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Luenberger Observers For Switching Discrete-Time Linear Systems

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***Abstract*— State estimation using Luenberger-like observers is considered for a class of switching discrete-time linear systems. The switching is assumed to be unknown among the various system modes described by known matrices. The convergence of the error dynamics is ensured, even in the presence of bounded noises, by conditions that can be expressed by means of Linear Matrix Inequalities (LMIs). The design of such observer may be accomplished by minimizing an upper bound on a quadratic cost function of the estimation error using LMI-based optimization techniques. Moreover, an improvement to the estimator is presented that is based on a projection technique.**

I. INTRODUCTION

In this paper, we address the problem of constructing Luenberger observers for switching discrete-time linear systems in the presence of unknown switching in a given finite set of admissible cases. Such a problem turns out to be more difficult than classical observer design. A Luenberger observer provides an estimation error convergent to zero if and only if the gain is chosen such that the poles of the error dynamics are in the strictly stable region [1]. In this context, the problem was faced by suitably extending the classical Luenberger observer [2], where the switching times are assumed to be known. Later on, other results were presented in [3], [4], [5], [6], where the hypothesis on the knowledge of the switching times is relaxed. Other investigations were focused on observability issues that arise for various classes of hybrid systems [7], [8], [9].

In [2], [10], the problem of constructing observers for switching systems has been solved by assuming the perfect knowledge of the switching times and modes, where design methods are proposed that consists in finding observer gains that admit a common Lyapunov function for the error dynamics. Here we take advantage of such results and propose an estimation scheme based on the combination of the identification of the discrete state with the estimation of the state variables by means of a Luenberger-like observer. In addition, the problem of determining the observer gains has been reduced to the fulfillment of LMIs. Thus, the performance of the estimator can be tuned by minimizing upper bound on a quadratic cost function of the estimation error using LMI-based optimization routines [11]. Moreover, the proposed observer may also be applied with success in the presence of bounded process and measurement noises, provided that the procedure for identifying the discrete state is suitably modified. Finally, along the lines of [2], [10], the observer is provided with an additional estimation update based on a simple projection technique. For the sake of brevity, all the proofs are omitted.

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Before concluding this section, let us introduce some notations and basic definitions. Given a generic vector v, $\|v\|$ denotes the Euclidean norm of v and, given a positive definite matrix P, $\|v\|_P$ denotes the weighted norm of v, $\|v\|_P \triangleq (v^\top P v)^{1/2}$. For a generic time-varying vector v_t, let us define $v_{t-N}^t \triangleq \text{col}\,(v_{t-N}, v_{t-N+1}, \ldots, v_t)$. For a symmetric positive or negative definite matrix D, $\sigma_{\min}\,(D)$ and $\sigma_{\max}\,(D)$ are the minimum and maximum eigenvalues of D, respectively. The norm of a matrix B is $\|\,B\,\| \triangleq \sqrt{\sigma_{\max}\,(B^\top B)}$. Given a generic matrix M, we denote as $\text{span}(M)$ the linear space generated as a linear combination of the columns of M.

II. OBSERVERS FOR SWITCHING DISCRETE-TIME LINEAR SYSTEMS

Let us consider a class of switching discrete-time linear systems described by

$$\begin{aligned} x_{t+1} &= A(\lambda_t)\, x_t \\ y_t &= C(\lambda_t)\, x_t \end{aligned} \tag{1}$$

where $t = 0, 1, \ldots$ is the time instant, $x_t \in \mathbb{R}^n$ is the continuous state vector (the initial state x_0 is unknown), $y_t \in \mathbb{R}^m$ is the measurement vector, and $\lambda_t \in \Lambda \triangleq \{1, 2, \ldots, M\}$ is the discrete state. $A(\lambda)$ and $C(\lambda)$, $\lambda \in \Lambda$, are $n \times n$ and $m \times n$ matrices, respectively.

If we assume to perfectly know λ_t, an asymptotic observer for (1) is the following:

$$\hat{x}_{t+1} = A(\lambda_t)\, \hat{x}_t + L(\lambda_t)\, [y_t - C(\lambda_t)\, \hat{x}_t] \tag{2}$$

where $t = 0, 1, \ldots$, $\hat{x}_t$ is the estimate of x_t, $\hat{x}_0$ is chosen "a priori", and $L(\lambda_t)$ is the observer gain at the time instant t, i.e., we require that the gain $L(\lambda)$ is associated with the couple $(A(\lambda), C(\lambda))$, $\lambda \in \Lambda$.

Under the knowledge of λ_t, the dynamics of the estimation error $e_t \triangleq x_t - \hat{x}_t$ behaves like a switching system and a common Lyapunov function can be searched in order to ensure the stability of the estimation error [2], [10]. Unfortunately, if the switching mode λ_t is unknown, the design of a Luenberger observer turns out to be much more difficult. With this respect, at any time stage t one can try to identify the discrete state λ_t on the basis of the observation of the output of the system over a certain interval "around" the current time t. Then the prediction $\hat{x}_{t+1}$ of the continuous state x_{t+1} can be obtained as

$$\hat{x}_{t+1} = A(\hat{\lambda}_t)\, \hat{x}_t + L(\hat{\lambda}_t)\, \left[y_t - C(\hat{\lambda}_t)\, \hat{x}_t \right] \tag{3}$$

where $\hat{\lambda}_t$ is some estimate of the discrete state λ_t. In the following, a possible approach for the choice of the estimate $\hat{\lambda}_t$ will be proposed, that ensures the convergence of the estimation error under suitable assumptions.

