Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

On Model Reduction of Polynomial Dynamical Systems

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Abstract— **In this paper, we develop a computational method for model reduction of polynomial dynamical systems. This is achieved using sum of squares relaxations on certain Lyapunov inequalities, which are the nonlinear counterparts of the Lyapunov controllability and observability linear matrix inequalities for linear systems. In our model reduction procedure, we use notions of balanced realization and balanced truncation for a polynomial model. In addition, we derive an a-priori error bound on the approximation error for balanced truncation.**

I. INTRODUCTION

Balanced realizations are commonly used for controllability and observability analysis of linear systems. Balanced truncation (see, e.g., [2], [4], [7]), i.e., truncation of a balanced realization, is a popular method for model reduction. This is because balanced truncation is relatively simple, and there are strong guarantees on the quality of the reduced model. For these reasons, many generalizations of the notion of a balanced realization have been made, for example, to uncertain systems [1], to linear time-varying and parameter-varying systems [11], [14], [15], and to nonlinear systems [3], [10], [13]. Unfortunately, while for linear systems finding a balancing coordinate transformation via solutions (the so-called Gramians) of the controllability and observability Lyapunov equations are relatively easy, for nonlinear systems these equations tend to be hard to solve and therefore balancing the system is in general not a simple task.

The problem of balancing nonlinear systems with polynomial vector fields is addressed in this paper. For this, we will search for Gramians that fulfill the corresponding Lyapunov inequalities (instead of equalities) either globally or locally around some equilibrium of interest. When the Gramians are affinely parameterized using some unknown coefficients, their computation can be cast as a polynomial programming problem, for which a relaxation method called sum of squares programming [8] can be employed to search for the unknown coefficients. The Gramians can be used for reachability and observability analysis, as well as for finding a balancing transformation when they are quadratic. Finally, by modifying the inequalities and using techniques similar to those in [9], it is possible to derive a procedure with local a-priori bound on the approximation error.

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The work of the first author was financially supported by the Army Institute for Collaborative Biotechnologies, the NSF Award CCF-0326635, and the AFOSR Award FA9550-05-1-0032. The work of the second author was financially supported by the Swedish Research Council (project 2000-5630) and the Swedish Foundation for Strategic Research (project CPDC).

II. PRELIMINARIES

The standard Euclidean norm is denoted by $|x| \triangleq (x^T x)^{1/2}$. The standard norm on $L_2[0,T]$ is denoted by $\|u\| \triangleq (\int_0^T |u(t)|^2 dt)^{1/2}$.

A. Polynomial Dynamical Systems

A dynamical system G:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t), \quad x(0) = x_0, \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, and $y(t) \in \mathbb{R}^p$ is the output, is called a *polynomial system* if $f(x)$, $g(x)$, and $h(x)$ are polynomials. Without loss of generality, we will assume that the origin is the equilibrium of interest, i.e., $f(0) = 0$, and that $h(0)$ is equal to zero as well. Note also that the system (1) can always be written as

$$\begin{aligned} \dot{x}(t) &= AZ(x(t)) + \sum_{i=1}^{m} B_i Z(x(t)) u_i(t), \quad x(0) = x_0, \\ y(t) &= CZ(x(t)), \end{aligned} \tag{2}$$

where $Z(x)$ is a column vector of monomials (say, of dimension q),

$$Z(x) = \begin{bmatrix} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \\ \vdots \end{bmatrix},$$

with the degrees α_i, $\beta_i, \ldots$ being nonnegative integers, and $A \in \mathbb{R}^{n \times q}$, $B_i \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{p \times q}$ are constant matrices.

B. Sum of Squares Programming

A polynomial $p(x)$ is said to be a sum of squares (SOS), if there exist polynomials $f_1(x), \ldots, f_m(x)$ such that $p(x) = \sum_{i=1}^m f_i^2(x)$. The existence of such a decomposition can be shown equivalent to the existence of a real positive semidefinite matrix Q such that $p(x) = Z^T(x) Q Z(x)$, where $Z(x)$ is the vector of monomials of degree less than or equal to degree$(p(x))/2$. This equivalence makes an SOS decomposition computable using semidefinite programming. Computation of SOS decompositions using semidefinite programming was first suggested in [6].

It is clear that an SOS polynomial is *globally nonnegative*. This is a property of SOS polynomials that is crucial in many control applications, where we can obtain a tractable computational relaxation by replacing various polynomial inequalities with SOS conditions. Although not all nonnegative polynomials are sums of squares, in many cases we are able to obtain solutions to computational problems that are

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otherwise at the moment unsolvable, simply by replacing the nonnegativity conditions with SOS conditions.

