Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Global asymptotic stabilization by output feedback under a state norm detectability assumption

Laurent Praly and Alessandro Astolfi

Abstract— **It is shown that, if the state norm is detectable, the result that global asymptotic stabilizability and complete observability imply semi-global stabilizability by output feedback can be transformed into a global result. Besides the use of a state norm observer, a key feature of our output feedback design is the exploitation of the recursive high gain observer design introduced in [1].**

I. Introduction

The problem of asymptotic stabilization by means of output feedback of general nonlinear systems has been widely studied in the last years. Following the fundamental semi-global existence result presented in [8], several attempts have been made, on one side to extend this result to obtain, for instance, global properties, and, on the other hand, to construct explicit output feedback algorithms for classes of nonlinear systems. Within this second research direction several results have been proposed for systems that are linear in the unmeasured states, see *e.g.* [6], [2] and references therein, or for systems in normal forms, see *e.g.* [4], [3], [5] and references therein. In this paper we focus on the first research direction, namely we aim at extending the result in [8] to achieve a global output feedback stabilization result. This is achieved by adding a further assumption, namely a state norm detectability assumption, and exploiting the novel observer design methodology developed in [1]. It must be noted that other extensions of the result of [8] have been proposed, see *e.g.* [3], [7] and the references therein. However, these results are still concerned with semiglobal properties, whereas we aim at obtaining a globally stabilizing output feedback.

II. Main result and discussion

A. Main result

The dynamics of the system under consideration are described by the equations :

$$\dot{x} \;=\; f(x,u) \quad , \qquad y \;=\; h(x) \tag{1}$$

with input $u \in \mathbb{R}$, state $x \in \mathbb{R}^n$ and measured output $y \in \mathbb{R}$, and where the functions f and h are sufficiently many times differentiable. Without loss of generality we choose the coordinates so that the origin is the desired equilibrium to be (globally) asymptotically stabilized by output feedback. For such a system we make the following assumptions.

Assumption A1 (Uniform observability) : *There exists a C^1 function Φ and an integer m such that, for any solution of the system :*

$$\left\{\begin{array}{rcl} \dot{x} &=& f(x,u_0)\ , \\ \dot{u}_0 &=& u_1\ , \\ &\vdots& \\ \dot{u}_{m-1} &=& u_m\ , \\ y &=& h(x) \end{array}\right. \tag{2}$$

with state $(x,u_0,\ldots,u_{m-1}) \in \mathbb{R}^{n+m}$ and input $u_m \in \mathbb{R}$, we have the following identity for each time t in its domain of definition :

$$x(t) = \Phi(y(t), y_1(t), \ldots, y_m(t), u_0(t), \ldots, u_{m-1}(t)) \tag{3}$$

where y_i denotes the i-th time derivative of the output y.

Assumption A2 (Stabilizability) : *There exists a C^{m+1} function φ such that the origin is a globally asymptotically stable solution of $\dot{x} = f(x,\varphi(x))$.*

Assumption A3 (State norm detectability) : *There exist C^1 functions W, α and β, such that α is non-increasing in its first argument, β is non-decreasing in its first argument and we have, for all (x,u),*

$$\dot{\overbrace{W(x)}} \;=\; \frac{\partial W}{\partial x}(x)\, f(x,u) \;\leq\; \alpha(W(x),u,h(x)) \tag{4}$$

and :

$$|x| \;\leq\; \beta(W(x),h(x)) \qquad \forall x\ . \tag{5}$$

Moreover, we have :

$$\alpha(0,u,h) \;\geq\; 0 \qquad \forall (u,h)\ , \tag{6}$$

and there exist a continuous function $\overline{\alpha}$, two non-negative real numbers c_1 and W_* and four strictly positive real numbers c_2, c_3, η and α_* satisfying, for all (W,u,h),

$$\alpha((1+c_3)W + c_1, u, h) \;+\; c_2 \;\leq\; [1+c_3]\,\alpha(W,u,h) \tag{7}$$

$$\alpha(W,u,h) \;\leq\; \overline{\alpha}(u,h)\ , \tag{8}$$

and, for all (W,u,h) satisfying $W \geq W_*$ and $|u|+|h| \leq \eta$,

$$\alpha(W,u,h) \;\leq\; -\alpha_*\ . \tag{9}$$

Theorem 1: Under Assumptions A1 to A3, there exists[1] a dynamic output feedback control law which renders all closed loop solutions defined and bounded on $[0,+\infty)$ and such that their x-component converges to the origin as t goes to $+\infty$.

Laurent Praly is with CAS - Ecole des Mines de Paris, 35, rue St-Honore, 77305 Fontainebleau CEDEX, France. E-mail: Laurent.Praly@ensmp.fr.

Alessandro Astolfi is with the Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, United Kingdom. E-mail: a.astolfi@imperial.ac.uk.

[1]See equations (37) for an expression.

B. Discussion

As already emphasized, the above result is an extension to the global case of the semi-global result established in [8] (see also [3], [7], and references therein, for other semi-global extensions). Moving to globality is a significant change (and challenge) since all bounds that we have a priori to design the feedback are no more available. As we shall see they can be recovered thanks to Assumption A3, which guarantees the existence of a dynamic system whose output is, after a finite time and in any case before any possible finite escape time, an upperbound of $|x(t)|$ for any solution.

III. Design of the output feedback

A. Design of a state norm estimator

With the functions α and β given by Assumption A3, consider the augmented system :

$$\dot{x} \;=\; f(x,u) \quad , \qquad \dot{\omega} \;=\; \alpha(\omega, u, h(x)) \; , \tag{10}$$

with the initial condition for ω taken non-negative. For such a system we recall a result established in [5].