Towards this end, let us consider a generic sequence $\pi \in \Lambda^N$ of N discrete states, i.e., $\pi \triangleq \left(\lambda^{(1)}, \ldots, \lambda^{(N)}\right)$, and define the observability matrix associated with such a sequence as

$$F(\pi) \triangleq \begin{bmatrix} C(\lambda^{(1)}) \\ C(\lambda^{(2)})A(\lambda^{(1)}) \\ \vdots \\ C(\lambda^{(N-1)})A(\lambda^{(N-2)}) \cdots A(\lambda^{(1)}) \\ C(\lambda^{(N)})A(\lambda^{(N-1)}) \cdots A(\lambda^{(1)}) \end{bmatrix} .$$

Furthermore, let $\Phi(\pi) \triangleq A(\lambda^{(N)}) \cdots A(\lambda^{(1)})$ be the transition matrix associated to π. Note that the time-invariance of system (1) with respect to the extended state (x_t, λ_t) ensures also the time-invariance of the matrices $F(\pi)$ and $\Phi(\pi)$. This will be true also for the other quantities defined in the following.

Given a switching pattern $\pi \in \Lambda^N$, let us denote by $\mathcal{S}(\pi)$ the set of all the possible vectors of observations associated with the switching pattern π, i.e.,

$$\mathcal{S}(\pi) \triangleq \left\{ y \in \mathbb{R}^{mN} : y = F(\pi)x \,,\ x \in \mathbb{R}^n \right\} .$$

Of course $\mathcal{S}(\pi)$ is the linear space generated by the columns of $F(\pi)$.

In order to identify the discrete state λ_t, a first very simple idea would consist in considering as possible estimates of λ_t only the discrete states $\hat{\lambda}_t$ such that $\hat{\lambda}_t \in \mathcal{S}(\lambda)$. Unfortunately – as shown in [12], [8] – in general this would not lead to a reliable estimate $\hat{\lambda}_t$, in that, unless the number of measures available at each time step is at least equal to the number of state variables (i.e., $m \geq n$), it may not be possible to detect switches that occur in the last or in the first instants of an observation window . With this respect and along the lines of [13], [12], one could try to identify the discrete state λ_t on the basis of the observations vector over an extended interval of the form $[t-\alpha, t+\omega]$. Of course, this causes a delay equal to ω in the computation of $\hat{\lambda}_t$ and so of $\hat{x}_{t+1}$. It is important to remark that such a delay is unavoidable in order to obtain a reliable information on the discrete state λ_t.

Suppose now that, at time instant t, the discrete state of the system is λ. Furthermore, given two switching patterns $\pi \in \Lambda^N$ and $\pi' \in \Lambda^{N'}$, let us denote as $\pi \otimes \pi' \in \Lambda^{N+N'}$ the switching pattern obtained from the concatenation of π and π'. Then it is immediate to verify that the observations vector $y_{t-\alpha}^{t+\omega}$ belongs to the set

$$\begin{aligned} \mathcal{S}^{\alpha,\omega}(\lambda) \quad &\triangleq \quad \Big\{ y \in \mathbb{R}^{m(1+\alpha+\omega)} : y = F(\pi \otimes \lambda \otimes \bar{\pi})x \,, \\ &\qquad x \in \mathbb{R}^n,\ \pi \in \Lambda^\alpha,\ \bar{\pi} \in \Lambda^\omega \Big\}. \\ &= \bigcup_{\pi \in \Lambda^\alpha, \bar{\pi} \in \Lambda^\omega} \mathcal{S}(\pi \otimes \lambda \otimes \bar{\pi}) \,. \end{aligned}$$

As a consequence, in the following we shall consider as possible estimates of λ_t only the discrete states $\hat{\lambda}_t$ belonging to the set of feasible discrete states $\Lambda_t^{\alpha,\omega}$ (i.e., the set of all the discrete states consistent with the observations vector $y_{t-\alpha}^{t+\omega}$), defined as

$$\Lambda_t^{\alpha,\omega} \triangleq \left\{ \lambda \in \Lambda : \quad y_{t-\alpha}^{t+\omega} \in \mathcal{S}^{\alpha,\omega}(\lambda) \right\} .$$