An SOS program is a convex optimization problem of the form

$$\begin{aligned} &\text{minimize } \sum_{j=1}^{m} w_j c_j \\ &\text{s.t.} \quad a_{i,0}(x) + \sum_{j=1}^{m} a_{i,j}(x) c_j \text{ is SOS, for } i = 1, ..., p, \end{aligned}$$

where the c_j's are scalar real decision variables, the w_j's are given real numbers, and the $a_{i,j}(x)$ are given polynomials (with fixed coefficients). See also another equivalent canonical form of SOS programs in [8]. Sum of squares programs can still be solved via semidefinite programming using the equivalence relation explained above. The software SOSTOOLS [8] in conjunction with a semidefinite programming solver such as SeDuMi [12] can be used to efficiently solve SOS programs.

III. Observability and Reachability Gramians

We will first present a polynomial inequality that bounds the output of the polynomial system (1) around the origin, under the assumption that the input is zero, and the system starts at a nonzero initial condition $x(0) = x_0$.

Proposition 1: Given the system (1), suppose that there exists a positive definite polynomial $W_o(x)$ such that $W_o(0) = 0$ and

$$-\frac{\partial W_o}{\partial x}(x) f(x) - h^T(x) h(x) \geq 0 \quad \forall x \in \boldsymbol{K} \subseteq \mathbb{R}^n, \tag{3}$$

where $\boldsymbol{K}$ is a neighborhood of the origin. Then, there exists a neighborhood of the origin $\boldsymbol{B}_o \subseteq \boldsymbol{K}$ such that if the system (1) starts at $x(0) = x_0 \in \boldsymbol{B}_o$, then for zero input the norm of the system output will satisfy $\|y\|^2 \leq W_o(x_0)$, where the norm is taken over any time interval $[0, T]$, $T \geq 0$.

Proof: First notice that $W_o(x)$ is a Lyapunov function for the system $\dot{x} = f(x)$, which follows from (3) and the positive definiteness of $W_o(x)$. Choose a small enough $\gamma > 0$ such that $\boldsymbol{B}_o \triangleq \{x \in \mathbb{R}^n : W_o(x) \leq \gamma\}$ is contained in $\boldsymbol{K}$. It follows that all trajectories of the system starting from $\boldsymbol{B}_o$ will stay in $\boldsymbol{B}_o$ forever. Now, if the system starts at $x(0) = x_0 \in \boldsymbol{B}_o$, we can integrate (3) over the time interval $[0, T]$ to obtain $\|y\|^2 = \int_0^T |y(t)|^2 dt \leq W_o(x_0) - W_o(x(T)) \leq W_o(x_0)$, where the end time T is arbitrary. ■

States x_0 for which $W_o(x_0)$ is small are weakly observable in y. This is why $W_o(x)$ is called an *observability Gramian* for (1). Note also that by applying LaSalle's invariance principle we can conclude that if the largest invariant set contained in $\{x \in \mathbb{R}^n : h(x) = 0\}$ is equal to $\{0\}$, then the equilibrium at the origin is asymptotically stable.

If the domain $\boldsymbol{K}$ is semialgebraic, e.g., given as $\boldsymbol{K} = \{x \in \mathbb{R}^n : g_K(x) \geq 0\}$ for some polynomial $g_K(x)$, then a polynomial observability Gramian $W_o(x)$ can be searched using SOS programming. To ensure that (3) holds, for example, we use an SOS multiplier $\sigma(x)$ and ask that the expression $-\frac{\partial W_o}{\partial x}(x) f(x) - h^T(x) h(x) - \sigma(x) g_K(x)$ is SOS as well. Both the Gramian $W_o(x)$ and the multiplier $\sigma(x)$ are parameterized in terms of some unknown coefficients, and the values of those coefficients which satisfy the SOS conditions are computed by the SOS solver. See [5] for more discussion on this in the context of Lyapunov functions.

Similar to the case of linear systems, in the nonlinear case there exists also a duality between observability and reachability analysis. For reachability analysis, the following result is available, which gives a lower bound on the input energy needed to reach a certain state.