Lemma 1: Let the input function u be in $L^\infty_{\mathtt{loc}}([0,\infty))$. Then, for any solution of (10), right maximally defined on $[0,T)$, there exists $T^* \in [0,T)$ such that we have,

$$|x(t)| \leq \beta([1+c_3]\omega(t) + c_1 + y(t)^2, y(t)) \quad \forall t \in [T^*, T) \; . \tag{11}$$

Moreover, if T is finite, we have $\lim_{t\to T} |x(t)| \;=\; +\infty$.

Note that, without loss of generality, we can assume :

$$W \;\leq\; \beta(W,h) \qquad \forall (W,h) \; . \tag{12}$$

B. Design of the controller part of the output feedback

In view of the uniform observability assumption A1, we consider system (2), which we rewrite in the compact form :

$$\dot{x}_v \;=\; f_v(x_v) \;+\; g\, u_m \tag{13}$$

with $u_v \;=\; (u_0, \ldots, u_{m-1})$ and $x_v \;=\; (x, u_v)$. In the following, depending on whether or not we need to exhibit their components, we will use equivalently the compact notation u_v or x_v or their expansions $(u_0, \ldots, u_{m-1})$ or (x, u_v).

Note now that, by the stabilizability assumption A2, there exists a globally asymptotically stabilizing feedback φ for the x-subsystem of system (13). As this function is assumed to be at least $m+1$ times differentiable, we can conclude, for instance by following a backstepping procedure, that there exists a function φ_v, which is at least C^1, such that the state feedback : $u_m \;=\; \varphi_v(x_v)$ renders the origin a globally asymptotically stable equilibrium for the extended system (13).

We have therefore obtained a dynamic state feedback for system (1) described by the equations :

$$u_m = \varphi_v(x,u_v), u = C_m\, u_v, \dot{u}_v = A_m\, u_v \;+\; B_m\, u_m \; ,$$

where we have introduced the notation[2] :

$$A_m = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix} \; , \; B_m = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{14}$$

$$C_m \;=\; \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} \; . \tag{15}$$

From the above stability property it follows that there exist a C^1, positive definite and radially unbounded (Lyapunov) function $V : \mathbb{R}^{n+m} \to \mathbb{R}_+$, a positive definite function $R : \mathbb{R}^{n+m} \to \mathbb{R}_+$ and a continuous positive definite (stability margin) function $\rho : \mathbb{R}^{n+m} \to \mathbb{R}_+$ such that :

$$\begin{aligned} &\frac{\partial V}{\partial x_v}(x_v)\,[f_v(x_v) + g\, u_m] \;<\; -R(x_v) \\ &\qquad\qquad \forall (x_v, u_m) : \, x_v \neq 0 \; , \; |u_m - \varphi_v(x_v)| \;\leq\; \rho(x_v) \; . \end{aligned} \tag{16}$$

For the subsequent developments we find convenient to introduce modifications $\widetilde{V}$ and $\widetilde{\rho}$ of V and ρ such that $\widetilde{\rho}$ is a C^1 non-increasing function from $\mathbb{R}_+$ to $(0,1]$ and, for some strictly positive real number v_*, there exists a strictly positive real number ρ_* such that :

$$\widetilde{V}(x_v) \;\;=\;\; \max\{V(x_v) - v_*, 0\}^2 \; , \tag{17}$$

$$\widetilde{\rho}(|x_v|) \;\;\geq\;\; \rho_* \qquad \forall x_v : \, \widetilde{V}(x_v) \;=\; 0 \; , \tag{18}$$

and :

$$\begin{aligned} &\frac{\partial \widetilde{V}}{\partial x_v}(x_v)\,[f(x_v) + g\, u_m] \;<\; 0 \\ &\qquad \forall (x_v, u_m) : \, \widetilde{V}(x_v) \neq 0 \; , \; |u_m - \varphi_v(x_v)| \;\leq\; \widetilde{\rho}(|x_v|^2) \; . \end{aligned} \tag{19}$$

Such modifications can always be obtained since ρ is positive definite. Note that $\widetilde{V}$ is C^1 and radially unbounded.

Note now that there exists a C^1 non-decreasing function $\mathfrak{b}_\varphi$, satisfying, for all x_v,

$$1 \;+\; |\varphi_v(x_v)| \;\leq\; \mathfrak{b}_\varphi(|x_v|^2) \; . \tag{20}$$

Using this function we have, trivially, for all x_v,

$$\varphi_v(x_v) \;=\; \mathfrak{b}_\varphi(|x_v|^2)\, \mathtt{sat}\left(\frac{\varphi_v(x_v)}{\mathfrak{b}_\varphi(|x_v|^2)}\right) \; , \tag{21}$$

where $\mathtt{sat}$ denotes the standard saturation function. However, by Lemma 1, we know that we can get an upperbound of $|x|$. These arguments motivate the following output (partial) feedback :

$$\begin{cases} \mathfrak{x}_v^2 \;= \beta\left([1+c_3]\omega + c_1 + y^2 + |u_v|^2, y\right)^2 + |u_v|^2 \\ u_m = \mathfrak{b}_\varphi\left(\mathfrak{x}_v^2\right)\, \mathtt{sat}\left(\frac{\varphi_v(\widehat{x}, u_v)}{\mathfrak{b}_\varphi(\mathfrak{x}_v^2)}\right) \; , \\ u \;= C_m\, u_v \; , \\ \dot{u}_v \;= A_m\, u_v \;+\; B_m\, u_m \; , \end{cases} \tag{22}$$

where $\widehat{x}$ represents an estimate of x to be delivered by an observer. Note that, compared to (11), $|u_v|^2$ has been