Remark 1: In order to define the sets $\mathcal{S}^{\alpha,\omega}(\lambda)$ and $\Lambda_t^{\alpha,\omega}$, no assumptions have been made on the evolution of the discrete state. In many practical cases, the a-priori knowledge of the system may include some constraints on the law governing such an evolution. Think, for example, of the case in which the discrete state is slowly varying, i.e., there exists a minimum number τ of steps between one switch and the following one (see [13]). Another possibility arises when the switches between different configurations of the matrix A are unpredictable and unknown but the switches in the measurement equation, i.e., in the matrix C, are supposed to be known (this happens, for example, when not all measures are available at each sample time as the operating frequencies of the sensors are different). Of course, such a-priori knowledge may make the task of recovering the discrete state from the measures $y_{t-\alpha}^{t+\omega}$ considerably simpler. As a consequence, at every time step, instead of considering all the possible switching patterns belonging to $\Lambda^{\alpha+\omega+1}$, one could consider a restricted set $\Pi_t^{\alpha,\omega} \subseteq \Lambda^{\alpha+\omega+1}$ of all the admissible switching patterns, i.e., of all the switching patterns consistent with the a-priori knowledge of the law governing the evolution of the discrete state. With this respect, it could be convenient to introduce a new time-varying set $\mathcal{S}_t^{\alpha,\omega}(\lambda)$, that is,

$$\mathcal{S}_t^{\alpha,\omega}(\lambda) \triangleq \bigcup_{\pi \in \Lambda^\alpha, \bar{\pi} \in \Lambda^\omega, \pi \otimes \lambda \otimes \bar{\pi} \in \Pi_t^{\alpha,\omega}} \mathcal{S}(\pi \lambda \bar{\pi}) \,,$$

and to update the definition of the set $\Lambda_t^{\alpha,\omega}$ accordingly. This would add no theoretical difficulty but some notational complication. Hence, not to complicate the presentation, in the following we shall always suppose the law governing the evolution of the discrete state completely unknown.

Of course, if no assumptions are made on system (1), it is quite possible that, at the generic time instant t, the cardinality of the set $\Lambda_t^{\alpha,\omega}$ is strictly greater than one. In this case, it is impossible to determine uniquely the current discrete state λ_t on the basis of the observations vector $y_{t-\alpha}^{t+\omega}$. As shown in [8], [12], the possibility of distinguishing between two different discrete states depends on the current continuous state. With this respect, let us define as $\mathcal{X}^{\alpha,\omega}(\lambda, \lambda')$ the set of all the continuous states x such that, if $x_t = x$ and $\lambda_t = \lambda$, then λ' *may* belong to $\Lambda_t^{\alpha,\omega}$. Such a set can be determined as the set of all the continuous states x of the form

$$x = \Phi(\pi)\tilde{x}$$

for all $\pi \in \Lambda^\alpha$ and $\tilde{x} \in \mathbb{R}^n$ such that there exist $\bar{\pi} \in \Lambda^\omega$, $\tilde{x}' \in \mathbb{R}^n$, $\pi' \in \Lambda^\alpha$, and $\bar{\pi}' \in \Lambda^\omega$ with

$$F(\pi \otimes \lambda \otimes \bar{\pi})\tilde{x} = F(\pi' \otimes \lambda' \otimes \bar{\pi}')\tilde{x}'.$$

Hence $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$ turns out to be the union of a finite number of linear subspace of $\mathbb{R}^n$. The following elementary example should clarify the previous definition.

Example 1: Consider a simple linear switching system described by equation (1) with

$$\begin{aligned} A(1) &\triangleq \begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix}, \quad A(2) \triangleq \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}, \\ C(1) &\triangleq [-1 \;\; -2], \quad C(2) \triangleq [-1 \;\; -2]. \end{aligned} \tag{4}$$

Suppose that, at every time step $t = 0, 1, \ldots,$ we would like to determine the discrete state λ_t on the basis of the observations vector y_t^{t+2} (this corresponds to the choices $\alpha = 0$ and $\omega = 2$). Since we have

$$\begin{aligned} F(1,1,1) &= F(1,1,2) = \begin{bmatrix} -1 & -2 \\ 0 & -2 \\ 1 & -2 \end{bmatrix}, \\ F(1,2,1) &= F(1,2,2) = \begin{bmatrix} -1 & -2 \\ 0 & -2 \\ 2 & -2 \end{bmatrix}, \\ F(2,1,1) &= F(2,1,2) = \begin{bmatrix} -1 & -2 \\ 1 & -2 \\ 4 & -2 \end{bmatrix}, \\ F(2,2,1) &= F(2,2,2) = \begin{bmatrix} -1 & -2 \\ 1 & -2 \\ 7 & -2 \end{bmatrix}, \end{aligned}$$

it is immediate to verify that, if and only if $x_t = [0 \;\; k]'$ for every $k \in \mathbb{R}$, then it is impossible to distinguish between the two discrete states. Hence in this case $\mathcal{X}^{0,2}(1,2) = \mathcal{X}^{0,2}(2,1) = \{x = [0 \;\; k]',\; k \in \mathbb{R}\}$.

By exploiting the definition of the sets $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$, it is possible to give sufficient condition that the gains of observer (3) have to satisfy in order to ensure the convergence of the estimation error. More specifically, the following theorem can be stated.