Proposition 2: Given the system (1), suppose that there exists a positive definite polynomial $W_c(x)$ such that $W_c(0) = 0$ and

$$-\frac{\partial W_c}{\partial x}(x)(f(x) + g(x)u) + u^T u \geq 0 \quad \forall x \in \boldsymbol{K} \text{ and } u \in \boldsymbol{L}, \tag{4}$$

where $\boldsymbol{K}$ and $\boldsymbol{L}$ are open sets containing the origin. Then, there exists a neighborhood of the origin $\boldsymbol{B}_c \subseteq \boldsymbol{K}$ such that if the system (1) starts at $x(0) = 0$ and the instantaneous input $u(t)$ is restricted to lie in the set $\boldsymbol{L}$, then the norm of the input needed to reach some state $x(T) \in \boldsymbol{B}_c$ will satisfy $\|u\|^2 \geq W_c(x(T))$.

Proof: Choose a sufficiently small $\gamma > 0$ such that $\boldsymbol{B}_c \triangleq \{x \in \mathbb{R}^n : W_c(x) \leq \gamma\}$ is contained in $\boldsymbol{K}$. Now suppose that an input $u : [0, T] \to \boldsymbol{L}$ is applied to the system, resulting in the state of the system at time T being $x(T) \in \boldsymbol{B}_c$. First, consider the case where $x(t) \in \boldsymbol{B}_c$ for all $t \in [0, T]$. By integrating (4) over the interval $[0, T]$, we obtain the inequality $\|u\|^2 = \int_0^T |u(t)|^2 dt \geq W_c(x(T)) - W_c(0) = W_c(x(T))$. On the other hand, if $x(t)$ is not contained in $\boldsymbol{B}_c$ on the whole time interval, then for some $\tilde{t} \in [0, T]$ we will have $x(\tilde{t}) \in \partial \boldsymbol{B}_c$ and $x(t) \in \boldsymbol{B}_c$ $\forall t \in [0, \tilde{t}]$. Thus, using the same argument for above, but for $x(\tilde{t})$, we obtain $\|u\|^2 \geq \int_0^{\tilde{t}} |u(t)|^2 dt \geq W_c(x(\tilde{t})) \geq W_c(x(T))$. This completes the proof of the proposition. ■

States $x(T)$ for which $W_c(x(T))$ is large cannot be reached with small inputs. This motivates us to call $W_c(x)$ a *reachability Gramian* for the system (1). Similar to the case of observability Gramian, when $\boldsymbol{K}$ and $\boldsymbol{L}$ are semialgebraic, a polynomial reachability Gramian $W_c(x)$ can be searched using SOS programming.

It is important to note that neither $W_o(x)$ nor $W_c(x)$ is unique. To obtain upper and lower bounds that are as tight as possible, $W_o(x_0)$ should be minimized and $W_c(x(T))$ should be maximized. While this is easy to do when we are only concerned with a single state x_0 or $x(T)$, generally we would be interested in obtaining tight bounds for a set of states, e.g., for all states in a neighborhood of the origin. In relation to this, a heuristics can be given as follows. First note that when computing $W_o(x)$ or $W_c(x)$ using SOS programming, we replace the inequality conditions by SOS conditions, and thus $W_o(x)$ and $W_c(x)$ will also be sums of squares. As mentioned in Section II, this is equivalent to the existence of a quadratic form

$$W_o(x) = Z_o^T(x) Q Z_o(x), \quad W_c(x) = Z_c^T(x) \tilde{P} Z_c(x)$$

where $Z_o(x)$ and $Z_c(x)$ are some vectors of monomials, and Q, $\tilde{P}$ are positive semidefinite matrices. To "minimize" $W_o(x)$ or "maximize" $W_c(x)$, we could then minimize the trace of Q, and maximize the trace of $\tilde{P}$, respectively. These objective functions can be easily included in the SOS program formulations.

Another important property of Gramians is that they are invariant under smooth invertible coordinate transformations (diffeomorphisms). Suppose that a transformation $\phi(z) = x$ is applied to the system (1). The system in the new coordinates z is given by

$$\begin{aligned}\dot{z}(t) &= \tilde{f}(z(t)) + \tilde{g}(z(t))u(t), \quad z(0) = \phi^{-1}(x(0)),\\ y(t) &= \tilde{h}(z(t)),\end{aligned} \tag{5}$$

where $\tilde{f}(z) = (\frac{\partial \phi}{\partial z})^{-1} f(\phi(z))$, $\tilde{g}(z) = (\frac{\partial \phi}{\partial z})^{-1} g(\phi(z))$, and $\tilde{h}(z) = h(\phi(z))$. It can then be shown that $\tilde{W}_o(z) = W_o(\phi(z))$ and $\tilde{W}_c(z) = W_c(\phi(z))$ satisfy the Lyapunov inequalities in Propositions 1 and 2.