[2]The subscript m is used to denote the dimensions of the given matrices, *e.g.* $A_m \in \mathbb{R}^{m \times m}$.

added in the argument of β (see Lemma 2 below). A direct consequence of (22) is that, from now on, we can assume that u_m satisfies the bound :

$$|u_m| \;\leq\; \mathfrak{b}_\varphi(\mathfrak{r}_v^2) \;. \tag{23}$$

C. Observability normal form

In view of the expression (3), which allows to reconstruct the state x from the output, the input and their derivatives, we define the following functions

$$\left\{\begin{array}{rcccl} y_0 &=& y &=& h(x)\;, \\ y_1 &=& \dot{y}_0 &=& h_1(x,u_0)\;, \\ && \vdots && \\ y_m &=& \dot{y}_{m-1} &=& h_m(x,u_0,\ldots,u_{m-1})\;, \\ && \dot{y}_m &=& h_{m+1}(x,u_0,\ldots,u_m)\;, \end{array}\right. \tag{24}$$

where the h_i's are at least C^1, since f and h are assumed sufficiently smooth, and are obtained from successive differentiations and direct substitution of $\dot{x}$ by $f(x,u_0)$.

Note that there exist C^1 non-decreasing functions $\mathfrak{b}_h$ and $\mathfrak{b}_{h\varphi}$ satisfying, for all $(x,u_0,\ldots,u_m)$ and $i \in \{0,\ldots,m\}$, the bounds

$$\begin{array}{r} |h_i(x,u_0,\ldots,u_{i-1})| \leq \mathfrak{b}_h(|x_v|^2)\;, \\ |h_{m+1}(x,u_0,\ldots,u_m)| \leq \mathfrak{b}_h(|x_v|^2) \;+\; \mathfrak{b}_{h\varphi}(u_m^2)\;. \end{array} \tag{25}$$

With this notation, the uniform observability assumption A1 can be equivalently rewritten as $x = \Phi(h_v(x,u_v),u_v)$ for all (x,u_v), where we have used the compact notation :

$$h_v(x,u_v) = (h(x),\ldots,h_m(x,u_0,\ldots,u_{m-1}))\;. \tag{26}$$

It follows that any solution of the system (2) (or (13)) is a solution of (the converse may not be true) :

$$\left\{\begin{array}{rcl} y &=& C_{m+1}\, y_v\;, \\ \dot{y}_v &=& A_{m+1}\, y_v \;+\; B_{m+1}\, f_m(y_v,u_v,u_m)\;, \end{array}\right. \tag{27}$$

where we have used the compact notation :

$$y_v \;=\; (y_0,\ldots,y_m) \tag{28}$$

$$f_m(y_v,u_v,u_m) \;=\; h_{m+1}(\Phi(y_v,u_v),u_v,u_m)\;. \tag{29}$$

Moreover, from the bounds (25), we have :

$$|f_m(h_v(x,u_v),u_v,u_m)| \;\leq\; \mathfrak{b}_h(|x_v|^2) \;+\; \mathfrak{b}_{h\varphi}(u_m^2) \tag{30}$$

and therefore :

$$\begin{aligned} &f_m(h_v(x,u_v),u_v,u_m) \\ &\quad= \left[\mathfrak{b}_h(|x_v|^2) \;+\; \mathfrak{b}_{h\varphi}(u_m^2)\right] \operatorname{sat}\left(\tfrac{f_m(h_v(x,u_v),u_v,u_m)}{\mathfrak{b}_h(|x_v|^2) \,+\, \mathfrak{b}_{h\varphi}(u_m^2)}\right)\;. \end{aligned} \tag{31}$$

Finally, we can find C^1 non-decreasing functions $\mathfrak{l}_f$, $\mathfrak{l}_{f\varphi}$ and $\mathfrak{l}_\varphi$ satisfying :

$$\mathfrak{l}_\varphi(s) \;\geq\; 1 \qquad \forall s \geq 0 \tag{32}$$

and, for all (x,u_v,u_m),

$$\begin{aligned} &\sup_{e:|e|\leq 1} \tfrac{|f_m(h_v(x,u_v)+e,u_v,u_m) \,-\, f_m(h_v(x,u_v),u_v,u_m)|}{|e|} \\ &\qquad\qquad \leq\; \mathfrak{l}_f(|x_v|^2) \;+\; \mathfrak{l}_{f\varphi}(u_m^2)\;, \end{aligned} \tag{33}$$

$$\begin{aligned} &\sup_{e:|e|\leq 1} \tfrac{|\varphi_v(\Phi(h_v(x,u_v)+e,u_v),u_v) \,-\, \varphi_v(\Phi(h_v(x,u_v),u_v),u_v)|}{|e|} \\ &\qquad\qquad \leq\; \mathfrak{l}_\varphi(|x_v|^2)\;. \end{aligned} \tag{34}$$