Theorem 1: Suppose that the gains $L(\lambda),\ \lambda \in \Lambda$ satisfy the following conditions:

(i) $[A(\lambda) - L(\lambda)\,C(\lambda)]^\top P\,[A(\lambda) - L(\lambda)\,C(\lambda)] - P < 0$, for $\lambda \in \Lambda$, where $P = P^\top > 0$;

(ii) $\{[A(\lambda) - A(\lambda')] - L(\lambda')\,[C(\lambda) - C(\lambda')]\}\,x = 0$, for every $x \in \mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$ and for every $\lambda \neq \lambda'$.

Furthermore, suppose that at any time instant $t = \alpha, \alpha+1, \ldots,$ the estimate $\hat{\lambda}_t$ is chosen inside the set $\Lambda_t^{\alpha,\omega}$. Then observer (3) involves an estimation error exponentially convergent to zero, i.e., there exist $h > 0$ and $0 < \beta < 1$ such that

$$\|e_t\| \leq h\,\beta^{t-\alpha}\,\|e_\alpha\| \quad , \quad t = \alpha, \alpha+1, \ldots \quad . \tag{5}$$

□

It is worth noting that Theorem 1 ensures the convergence of the estimation error, regardless of the values of the estimates $\hat{\lambda}_t,\ t = \alpha, \alpha+1, \ldots,$ as long as they are chosen inside the sets $\Lambda_t^{\alpha,\omega}$. However, it should be clear that a sensible choice of such estimates could improve the performance of the proposed observer. With this respect, a reasonable choice consists in the minimum residual evaluation test:

$$\hat{\lambda}_t = \arg\min_{\lambda \in \Lambda_t^{\alpha,\omega}} \|y_t - C(\lambda)\hat{x}_t\|^2.$$

The basic idea behind Theorem 1 is quite simple: condition (i) is quite classical and ensures the existence of a quadratic Lyapunov function for the error dynamics; condition (ii) is needed to decouple the error dynamics from that of the system. At first glimpse, condition (ii) may look quite restrictive, however it is important to remark that there are some special non-trivial cases in which it automatically holds, regardless of the choice of the gains $L(\lambda)$. More specifically, condition (ii) holds if either

(a) $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda') = \{0\}$ for every $\lambda \neq \lambda'$ or

(b) $[A(\lambda) - A(\lambda')]\,x = 0$ and $[C(\lambda) - C(\lambda')]\,x = 0$ for every $x \in \mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$ and for every $\lambda \neq \lambda'$.

Clearly, case (a) is actually a subcase of (b). However, for the sake of clarity, we prefer to consider such two cases separately. First note that (a) corresponds to the complete identifiability of the discrete state λ_t, in that, in this case, unless $x_t = 0$, the set of feasible discrete states $\Lambda_t^{\alpha,\omega}$ has always cardinality 1 and therefore the current discrete state λ_t can be determined uniquely on the basis of the observations vector $y_{t-\alpha}^{t+\omega}$. As shown in [8], [13], where the results presented in [7] are extended, a necessary and sufficient condition for (a) to hold is that the rank of the joint observability matrix $[F(\pi \otimes \lambda \otimes \bar{\pi}) \;\; F(\pi' \otimes \lambda' \otimes \bar{\pi}')]$ is equal to $2n$ for every $\lambda \neq \lambda'$ and every $\pi, \pi' \in \Lambda^\alpha$ and $\bar{\pi}, \bar{\pi}' \in \Lambda^\omega$. It is worth noting that the convergence results of [13], [12] in the framework of receding-horizon estimation have been derived under an assumption similar to (a), while, in the light of Theorem 2, the proposed Luenberger-like estimation scheme can be applied to a broader class of switching systems. With this respect, it is immediate to verify that the simple system considered in Example 1, for which conditions (a) do not hold, falls within case (b), in that

$$\begin{aligned} [A(1) - A(2)] \begin{bmatrix} 0 \\ k \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ [C(1) - C(2)] \begin{bmatrix} 0 \\ k \end{bmatrix} &= 0. \end{aligned}$$

Let us now consider condition (i) of Theorem 1. As it is well known, necessary conditions for the Lyapunov inequalities to hold is that the pairs $(A(\lambda), C(\lambda))$ are detectable. Note that, though each inequality in (i) separately admits a solution if and only if the pairs $(A(\lambda), C(\lambda))$ are detectable, indeed, in order to ensure stability, a more restrictive condition is required, i.e., the existence of a matrix P satisfying all the inequalities. As it is difficult to find a common Lyapunov function once the gains have been selected, it is preferable to simultaneously choose the matrices $L(\lambda)$ and P. This problem may be reduced to a simpler form that is well-suited to being solved by means of an LMI method. Likewise in [2], using the Schur lemma, each inequality in (i) turns out to be equivalent to

$$\begin{bmatrix} P & (P\,A(\lambda) - Y(\lambda)\,C(\lambda))^\top \\ P\,A(\lambda) - Y(\lambda)\,C(\lambda) & P \end{bmatrix} > 0 \tag{6}$$

where $L(\lambda) = P^{-1}\,Y(\lambda)$. As to (ii), recall that each set $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$ is the union of a finite number, say $N_s(\lambda,\lambda')$, of linear subspaces of $\mathbb{R}^n$ and, consequently, can be written as

$$\mathcal{X}^{\alpha,\omega}(\lambda,\lambda') = \bigcup_{i=1}^{N_s(\lambda,\lambda')} \mathrm{span}\,(B_i(\lambda,\lambda'))\ .$$

where the matrix $B_i(\lambda,\lambda')$ represents a base of the i-th linear space in $\mathcal{X}^{\alpha,\omega}(\lambda,\lambda')$. Thus each condition in (i) turns out to be equivalent to the $N_s(\lambda,\lambda')$ conditions:

$$\{P[A(\lambda) - A(\lambda')] - Y(\lambda')[C(\lambda) - C(\lambda')]\}\,B_i(\lambda,\lambda') = 0 \tag{7}$$

for $i = 1,\ldots,N_s(\lambda,\lambda')$.