IV. Input–Output Analysis Using Gramians

Once we find Gramians $W_o(x)$ and $W_c(x)$, we can conclude that the origin is stable under zero input, and that for sufficiently small inputs, the trajectory of the system will not leave a neighborhood of the origin. These observations are summed up in the following proposition.

Proposition 3: Suppose that there exist Gramians $W_o(x)$ and $W_c(x)$ satisfying the conditions of Propositions 1 and 2 for the system (1). Then there exist a neighborhood of the origin $\boldsymbol{B}_R$ and a positive constant γ_R, such that if

$$\|u\| \le \gamma_R, \quad u \in \boldsymbol{L}, \quad x(0) = 0,$$

then $x(t) \in \boldsymbol{B}_R \subseteq \boldsymbol{K}$ for all $t \ge 0$.

Proof: Let $\boldsymbol{B}_R = \boldsymbol{B}_o$ (as in Proposition 1), and let γ_R be the maximum γ such that $\{x \in \mathbb{R}^n : W_c(x) \le \gamma^2\}$ is contained in $\boldsymbol{B}_R$. Now suppose that an input $u(t)$ for which $\|u\| \le \gamma_R$ and $u \in \boldsymbol{L}$ is given. Assume that $u(t) = 0$ for $t \ge T$, where T can be infinite if $u(t)$ never becomes identically equal to zero. Then Proposition 2 can be used to show that for $t \in [0, T]$ the state $x(t)$ is in $\{x \in \mathbb{R}^n : W_c(x) \le \gamma_R^2\} \subseteq \boldsymbol{B}_R$. Finally, since $\boldsymbol{B}_R$ is an invariant set for the system with zero input, for all $t \ge T$ the state $x(t)$ is also in $\boldsymbol{B}_R$, thus proving the proposition. ■

Using Proposition 3, we can measure the interaction between u and y. Define a map Γ_G that takes inputs $u \in L_2[0, T]$, $\|u\| < \gamma_R$, and maps it through the system G in (1) into the truncated output $y \in L_2[T, \infty)$. The map Γ_G is a nonlinear version of the Hankel operator. The Hankel operator has been studied extensively in the past, and has been one of the main tools used in model reduction of linear systems [2]. The Hankel norm is defined as

$$\|G\|_{H,\gamma_r} \triangleq \sup_{\|u\| < \gamma_R,\ u \in L} \frac{\|\Gamma_G(u)\|}{\|u\|},$$

and an upper bound on the Hankel norm can be provided using the following proposition. It shows how the observability and reachability Gramians give a measure of the interaction between u and y in G.

Proposition 4: Assume that the Gramians $W_o(x)$ and $W_c(x)$ for the polynomial system G as in Propositions 1 and 2 are given. Suppose that the inequality

$$W_o(x) \le \gamma_H^2 W_c(x) \quad \forall x \in \boldsymbol{B}_H \tag{6}$$

holds, where $\boldsymbol{B}_H = \{x \in \mathbb{R}^n : W_c(x) \le \gamma_R^2\}$ and γ_R is as in Proposition 3. Then $\|G\|_{H,\gamma_r} \le \gamma_H$.

Proof: Suppose that an input $u : [0, T] \to \boldsymbol{L}$ with $\|u\| \le \gamma_R$ brings the system state to some $x(T) \in \boldsymbol{B}_H$. Then from Propositions 1 and 2 it follows that $\|\Gamma_G(u)\|^2 \le W_o(x(T)) \le \gamma_H^2 W_c(x(T)) \le \gamma_H^2 \|u\|^2$. Therefore the norm bound follows. ■

Remark 5: The Hankel norm bound γ_H can be computed using SOS programming, by formulating (6) as an SOS condition, and minimizing γ_H^2 to obtain an upper bound that is as tight as possible.