D. Design of the observer

To complete the design of the output feedback given in (22) we need to obtain an estimate $\widehat{x}$ of x. Thanks to the uniform observability assumption A1, we know that x can be reconstructed from u_v and y_v. It follows that an estimate of x can be given if we have one for y_v. This motivates the design of the following observer. It is a direct application of the recursive high gain observer proposed in [1]. Its interest is that, by exploiting the state norm detectability property, it gives after a finite time and in any case before any possible finite escape time, an estimate, as precise as we need. This observer takes the form :

$$\begin{aligned} \dot{\widehat{y}}_v \;=\; A_{m+1}\,\widehat{y}_v \;+\; k\,(y - C_{m+1}\widehat{y}_v)& \\ +\; B_{m+1}\,\mathfrak{f}\operatorname{sat}\left(\frac{f_m(\widehat{y}_v,u_v,u_m)}{\mathfrak{f}}\right)& \end{aligned} \tag{35}$$

There are two specific points in this observer. The first one is in the use of the saturation function, motivated by equation (31). Exploiting the property stated in Lemma 1, we choose the gain $\mathfrak{f}$ as :

$$\left\{\begin{array}{l} \dot{\omega} \;= \alpha(\omega,u_0,y)\;, \\ \mathfrak{r}_v^2 = \beta([1+c_3]\omega + c_1 + y^2 + |u_v|^2, y)^2 \;+\; |u_v|^2 \\ \mathfrak{f} \;\;= \mathfrak{b}_h(\mathfrak{r}_v^2) \;+\; \mathfrak{b}_{h\varphi}(\mathfrak{b}_\varphi(\mathfrak{r}_v^2)^2)\;. \end{array}\right. \tag{36}$$

The second point is in the choice of the gain vector k. It is taken as a C^1 function of the signal $\mathfrak{r}_v^2$, given above, *i.e.* of (ω,u_v,y). More information on this function will be given in Section IV-C. Summarizing our design, we have obtained the following output feedback :

$$\left\{\begin{array}{ll} \mathfrak{r}_v^2 &= \beta\left([1+c_3]\omega + c_1 + y^2 + |u_v|^2, y\right)^2 + |u_v|^2\;, \\ \mathfrak{f} &= \mathfrak{b}_h(\mathfrak{r}_v^2) \;+\; \mathfrak{b}_{h\varphi}(\mathfrak{b}_\varphi(\mathfrak{r}_v^2)^2)\;, \\ \widehat{x} &= \Phi(\widehat{y}_v,u_v)\;, \\ u &= C_m\, u_v\;, \\ u_m &= \mathfrak{b}_\varphi\left(\mathfrak{r}_v^2\right) \operatorname{sat}\left(\frac{\varphi_v(\widehat{x},u_v)}{\mathfrak{b}_\varphi(\mathfrak{r}_v^2)}\right)\;, \\ \dot{\omega} &= \alpha(\omega,u,y)\;, \\ \dot{\widehat{y}}_v &= A_{m+1}\,\widehat{y}_v \;+\; k\,(y - C_{m+1}\widehat{y}_v) \\ & \qquad\qquad +\; B_{m+1}\,\mathfrak{f}\operatorname{sat}\left(\dfrac{f_m(\widehat{y}_v,u_v,u_m)}{\mathfrak{f}}\right)\;, \\ \dot{u}_v &= A_m\, u_v \;+\; B_m\, u_m\;. \end{array}\right. \tag{37}$$

IV. Analysis of the closed loop system

A. State norm estimation in finite time

The use of saturation functions, both in the controller and the observer, and the structure of the observer itself, with gains depending only on (ω,u_v,y), guarantee that the result of Lemma 1 extends to the closed loop system. Specifically, the following fact holds.

Lemma 2: Let k be a continuous function of (ω,u_v,y) only. Then, for any closed loop solution of (1), (37), right maximally defined on $[0,T)$, there exists $T_0^* \in [0,T)$ such that, for all $t \in [T_0^*,T)$,

$$|x(t)| \leq \beta([1+c_3]\omega(t) + c_1 + y(t)^2 + |u_v(t)|^2, y(t)). \quad (38)$$

Moreover, if T is finite :

$$\lim_{t\to T} |x_v(t)| \;=\; +\infty \;. \qquad (39)$$

Proof : The state of the closed loop system is $(\omega, x_v, \widehat{y}_v)$. Let $(\omega(t), x_v(t), \widehat{y}_v(t))$ be a solution right maximally defined on $[0,T)$. We observe that (6) implies $\omega(t) \geq 0$ for all $t \in [0,T)$. To prove condition (39) we first remark that, since k, $\mathfrak{f}$ and y are functions of (ω, x_v) only, and because of the saturation function, there exist continuous functions $\mathfrak{b}_{y1}$ and $\mathfrak{b}_{y2}$ such that $|\dot{\widehat{y}}_v| \;\leq\; \mathfrak{b}_{y1}(\omega,x_v)\,|\widehat{y}_v| \;+\; \mathfrak{b}_{y2}(\omega,x_v)$. It follows that, for all $t \in [0,T)$, we have :

$$|\widehat{y}_v(t)| \;\leq\; |\widehat{y}_v(0)| \,\exp\left(\int_0^t \mathfrak{b}_{y1}(\omega(s),x_v(s))\,ds\right) \qquad (40)$$
$$+\int_0^t \mathfrak{b}_{y2}(\omega(s),x_v(s))\,\exp\left(\int_s^t \mathfrak{b}_{y1}(\omega(r),x_v(r))dr\right)ds \;.$$

As a result, if T is finite and the component (ω, x_v) of the solution is bounded on $[0,T)$, so is the component $\widehat{y}_v$. This is a contradiction. Hence, if T is finite, we have $\lim_{t\to T} |(\omega(t), x_v(t))| \;=\; +\infty$. But then :

$$\omega(t) \;=\; \omega(0) \;+\; \int_0^t \alpha(\omega(s),u(s),y(s))\,ds \;. \qquad (41)$$