By exploiting (6) and (7), observer (3) can be constructed by solving the following LMI problem.

Problem 1: Find $P = P^\top > 0$ and $Y(\lambda)$, $\lambda \in \Lambda$, such that conditions (6) and (7) are satisfied for any $\lambda, \lambda' \in \Lambda$ and take the observer gains $L(\lambda) = P^{-1}\,Y(\lambda)$. □

We would like once more to point out that conditions (7) may be in general quite restrictive and may make Problem 1 unfeasible. However, if either (a) or (b) holds, then conditions (7) are automatically verified, and only (6) have to be satisfied.

The satisfaction of the Lyapunov inequalities (6) guarantees to get a stable error dynamics. In addition, along the lines of previous results (see, e.g., [10]), an upper bound on a quadratic cost function of the estimation error can be found and, consequently, the gains of observer (3) may be selected so as to minimize it. More specifically, if one consider the performance index

$$J_N = \sum_{t=0}^{N} e_t^\top\,Q\,e_t \tag{8}$$

where the weight matrix $Q > 0$ can be chosen arbitrarily, an upper bound on the asymptotic value of J_N can be minimized by solving the following problem (see [10]).

Problem 2: Given a symmetric positive definite matrix Q, find $\nu > 0$, $\delta > 0$, $P = P^\top > 0$, and $Y(\lambda)$, $\lambda \in \Lambda$, that minimize ν under the constraints

$$\delta I - P > 0$$

$$\begin{bmatrix} \nu\,P - \delta\,Q & (\nu P\,A(\lambda) - \nu Y(\lambda)\,C(\lambda))^\top \\ \nu P\,A(\lambda) - \nu Y(\lambda)\,C(\lambda) & \nu P \end{bmatrix} > 0$$

for $\lambda \in \Lambda$, and (7) for any $\lambda, \lambda' \in \Lambda$. Then, take the observer gains $L(\lambda) = P^{-1}\,Y(\lambda)$. □

Problem 2 can be solved by using LMI-based iterative optimization methods as the conditions are LMIs if ν is kept constant.

Example 1 (continued): Let us now consider once again the simple system described in Example 1. Furthermore, suppose that the weight matrix Q is chosen equal to I. By using the routines of the Matlab LMI Toolbox, the following solution of Problem 2 was obtained:

$$L(1) = \begin{bmatrix} 1.3596 \\ -1.8597 \end{bmatrix}, \quad L(2) = \begin{bmatrix} 4.0815 \\ -3.9012 \end{bmatrix}.$$

It is immediate to verify that such gains satisfy condition (i) of Theorem 2 with the Lyapunov matrix

$$P = \begin{bmatrix} 212.2196 & 242.1431 \\ 242.1431 & 281.5651 \end{bmatrix}.$$

Since for the considered system condition (ii) is automatically verified, such gains involve an estimation error exponentially convergent to zero, provided that the estimates $\hat{\lambda}_t$ are chosen inside the sets $\hat{\Lambda}_t^{\alpha,\omega}$, $\quad t = \alpha, \alpha+1, \ldots$.

III. Design of the observer in the presence of bounded noises

It is natural to ask whether the estimation scheme proposed in the previous section can be applied with success even in the presence of noises affecting the system and the measurement equations. Towards this end, let us now suppose that system (1) is affected by noises, i.e., let us consider the noisy discrete-time linear systems

$$\begin{aligned} x_{t+1} &= A(\lambda_t)\,x_t + w_t \\ y_t &= C(\lambda_t)\,x_t + v_t \end{aligned} \tag{9}$$

where $w_t \in \mathcal{W} \subset \mathbb{R}^n$ is the system noise vector and $v_t \in \mathcal{V} \subset \mathbb{R}^m$ is the measurement noise vector. We assume the statistics of w_t, and v_t to be unknown.

First, it is important to remark that the definition of $\Lambda_t^{\alpha,\omega}$ given in the previous section should be updated in order to take into account the presence of noises. As a consequence, a new set $\bar{\Lambda}_t^{\alpha,\omega}$ of feasible discrete states in the presence of noises should be defined.

