = 0$ and $H_2(\mathbf{x}) := x_2 - m x_1 = 0$. We fix $m = 0.001$. Note that when $b = -0.9$ a corner-collision occurs, as the limit cycle of radius 1 hits the tip of the corner defined above at the point $(0, 0)$. Varying the control parameter b will cause the corner to penetrate the limit cycle and hence change its properties.

By varying m and b, we expect the properties of the Poincaré map to change and hence those of the limit cycle.

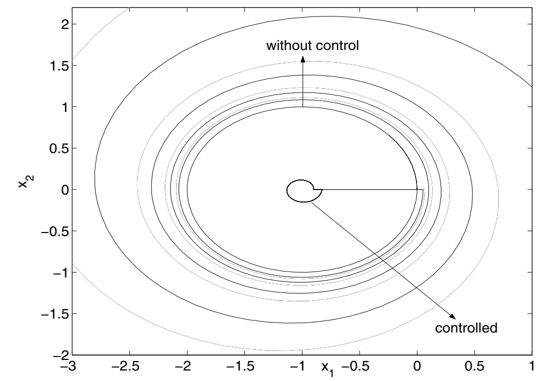


Fig. 6. Method I. The original limit cycle is plotted together with the actual limit cycle of the controlled system. The amplitude is reduced.

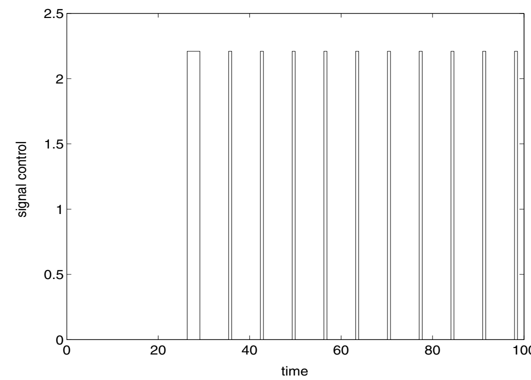


Fig. 7. Method I: Control signal.

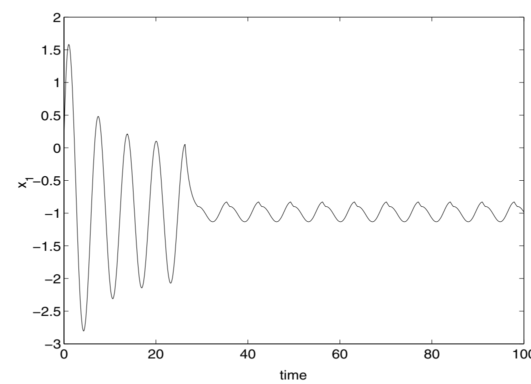


Fig. 8. Method I: Dynamics of x_1.

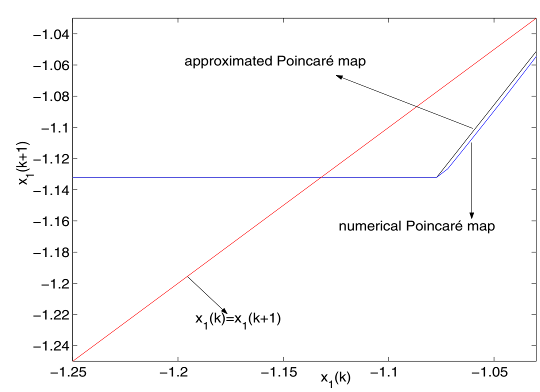


Fig. 9. Method I: Poincaré map.

For example, Fig.6 shows how the amplitude of the original limit cycle is reduced. Figure7 shows the control signal, which is only non-vanishing for small intervals. Figure8 shows the waveform of the first variable. The ripple is considerably reduced, as desired.

Fig. 9 shows the piecewise-linear map for $b = -0.9, m = 0.001$, both the approximated computing P, and the numerically computed. As it can be seen, the PWL map is non-invertible on one part of it, as it is predicted by the theory. The non-invertible part agrees completely both with the approximated and numerically computed maps.

Note that m can be reduced and thus the corner region can be designed as thin as desired without interfering with the target dynamics. Here we can observe that the controller is indeed effective in varying the map and therefore reduce the amplitude of the limit cycle or even make it disappear.

B. Method II

For method 2, we choose the region S_1 as the phase space corresponding to an upper half-plane, as specified in the previous section. For $m = 1$ a grazing-sliding bifurcation occurs, and the cycle is modified in its sliding part. The amplitude can be changed reducing the value of m, and even the cycle itself can be suppressed for $m = 0$.