But, if the x_v component is bounded, so are u and y. Hence, by (8), it follows that α is bounded on $[0,T)$ along the solution. Thus, if T is finite, the ω-component must also be bounded. Again a contradiction. So (39) is established. The proof of (38) follows from exactly the same arguments as those given in [5]. $\square$

B. Existence of a bounding function

Let σ be the C^1 function defined by the ratio :

$$\sigma \;=\; \widetilde{\rho}/\mathfrak{l}_\varphi \;. \qquad (42)$$

The properties of the functions $\mathfrak{l}_\varphi$ and $\widetilde{\rho}$ imply that σ is non-increasing and upperbounded by one. Also, we can find C^1 non-decreasing functions $\mathfrak{b}_\sigma$ and $\mathfrak{b}_{\sigma\varphi}$ satisfying, for all (x_v, u_m),

$$2\sigma'(|x_v|^2)\,|x_v|\,[|f_v(x_v)| + |g||u_m|] \qquad (43)$$
$$\leq\; \mathfrak{b}_\sigma(|x_v|^2) \;+\; \mathfrak{b}_{\sigma\varphi}(|u_m|^2) \;.$$

From the various functions σ, $\widetilde{\rho}$, $b_{\dots}$ and $\mathfrak{l}_{\dots}$ introduced (see (30), (33)), (34) and (43)), we select a C^1 non-decreasing function γ satisfying :

$$\gamma(s) \;\geq\; s \quad \forall s \geq 0 \qquad (44)$$

and :

$$\gamma(s) \;\geq\; \max\left\{1\,,\; 4\tfrac{\mathfrak{b}_\sigma(s)+\mathfrak{b}_{\sigma\varphi}(\mathfrak{b}_\varphi(s)^2)}{\sigma(s)}\,, \right. \qquad (45)$$
$$\left. 4\mathfrak{l}_\varphi(s)\tfrac{\mathfrak{b}_h(s)+\mathfrak{b}_{h\varphi}(\mathfrak{b}_\varphi(s)^2)}{\widetilde{\rho}(s)}\,,\; \mathfrak{l}_f(s)+\mathfrak{l}_{f\varphi}(\mathfrak{b}_\varphi(s)^2)\right\}$$

To appreciate the interest of this function γ, observe that, for all $(x, u_v, \mathfrak{x}_v)$ and all u_m satisfying (23), the inequalities given below hold. Therein we use "$[\delta]$" to denote, generically, a term of the form

$$\max\left\{0\,,\; \frac{p(x_v,\mathfrak{x}_v)[q(|x_v|^2) - q(\mathfrak{x}_v^2)]}{\gamma(\mathfrak{x}_v^2)}\right\} \;.$$

According to Lemma 2, when p is non-negative and q is non-decreasing (resp. p is non-positive and q is non-increasing), such a term is known to be zero for each solution after time T_0^*, when we pick $\mathfrak{x}_v^2$ as (see (37)) :

$$\mathfrak{x} \;=\; \beta([1+c_3]\omega + c_1 + y^2 + |u_v|^2, y) \;, \qquad (46)$$
$$\mathfrak{x}_v^2 \;=\; \mathfrak{x}^2 \;+\; u_v^2 \;. \qquad (47)$$

• **Inequality for $\dot{\overbrace{\sigma(|x_v|^2)}}$:**

$$\begin{aligned}\frac{\left|\dot{\overbrace{\sigma(|x_v|^2)}}\right|}{\gamma(\mathfrak{x}_v^2)} \;&\leq\; \frac{2\,\sigma'(|x_v|^2)|x_v|\,|f(x_v)+gu_m|}{\gamma(\mathfrak{x}_v^2)}\;,\\ &\leq\; \frac{\mathfrak{b}_\sigma(|x_v|^2) + \mathfrak{b}_{\sigma\varphi}(|u_m|^2)}{\gamma(\mathfrak{x}_v^2)}\;,\\ &\leq\; \frac{\sigma(\mathfrak{x}_v^2)}{4} \;+\; \frac{\mathfrak{b}_\sigma(|x_v|^2) - \mathfrak{b}_\sigma(\mathfrak{x}_v^2)}{\gamma(\mathfrak{x}_v^2)}\;,\\ &\leq\; \frac{\sigma(|x_v|^2)}{4} \;+\; [\delta]\;, \end{aligned} \qquad (48)$$

• **General inequality for Δf :** To write the following expressions in compact form, let :

$$\Delta f \;=\; \mathfrak{f}\,\mathtt{sat}\left(\frac{f_m(\widehat{y}_v,u_v,u_m)}{\mathfrak{f}}\right) - f_m(h_v(x,u_v),u_v,u_m)\;. \qquad (49)$$

By (30) and the expression of $\mathfrak{f}$ in (37), we obtain :

$$\frac{|\Delta f|}{\gamma(\mathfrak{x}_v^2)} \;\leq\; \frac{\mathfrak{b}_h(|x_v|^2) + \mathfrak{b}_h(\mathfrak{x}_v^2) + 2\,\mathfrak{b}_{h\varphi}(\mathfrak{b}_\varphi(\mathfrak{x}_v^2))}{\gamma(\mathfrak{x}_v^2)}\;. \qquad (50)$$