In order to define formally the set $\bar{\Lambda}_t^{\alpha,\omega}$, some preliminary definitions are needed. Given a generic sequence $\pi \in \Lambda^N$ of N discrete states, i.e., $\pi \triangleq \left(\lambda^{(1)},\ldots,\lambda^{(N)}\right)$, let us define

$$H(\pi) \triangleq \begin{bmatrix} 0 & \cdots & 0 \\ C(\lambda^{(2)}) & \cdots & 0 \\ C(\lambda^{(3)})A(\lambda^{(2)}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ C(\lambda^{(N)}) \prod_{i=1}^{N-2} A(\lambda^{(N-i)}) & \cdots & C(\lambda^{(N)}) \end{bmatrix}.$$

Then the set $\bar{\mathcal{S}}(\pi)$ of all the possible vectors of observations associated with the switching pattern $\pi \in \Lambda^N$ in the presence of noises can be defined as

$$\bar{\mathcal{S}}(\pi) \triangleq \Big\{ y \in \mathbb{R}^{mN} : y = F(\pi)x + H(\pi)w + v\,, \\ x \in \mathbb{R}^n,\, w \in \mathcal{W}^{N-1},\, \in \mathcal{V}^N \Big\}.$$

Remark 2: Generally speaking, if no "a priori" assumption is made on the form of the sets $\mathcal{W}$ and $\mathcal{V}$, determining the set $\bar{\mathcal{S}}(\pi)$ can be a difficult task. However, if the sets $\mathcal{W}$ and $\mathcal{V}$ are polytopes, then also the set $\tilde{\mathcal{S}}(\pi) \triangleq \left\{ y \in \mathbb{R}^{mN} : y = H(\pi)w + v\,,\ w \in \mathcal{W}^{N-1}, v \in \mathcal{V}^N \right\}$ is a polytope. Hence the set $\bar{\mathcal{S}}(\pi)$, that is obtained as the Minkowski sum of the linear space $\mathcal{S}(\pi)$ and the polytope $\tilde{\mathcal{S}}(\pi)$, is a polyhedron. Therefore, it is possible to find a suitable matrix $\Psi(\pi)$ and a suitable vector $\rho(\pi)$ such that

$$\bar{\mathcal{S}}(\pi) = \left\{ y \in \mathbb{R}^{mN} :\ \Psi(\pi)y \geq \rho(\pi) \right\}\,.$$

Suppose now that, at time instant t, the discrete state of the system is λ. Then, it is immediate to verify that the observations vector $y_{t-\alpha}^{t+\omega}$ belongs to the set

$$\bar{\mathcal{S}}^{\alpha,\omega}(\lambda) \triangleq \bigcup_{\pi \in \Lambda^\alpha, \bar{\pi} \in \Lambda^\omega} \bar{\mathcal{S}}(\pi \otimes \lambda \otimes \bar{\pi})\,.$$

Accordingly, the set $\bar{\Lambda}_t^{\alpha,\omega}$ of all the discrete states consistent with the observations vector $y_{t-\alpha}^{t+\omega}$ in the presence of noises is given by

$$\bar{\Lambda}_t^{\alpha,\omega} \triangleq \left\{ \lambda \in \Lambda : \quad y_{t-\alpha}^{t+\omega} \in \bar{\mathcal{S}}^{\alpha,\omega}(\lambda) \right\}\,.$$

Note that, in the light of Remark 2, if the sets $\mathcal{W}$ and $\mathcal{V}$ are polytopes, then in order to find the set $\bar{\Lambda}_t^{\alpha,\omega}$ it is sufficient to find the set of discrete states $\lambda \in \Lambda$ such that $\Psi(\pi \otimes \lambda \otimes \bar{\pi}) y_{t-\alpha}^{t+\omega} \geq \rho(\pi \otimes \lambda \otimes \bar{\pi})$ for some $\pi \in \Lambda^\alpha$ and $\bar{\pi} \in \Lambda^\omega$.

By exploiting the foregoing definitions, the prediction $\hat{x}_{t+1}$ of the continuous state x_{t+1} in the noisy case can be obtained as

$$\begin{aligned} &\hat{x}_{t+1} = A(\hat{\lambda}_t)\, \hat{x}_t + L(\hat{\lambda}_t) \left[y_t - C(\hat{\lambda}_t)\, \hat{x}_t \right] \\ &\hat{\lambda}_t \in \bar{\Lambda}_t^{\alpha,\omega}(t) \end{aligned} \tag{10}$$

for $t = \alpha, \alpha+1, \ldots$.

In order to proceed to the derivation of the convergence properties of the proposed observer in the presence of bounded noises, let us now define $\bar{\mathcal{X}}^{\alpha,\omega}(\lambda, \lambda')$ as the sets of all the vectors x such that, if $x_t = x$ and $\lambda_t = \lambda$, then λ' may belong to $\bar{\Lambda}_t^{\alpha,\omega}$. The following technical lemma gives a characterization of such a set.

Lemma 1: Suppose that the sets $\mathcal{W}$ and $\mathcal{V}$ are bounded. Then, each vector $x \in \bar{\mathcal{X}}^{\alpha,\omega}(\lambda, \lambda')$ can be written as

$$x = x^h + x^b \tag{11}$$

where $x^h \in \mathcal{X}^{\alpha,\omega}(\lambda, \lambda')$ and x^b is norm-bounded, i.e., there exists a suitable constant $k(\lambda, \lambda')$ such that $\|x^b\| \leq k(\lambda, \lambda')$. □

Lemma 1 ensures that, as long as the current state $x_t = x$ is "far enough" from the set $\mathcal{X}^{\alpha,\omega}(\lambda, \lambda')$, then, even in the presence of bounded noises, it is possible to distinguish between the discrete states λ and λ' on the basis of the observations vector $y_{t-\alpha}^{t+\omega}$. In the light of such result, the following theorem can be stated.