The balanced truncation approach to model reduction is based on the idea of truncating states that are least observable *and* hardest to reach. Let us first recall what happens in the case of linear systems. For linear systems that are stable and controllable, (3) and (4) can always be solved for $W_o(x)$ and $W_c(x)$ with equality instead of inequality. Moreover, it is enough to consider quadratic functions for these Gramians. In this case, $W_o(x)$ provides the exact value of the output norm in Proposition 1, and $W_c(x)$ the exact value of the optimal input norm in Proposition 2. Now, suppose that for the Gramians $W_o(x)$ and $W_c(x)$, the following inequalities hold:

$$W_o(e_1) \ge W_o(e_2) \ge \ldots \ge W_o(e_n), \tag{7}$$

$$W_c(e_1) \le W_c(e_2) \le \ldots \le W_c(e_n), \tag{8}$$

where e_i is the unit vector along the i-th coordinate axis. Since the Gramians are quadratic, the inequalities are still valid when the e_i's are replaced by $x_i = \lambda e_i$, for arbitrary $\lambda \in \mathbb{R}$. It can then be argued that the direction along the n-th coordinate axis, for which the observability Gramian is the smallest and the controllability Gramian is the largest, is the coordinate that is least observable and hardest to reach, thus indicating that the n-th state is the state to truncate.

A similar heuristics can be suggested for nonlinear systems. Assume that the Gramians are homogeneous, and that either (7)–(8) hold, or for some $i \in \{1, \ldots n\}$ we have

$$W_o(e_i) \le W_o(e_j) \quad \forall j \in \{1, \ldots n\} \setminus \{i\}, \tag{9}$$

$$W_c(e_i) \ge W_c(e_j) \quad \forall j \in \{1, \ldots n\} \setminus \{i\}. \tag{10}$$

Note that because of the homogeneity of the Gramians, the above inequalities are also satisfied by λe_i and λe_j, for any $\lambda \in \mathbb{R}$. It is then reasonable to truncate the state x_i, based on arguments similar to the above. If the Gramians are not homogeneous, then the criteria (9)–(10) can for example be replaced by

$$\int_{\lambda_1}^{\lambda_2} W_o(\lambda e_i) d\lambda \leq \int_{\lambda_1}^{\lambda_2} W_o(\lambda e_j) d\lambda \quad \forall j \in \{1, \dots n\} \setminus \{i\},$$
$$\int_{\lambda_1}^{\lambda_2} W_c(\lambda e_i) d\lambda \geq \int_{\lambda_1}^{\lambda_2} W_c(\lambda e_j) d\lambda \quad \forall j \in \{1, \dots n\} \setminus \{i\},$$

which compare the average value of the Gramians along segments of the coordinate axes around the equilibrium, for some $\lambda_1 \leq 0$ and $\lambda_2 \geq 0$.

When none of the i's satisfy (9)–(10), the choice of state to truncate is no longer clear. Although we can for example simply truncate the state x_i for which $W_o(e_i)/W_c(e_i)$ is the smallest, it is beneficial to first consider if a coordinate transformation can be performed, such that in the new coordinates the Gramians satisfy (7)–(8) or (9)–(10). When the Gramians are quadratic and positive definite, a coordinate transformation which achieves (7)–(8) always exists, and there is a constructive procedure for computing such a transformation. Applying this transformation to the system is referred to as balancing the system [4]; see also Section V-A. Moreover, the Gramians in the new balanced coordinates will satisfy an additional property that $W_o(e_i) W_c(e_i) = 1$.

In the case of non-quadratic Gramian, the procedure for balancing coordinate transformation mentioned above no longer applies. Although there exists a result stating that such a transformation always exists [10], to the best of the authors' knowledge, so far no constructive procedure has been proposed. This will be a subject of future investigation.

V. Incremental Gramians and Truncation Error

It turns out that it is theoretically easier to bound the truncation error in an incremental framework, as we shall see in this section. The problem with incremental Gramians is that they are harder to compute in practice. Some possible ways around this and the relation between the incremental and the standard Gramians are discussed in Section V-B.

The *incremental* observability and reachability Gramians $W_{i,o}(x)$ and $W_{i,c}(x)$ fulfill

$$-\left.\frac{\partial W_{i,o}}{\partial x}\right|_{x-\hat{x}} [f(x) + g(x)u - f(\hat{x}) - g(\hat{x})u] - |h(x) - h(\hat{x})|^2 \geq 0 \quad (11)$$

$$-\left.\frac{\partial W_{i,c}}{\partial x}\right|_{x+\hat{x}} [f(x) + g(x)u + f(\hat{x}) + g(\hat{x})u] + 4|u|^2 \geq 0 \quad (12)$$

for all $x, \hat{x} \in \boldsymbol{K} \subseteq \mathbb{R}^n$ and $u \in \boldsymbol{L} \subseteq \mathbb{R}^m$. Here, $\boldsymbol{K}$ and $\boldsymbol{L}$ are open sets containing the origin as before. Notice that $W_{i,o}$ fulfills (3) when $\hat{x}$ and u are zero, and that $W_{i,c}$ fulfills (4) if we put $\hat{x} = x$ and define $W_c(x) \triangleq W_{i,c}(2x)/4$. Hence, (11) and (12) are more restrictive than (3) and (4).