The inequality (45) satisfied by γ gives :

$$\mathfrak{l}_\varphi(|x_v|^2)\,\frac{\mathfrak{b}_h(|x_v|^2) + \mathfrak{b}_h(\mathfrak{x}_v^2) + 2\mathfrak{b}_{h\varphi}(u_m^2)}{\gamma(\mathfrak{x}_v^2)} \qquad (51)$$
$$\leq\; \tfrac{\widetilde{\rho}(|x_v|^2)}{2} \;+\; [\delta]\;,$$

Therefore, by (42), we obtain :

$$\frac{|\Delta f|}{\gamma(\mathfrak{x}_v^2)} \;\leq\; \frac{\sigma(|x_v|^2)}{2} \;+\; [\delta]\;. \qquad (52)$$

• **Inequality for Δf for the case $|\widehat{y}_v - h_v(x,u_v)| \leq 1$:** Using (30) and exploiting the fact that the saturation function is "pushing" $\widehat{f_m}$ in the interval $[-\mathfrak{f},\mathfrak{f}]$, we obtain :

$$|\Delta f| \;\leq\; |f_m(\widehat{y}_v,u_v,u_m) - f_m(h_v(x,u_v),u_v,u_m)| + [\delta]\;. \qquad (53)$$

But, if $|\widehat{y}_v - h_v(x,u_v)| \leq 1$, with (33) and (45), we obtain :

$$\frac{|f_m(\widehat{y}_v,u_v,u_m) - f_m(h_v(x,u_v),u_v,u_m)|}{\gamma(\mathfrak{x}_v^2)} \qquad (54)$$
$$\leq\; |\widehat{y}_v - h_v(x,u_v)| \;+\; [\delta]\;.$$

This yields the implication :

$$|\widehat{y}_v - h_v(x,u_v)| \ \leq \ 1 \tag{55}$$
$$\Rightarrow \qquad \tfrac{|\Delta f|}{\gamma(\mathfrak{r}_v^2)} \ \leq \ |\widehat{y}_v - h_v(x,u_v)| + [\delta] \ .$$

• **Inequality for $\overset{\cdot}{\overbrace{\gamma(\mathfrak{r}_v^2)}}$:** Note that :

$$\overset{\cdot}{\overbrace{\gamma(\mathfrak{r}_v^2)}} \ = \ 2\,\gamma'(\mathfrak{r}_v^2)\left[\mathfrak{r}\dot{\mathfrak{r}} + u_v^T\dot{u}_v\right] \tag{56}$$

where, from the definition (46), it can be seen that $\dot{\mathfrak{r}}$ is a function of $[1+c_3]\omega + c_1 + y^2 + |u_v|^2$ and x_v. By (12), (44) and (47), it follows that we can find C^1 non-decreasing functions $\mathfrak{b}_\gamma$ and $\mathfrak{b}_{\gamma x}$ such that :

$$|\overset{\cdot}{\overbrace{\gamma(\mathfrak{r}_v^2)}}| \ \leq \ \mathfrak{b}_\gamma(\gamma(\mathfrak{r}_v^2)) \ + \ \mathfrak{b}_{\gamma x}(\mathfrak{r}_v^2) \ + \ [\delta] \ . \tag{57}$$

In view of the above inequalities, $\gamma(\mathfrak{r}_v^2)$ is called a bounding function.

C. Properties of the estimation error

This section is mainly a reproduction from [1]. To simplify notation, indices of vectors and matrices start from 0. This means, in particular, that the entry in the first column and first line of a matrix is called its (00) entry.

Let $e = \widehat{y}_v - h_v(x,u_v)$ and let $\mathcal{O} = A_{m+1} + k\,B_{m+1}$. The observer and system dynamics give simply :

$$\dot{e} \ = \ \mathcal{O}\,e \ + \ B_{m+1}\,\Delta f \ , \tag{58}$$

with Δf defined in equation (49). To study system (58) and design the gain vector k, we introduce $m+1$ functions a_i of the bounding function $\gamma(\mathfrak{r}_v^2)$, to be made precise later on. Let L be the following lower triangular matrix :

$$L = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ -a_1 & 0 & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (-a_m)^m & (-a_m)^{m-1} & \dots & -a_m & 0 \end{pmatrix} . \tag{59}$$

The vector k is chosen as :

$$\begin{pmatrix} -\,a_0 \\ {a_1}^2 \\ \vdots \\ (-a_m)^{m+1} \end{pmatrix} \ = \ (I+L)\,k \ . \tag{60}$$

Then let $\varepsilon = (I+L)\,e$, and let U be the positive definite and radially unbounded function of ε defined as :

$$U \ = \ \frac{1}{2}\sum_{j=0}^{m}(\mu_j\varepsilon_j)^2 \ , \tag{61}$$

where :

$$\begin{cases} \mu_m^2 = 4 \ , \\ \mu_j^2 = 4\ \left(1 \ + \ \displaystyle\sum_{i=j+1}^{m} i\,\ell_{ij}^2\right) \ , \quad j\in\{0,\dots,(m-1)\} \end{cases} \tag{62}$$

and ℓ_{ij} is the (i,j) entry of $(I+L)^{-1}$. This definition of U implies :

$$|e|^2 \ \leq \ U \ . \tag{63}$$

Following the same arguments as in [1] and using the properties of the function σ in (42), we can prove the following statement.