Theorem 2: Suppose that the sets $\mathcal{W}$ and $\mathcal{V}$ are bounded. Furthermore, suppose that the gains $L(\lambda)$, $\lambda \in \Lambda$ satisfy conditions (i) and (ii) of Theorem 1, i.e.,

(i) $[A(\lambda) - L(\lambda)\, C(\lambda)]^\top P\, [A(\lambda) - L(\lambda)\, C(\lambda)] - P < 0$ for every $\lambda \in \Lambda$;
(ii) $\{[A(\lambda) - A(\lambda')] - L(\lambda')\, [C(\lambda) - C(\lambda')]\}\, x = 0$, for every $x \in \mathcal{X}^{\alpha,\omega}(\lambda, \lambda')$ and for every $\lambda \neq \lambda'$.

Then observer (10) involves an estimation error that can be upper bounded as

$$\|e_t\| \leq h\, \beta^{t-\alpha}\, \|e_\alpha\| + \frac{1 - \beta^{t-\alpha}}{1 - \beta}\, \gamma\,, \quad t = \alpha, \alpha+1, \ldots \tag{12}$$

for some $0 < \beta < 1$, $h > 0$, and $\gamma > 0$. □

Note that, since $\beta < 1$, the upper bound on the estimation error given in Theorem 2 converges exponentially to the asymptotic value $\gamma/(1-\beta)$.

IV. An enhanced projection-based observer for switching systems

Observer (3) provides an estimate of the state at time $t+1$ using the measures available at time t by means of y_t. As a matter of fact, one could aim at determining the estimate $\hat{x}_{t+1}$ using also y_{t+1}. To this end, following the lines of [2], [10], a method is proposed and consists in updating the estimate given by observer (3) by means of a projection technique, which allows one to take y_{t+1} into account. More specifically, we regard $\hat{x}_{t+1}$ as an "a priori" estimate of x_{t+1} at time $t+1$ and want to determine a new estimate $\hat{x}_{t+1}^+$ having estimation error $e_{t+1}^+ \triangleq x_{t+1} - \hat{x}_{t+1}^+$.

Towards this end, the state space is decomposed into two orthogonal subspaces, like, for example, the null space of $C(\lambda_{t+1})$ (i.e., $N\left(C(\lambda_{t+1})\right) \triangleq \{\, x \in \mathbb{R}^n : C(\lambda_{t+1})\, x = 0 \,\}$) and its orthogonal space $N\left(C(\lambda_{t+1})\right)^{\perp P}$ using the scalar product $< x, z >_P \triangleq x^\top P\, z$, $x, z \in \mathbb{R}^n$ (this scalar product is well-defined as the matrix P is positive definite). If P is taken equal to the identity matrix, it is easy to verify that $N\left(C(\lambda_{t+1})\right)^\perp$ is $\mathrm{span}\left(C(\lambda_{t+1})^\top\right)$.

The decomposition can be accomplished by means of the subspaces given by $\mathrm{span}(P^{-1}\, C(\lambda_{t+1})^\top)$ and its orthogonal complement, instead of $N\left(C(\lambda_{t+1})\right)^\perp$ and $N\left(C(\lambda_{t+1})\right)$. The reason for using this subspace decomposition concerns the stability of the estimation error as will be clarified in the following.

Let η be the components of the projection of x on $\mathrm{span}(P^{-1}\, C(\lambda_{t+1})^\top)$, i.e., for definition, $x - P^{-1}\, C(\lambda_{t+1})^\top \eta$ is orthogonal to $\mathrm{span}(P^{-1}\, C(\lambda_{t+1})^\top)$ with respect to the scalar product $< \cdot, \cdot >_P$. It turns out that

$$\left(P^{-1}\, C(\lambda_{t+1})^\top z\right)^\top P\, \left(x - P^{-1}\, C(\lambda_{t+1})^\top \eta\right) = 0\,,$$

for all $z \in \mathbb{R}^m$ and we obtain

$$\eta = \left(C(\lambda_{t+1})\, P^{-1}\, C(\lambda_{t+1})^\top\right)^{-1} C(\lambda_{t+1})\, x \quad .$$

Thus, the projection of x_{t+1} on $\mathrm{span}\left(P^{-1}\, C(\lambda_{t+1})^\top\right)$ using the scalar product $< \cdot, \cdot >_P$ is given by $P^{-1}C(\lambda_{t+1})^\top \left(C(\lambda_{t+1})P^{-1}C(\lambda_{t+1})^\top\right)^{-1} C(\lambda_{t+1})\, x_{t+1}$ i.e., $P^{-1}C(\lambda_{t+1})^\top \left(C(\lambda_{t+1})P^{-1}C(\lambda_{t+1})^\top\right)^{-1} y_{t+1}$ Note that the projection matrix $P^{-1}C(\lambda_{t+1})^\top \left(C(\lambda_{t+1})P^{-1}C(\lambda_{t+1})^\top\right)^{-1}$ is well-defined as the matrix $C(\lambda_{t+1})$ is assumed to have full row rank m. In practice, the estimate of x_{t+1} is obtained by projecting $\hat{x}_{t+1}$ on the subspace corresponding to the new measure y_{t+1}, which provides a new estimate $\hat{x}^+_{t+1}$ such that e^+_{t+1} is smaller than that of the previous error, i.e., $\left\|e^+_{t+1}\right\|_P \le \|e_{t+1}\|_P$.