Fig.10 shows how the amplitude of the original limit cycle is reduced. Figure11 shows the control signal, which is only non-vanishing for small intervals. Figure12 shows the waveform of the first variable. The ripple is considerably reduced, as desired.

Fig. 13 shows the piecewise-linear map for $m = 0.1$, both the approximated computing P, and the numerically computed. As it can be seen, the PWL map is non-invertible on one part of it, as it is predicted by the theory. The non-invertible part agrees completely both with the approximated and numerically computed maps.

VI. CONCLUSIONS

In this paper we have shown that it is possible to synthesize a switching control law to change the amplitude of a target limit cycle in a planar smooth dynamical system. In so doing, the theory of nonsmooth bifurcations can be explicitly used in the design process. Namely, by appropriately selecting the control switching manifolds, it is possible to change the properties of the Poincaré map associated to the cycle of interest. The resulting control action is acting on the system in a relatively small neighborhood of the bifurcation point and hence guarantees the achievement of the control energy with a minimal control expenditure. We wish to emphasise that rather than being yet another technique for the control of bifurcations in nonlinear systems, the strategy presented

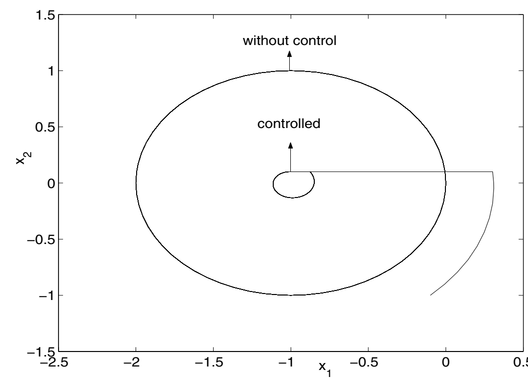


Fig. 10. Method II. The original limit cycle is plotted together with the actual limit cycle of the controlled system. The amplitude is reduced.

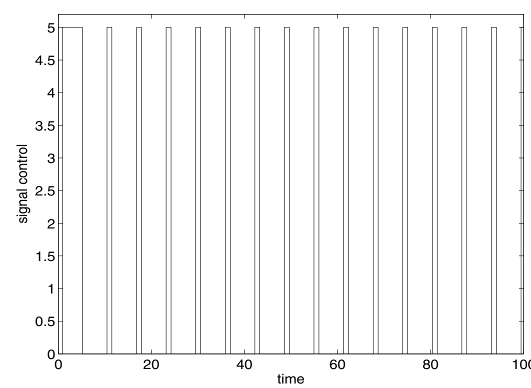


Fig. 11. Method II: Control signal.

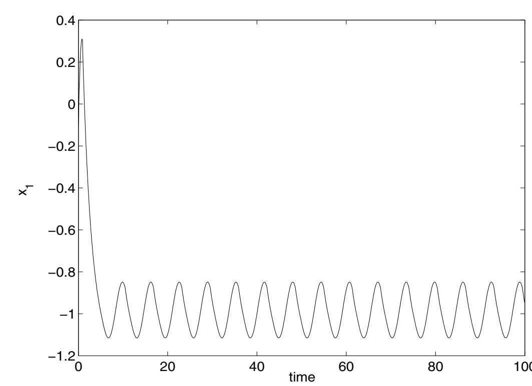


Fig. 12. Method II: Dynamics of x_1.

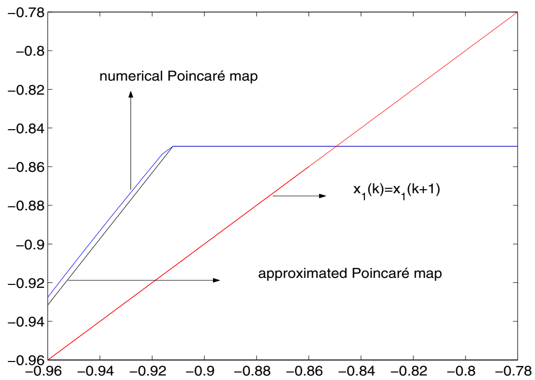


Fig. 13. Method II: Poincaré map.

here aims at exploiting the theory of nonsmooth bifurcations for control system design.

Ongoing research is aimed at further exploring the ideas presented in this paper and establish formal links between the controller gains and the properties of Ω-limit set of the closed-loop system.

VII. ACKNOWLEDGMENTS

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