Lemma 3: The functions a_i's can be chosen as C^1 functions of the bounding function $\gamma(\mathfrak{r}_v^2)$ in such a way that, for any closed loop solution, right maximally defined on $[0,T)$, there exists $T_1^* \in [T_0^*,T)$, where T_0^* is as given by Lemma 2, such that we have (without loss of generality)

$$\sigma(|x_v(T_1^*)|^2) \ > \ 0 \tag{64}$$

and, for all $t\in[T_1^*,T)$,

$$\overset{\cdot}{\overbrace{U(t)}} \ \leq \ -\gamma(\mathfrak{r}_v(t)^2)\left[\frac{5}{2}\,U(t) - \left(\frac{\Delta f(t)}{\gamma(\mathfrak{r}_v(t)^2)}\right)^2\right] \ , \tag{65}$$

and

$$\left|\overset{\cdot}{\overbrace{\sigma(|x_v(t)|^2)}}\right| \ \leq \ \gamma(\mathfrak{r}_v(t)^2)\,\frac{\sigma(|x_v(t)|^2)}{4} \ . \tag{66}$$

D. Proof of Theorem 1

Let $(\omega(t), x_v(t), \widehat{y}_v(t))$ be a solution of the closed loop system, right maximally defined on $[0,T)$. From Lemmas 2 and 3 we infer that there exists $T_1^* \in [0,T)$ such that the solution is bounded on $[0,T_1^*]$ (of course!), inequalities (64), (65) and (66) hold, and :

$$|x_v(t)|^2 \ \leq \ \mathfrak{r}_v(t)^2 \qquad \forall t\in[T_1^*,T) \ . \tag{67}$$

Moreover the following fact holds true.

Lemma 4: There exists $T_2^* \in [T_1^*,T)$, satisfying, for all $t\in[T_2^*,T)$,

$$\overset{\cdot}{\overbrace{U(t)}} \quad \leq \quad -\frac{3}{2}\,\gamma(\mathfrak{r}_v(t)^2)\,U(t) \ , \tag{68}$$

$$|e(t)| \quad \leq \quad \frac{\widetilde{\rho}(|x_v(t)|^2)}{\mathfrak{l}_\varphi(|x_v(t)|^2)} \ \leq \ 1 \ . \tag{69}$$

Proof : From (66) and (52), we get, for all $t\in[T_1^*,T)$,

$$-\gamma(\mathfrak{r}_v(t)^2)\,\frac{\sigma(|x_v(t)|^2)}{4} \ \leq \ \overset{\cdot}{\overbrace{\sigma(|x_v(t)|^2)}} \tag{70}$$

and :

$$\frac{|\Delta f(t)|}{\gamma(\mathfrak{r}_v(t)^2)} \ \leq \ \frac{\sigma(|x_v(t)|^2)}{2} \ . \tag{71}$$

Therefore, from the inequality (65), we obtain :

$$\overset{\cdot}{\overbrace{U(t) - \sigma(|x_v(t)|^2)^2}} \tag{72}$$
$$\leq -\gamma(\mathfrak{r}_v(t)^2)\left[\tfrac{5}{2}\left[U(t) - \sigma(|x_v(t)|^2)^2\right] + \tfrac{7}{4}\sigma(|x_v(t)|^2)^2\right]$$

Let τ be the function defined as :

$$\tau(t) \ = \ \int_0^t \gamma(\mathfrak{r}_v(s)^2)ds \ . \tag{73}$$

Note that, since γ is larger than one and satisfies (44), by (39), we have :

$$\lim_{t\to T}\tau(t) \ = \ +\infty \ . \tag{74}$$

Then, the above inequalities imply that, for all $t \in [T_1^*, T)$, we have :

$$\sigma(|x_v(t)|^2) \;\geq\; \exp(-\tfrac{\tau(t)-\tau(T_1^*)}{4})\,\sigma(|x_v(T_1^*)|^2)\ , \qquad (75)$$

and :

$$\begin{aligned} &U(t) - \sigma(|x_v(t)|^2)^2 && (76)\\ &\quad\leq\; \exp(-\tfrac{5(\tau(t)-\tau(T_1^*))}{2})\,\left[U(T_1^*)) - \sigma(|x_v(T_1^*))|^2)^2\right]\\ &\qquad -\tfrac{7}{8}\,\sigma(|x_v(T_1^*)|^2)^2\\ &\qquad\quad \times\left[\exp(-\tfrac{(\tau(t)-\tau(T_1^*))}{2}) - \exp(-\tfrac{5(\tau(t)-\tau(T_1^*))}{2})\right]. \end{aligned}$$

Therefore, for all $t \in [T_1^*, T)$, we have :

$$\begin{aligned} U(t) - \sigma(|x_v(t)|^2)^2 \;\leq\; &\exp(-\tfrac{(\tau(t)-\tau(T_1^*))}{2}) \qquad (77)\\ \times\Big[\big(U(T_1^*)) - \tfrac{1}{8}\sigma(|x_v(T_1^*))|^2)^2\big)&\exp(-2(\tau(t)-\tau(T_1^*)))\\ &-\tfrac{7}{8}\sigma(|x_v(T_1^*))|^2)^2\Big] \end{aligned}$$

Since, from (64), $\sigma(|x_v(T_1^*))|^2)^2$ is strictly positive, by equation (74), the right hand side of (77) is negative after some time T_2^* in $[T_1^*, T)$. As a result, condition (69) follows, by equation (63) and the fact that the function σ defined in (42) is upperbounded by one.