Of course, the discrete state λ_{t+1} is not available, therefore one must use some estimate $\hat{\lambda}_{t+1}$ chosen inside the set $\Lambda^{\alpha,\omega}_{t+1}$. With this respect, at every time step $t = \alpha, \alpha+1, \ldots,$ estimation may be performed as follows:

$$\begin{cases} \hat{x}_{t+1} = A(\hat{\lambda}_t)\, \hat{x}^+_t + L(\hat{\lambda}_t)\, \left(y_t - C(\hat{\lambda}_t)\, \hat{x}^+_t\right) \quad \hat{\lambda}_t \in \Lambda^{\alpha,\omega}_t \\ \hat{x}^+_{t+1} = \hat{x}_{t+1} + P^{-1}C(\hat{\lambda}_{t+1})^\top \times \\ \times \left(C(\hat{\lambda}_{t+1})P^{-1}C(\hat{\lambda}_{t+1})^\top\right)^{-1} \left(y_{t+1} - C(\hat{\lambda}_{t+1})\, \hat{x}_{t+1}\right) \\ \hfill \hat{\lambda}_{t+1} \in \Lambda^{\alpha,\omega}_{t+1} \end{cases} \tag{13}$$

where $\hat{x}^+_\alpha$ is chosen "a priori".

Like for observer (3), the gains have to be chosen to ensure a stable estimation error by looking for a common Lyapunov function P. Furthermore, in order to decouple the estimation error dynamics from the system dynamics, either conditions (a) or (b) given in Section II have to be verified. More specifically, the following theorem can be stated.

Theorem 3: Suppose that the gains $L(\lambda)$, $\lambda \in \Lambda$ satisfy the following conditions:

$$P - [A - L(\lambda)\, C(\lambda)]^\top \left\{P - C(\lambda')^\top \left[C(\lambda')P^{-1}C(\lambda')^\top\right]^{-1} \times C(\lambda')\right\}[A(\lambda) - L(\lambda)\, C(\lambda)] > 0 \tag{14}$$

for every $\lambda, \lambda' \in \Lambda$ where $P = P^\top > 0$.

Furthermore, suppose that either

(a) $\mathcal{X}^{\alpha,\omega}(\lambda, \lambda') = \{0\}$ for every $\lambda \neq \lambda'$ or
(b) $[A(\lambda) - A(\lambda')]\, x = 0$ and $[C(\lambda) - C(\lambda')]\, x = 0$ for every $x \in \mathcal{X}^{\alpha,\omega}(\lambda, \lambda')$ and for every $\lambda \neq \lambda'$.

Then observer (13) involves an estimation error exponentially convergent to zero. □

The design of observer (13) may be accomplished by means of LMIs, as the following lemma holds (see [10]).

Lemma 2: Given a symmetric positive definite matrix P, the inequalities (14) are implied by the conditions

$$\begin{bmatrix} P & * & * & * \\ P\, A(\lambda) - Y(\lambda)\, C(\lambda) & P & * & * \\ 0 & C(\lambda') & \gamma I & * \\ 0 & 0 & \kappa\, C(\lambda')^\top & P \end{bmatrix} > 0\,, \quad \lambda, \lambda' \in \Lambda \tag{15}$$

where $\gamma > 0$, $\kappa \neq 0$, $L(\lambda) = P^{-1}\, Y(\lambda)$ and the blocks replaced by $*$ can be readily inferred by imposing the symmetry of the matrix. □

As pointed out in [10], it is worth noting that (15) is a sufficient but not necessary condition for (14) to hold. However, the former is easier than the latter to handle for the construction of observer (13). Therefore, a more feasible approach to the design of such observer consists in solving the following LMI problem.

Problem 3: Find $\gamma > 0$, $\kappa \neq 0$, $P = P^\top > 0$, and $Y(\lambda)$, $\lambda \in \Lambda$, such that conditions (15) are satisfied and take the observer gains $L(\lambda) = P^{-1}\, Y(\lambda)$. □

As a final remark, by exploiting the results of Section III, it is straightforward to extend the modified estimation scheme (13) in order to deal with unknown but bounded noises by choosing the estimates $\hat{\lambda}_t$, $t = \alpha, \alpha+1, \ldots,$ inside the sets $\bar{\Lambda}^{\alpha,\omega}_t$. Even in this case, in the light of Lemma 1, convergence results similar to those of Theorem 2 could be easily derived.

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