Therefore, from equation (55), we conclude :

$$\frac{|\Delta f(t)|}{\gamma(\mathfrak{r}_v(t)^2)} \;\leq\; |e(t)| \;\leq\; \sqrt{U(t)} \qquad \forall t \in [T_2^*, T)\ . \qquad (78)$$

Thus equation (68) follows from equation (65). $\square$

As a result of the above discussion, we know that :

$$|e(t)| \;\leq\; \frac{\widetilde{\rho}(|x_v(t)|^2)}{\mathfrak{l}_\varphi(|x_v(t)|^2)} \;\leq\; 1 \qquad \forall t \in [T_2^*, T)\ . \qquad (79)$$

Then, by :

$$\widehat{x} \;=\; \Phi(h_v(x,u_v)+e, u_v)\ ,\ x \;=\; \Phi(h_v(x,u_v),u_v)\ , \quad (80)$$

and equation (34), we have, for all $t \in [T_2^*, T)$,

$$\begin{aligned} &|\varphi_v(\widehat{x}(t),u_v(t)) \,-\, \varphi_v(x(t),u_v(t))| && (81)\\ &\qquad\qquad \leq\; |e(t)|\,\mathfrak{l}_\varphi(|x_v(t)|^2) \;\leq\; \widetilde{\rho}(|x_v(t)|^2)\ . \end{aligned}$$

Since $\widetilde{\rho}$ is upperbounded by one, it follows from (20) and (67) that we have for all $t \in [T_2^*, T)$,

$$|\varphi_v(\widehat{x}(t),u_v(t))| \;\leq\; 1 \,+\, |\varphi_v(x(t),u_v(t))| \;\leq\; \mathfrak{b}_\varphi(\mathfrak{r}_v(t)^2)\ . \quad (82)$$

This implies that the saturation is not acting in the expression of u_m, *i.e.* :

$$u_m(t) \;=\; \varphi_v(\widehat{x}(t),u_v(t)) \qquad \forall t \in [T_2^*, T)\ . \qquad (83)$$

Hence (81) gives, for all $t \in [T_2^*, T)$,

$$|u_m(t) - \varphi_v(x(t),u_v(t))| \;\leq\; \widetilde{\rho}(|x_v(t)|^2)\ . \qquad (84)$$

Thus, by equation (19), we get :

$$\dot{\overbrace{\widetilde{V}(x_v(t))}} \;<\; 0 \qquad \forall t \in [T_2^*, T)\ . \qquad (85)$$

The function $\widetilde{V}$ being radially unbounded, it follows that the x_v-component of any closed loop solution is bounded. Moreover, we also know from (68) that the error e is bounded. So the component $(x_v, \widehat{y}_v)$ of the closed loop solution is bounded. Finally, as far as ω is concerned, we observe that equations (41), (8) and the above boundedness property imply that it cannot escape in finite time. Hence $T = +\infty$, *i.e.* the closed loop solution is right maximally defined on $[0, +\infty)$.

Then, from Lemma 4, we know that $U(t)$ and therefore $|e(t)|$ and $|u_m(t) - \varphi_v(x_v(t))|$ tend to zero as t goes to infinity. On the other hand, (16) yields, for all $t \in [0, +\infty)$,

$$\dot{\overbrace{V(t)}} \;\leq\; -R(x_v(t)) \,+\, \frac{\partial V}{\partial x_v}(x_v(t))g\left[u_m(t) - \varphi_v(x_v(t))\right]\ . \quad (86)$$

Since we have shown that $\frac{\partial V}{\partial x_v}$ is bounded along any closed loop solution, and R and V are positive definite, we conclude that $|x_v(t)|$, $|\widehat{y}_v(t)|$ and therefore $|u(t)|$ and $|y(t)|$ tend to zero as t goes to $+\infty$. Therefore, there exists $T_3^* \in [0, +\infty)$ such that :

$$|u(t)| \,+\, |y(t)| \;\leq\; \eta \qquad \forall t \in [T_3^*, +\infty)\ . \qquad (87)$$

Finally, equation (41) and (9) imply that the ω component of the closed loop solution is also bounded. This completes the proof.

V. Conclusions

It has been shown that uniform observability, state feedback stabilizability and a state norm detectability assumption imply the existence of a globally asymptotically stabilizing output feedback control law. This has been explicitly constructed exploiting the observer design tool proposed in [1], and the properties of the state norm estimator in [5].

References

[1] A. Astolfi, L. Praly Global complete observability and output-to-state stability imply the existence of a globally convergent observer. Proceedings of the 42nd IEEE Conference on Decision and Control, pp. 1562–1567, December 2003. Submitted for publication in MCSS.

[2] R.A. Freeman and P.V. Kokotović. *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. Birkhäuser, 1996.

[3] J-P. Gauthier and I. Kupka. Deterministic observation theory and applications. Cambridge University Press, 2001.

[4] A. Isidori. A tool for semiglobal stabilization of uncertain non-minimum-phase nonlinear systems via output feedback. *IEEE Trans. Autom. Control*, 45:1817–1827, 2000.

[5] G. Kaliora, A. Astolfi and L. Praly, Norm estimators and global output feedback stabilization of nonlinear systems with ISS inverse dynamics. Proceedings of the 43rd Conf. on Decision and Control, 2004.

[6] R. Marino and P. Tomei. *Nonlinear Control Design: Geometric, Adaptive and Robust*. Prentice Hall, 1995.

[7] H. Shim, A. Teel, Asymptotic controllability and observability imply semi-global practical asymptotic stabilizability by sampled-data output feedback. Automatica 39(3):441-454, 2003.

[8] A. Teel, L. Praly, Global stabilizability and observability imply semi-global stabilizability by output feedback, *Systems & Control Letters*, 22:313-325, 1994.