Integration over $[0, T]$ of (11) yields $\|y - \hat{y}\|^2 \leq W_{i,o}(x(0) - \hat{x}(0))$, assuming that the states do not leave $\boldsymbol{K}$ and that $u \in \boldsymbol{L}$. This can be guaranteed by an analysis similar to the one in Proposition 3. Hence, (11) bounds the difference in the outputs for the systems G and $\hat{G}$ of identical dynamics with the same input, but with different initial states. Similarly, integration over $[0, T]$ of (12) yields $W_{i,c}(x(T) + \hat{x}(T)) \leq 4\|u\|^2$, if $x(0) = \hat{x}(0) = 0$. These relations will be interesting if we force some of the states in $\hat{x}$ to become zero. We use the following partition of the state vector:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \hat{x} = \begin{bmatrix} \hat{x}_1 \\ 0 \end{bmatrix}, \quad x_1, \hat{x}_1 \in \mathbb{R}^{\hat{n}}, \quad (13)$$

and similarly for the vector fields, $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$. The reduced-order model $\hat{G}$ is then given by

$$\begin{aligned} \dot{\hat{x}}_1(t) &= f_1(\hat{x}_1(t), 0) + g_1(\hat{x}_1(t), 0)u(t), \quad \hat{x}_1(0) = \hat{x}_{10}, \\ \hat{y}(t) &= h(\hat{x}_1(t), 0). \end{aligned} \quad (14)$$

We also introduce the auxiliary signal $\hat{z}(t) = f_2(\hat{x}_1(t), 0) + g_2(\hat{x}_1(t), 0)u(t) \in \mathbb{R}^{n-\hat{n}}$. Let us now assume that we have solutions $W_{i,o}(x)$ and $W_{i,c}(x)$ to the inequalities (11) and (12). Using (13) and (14), we obtain

$$-\frac{d}{dt} W_{i,o}(x - \hat{x}) + \left(\left.\frac{\partial W_{i,o}}{\partial x_2}\right|_{x-\hat{x}} \right) \cdot \hat{z} - |y - \hat{y}|^2 \geq 0, \quad (15)$$

$$-\frac{d}{dt} W_{i,c}(x + \hat{x}) - \left(\left.\frac{\partial W_{i,c}}{\partial x_2}\right|_{x+\hat{x}} \right) \cdot \hat{z} + 4|u|^2 \geq 0, \quad (16)$$

where $\frac{\partial W_{i,o}}{\partial x_2}, \frac{\partial W_{i,c}}{\partial x_2} \in \mathbb{R}^{1 \times (n-\hat{n})}$. Under the assumption that the *matching condition*

$$\left.\frac{\partial W_{i,o}}{\partial x_2}\right|_{x-\hat{x}} = \sigma^2 \left.\frac{\partial W_{i,c}}{\partial x_2}\right|_{x+\hat{x}} \quad (17)$$

holds, for some positive number σ, the error between the model G in (1) and $\hat{G}$ in (14) is bounded by

$$\|y - \hat{y}\| \leq 2\sigma \|u\|, \quad (18)$$

if $x_0 = 0$, $\hat{x}_{10} = 0$, $\|u\| \leq \gamma_R$, and $u(t) \in \boldsymbol{L}$. This follows because the terms containing $\hat{z}$ in (15)–(16) cancel. The matching condition (17) is a severe restriction, but as we shall see next, it can always be fulfilled if the incremental Gramians are quadratic.

A. Quadratic Gramians and Balancing

If the Gramians $W_{i,o}(x)$ and $W_{i,c}(x)$ are quadratic, i.e., $W_{i,o}(x) = x^T Q x$, $W_{i,c}(x) = x^T \tilde{P} x$, with symmetric positive-definite matrices

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{12}^T & \tilde{P}_{22} \end{bmatrix},$$

then we can check that the matching condition (17) means that $\left.\frac{\partial W_{i,o}}{\partial x_2}\right|_{x-\hat{x}} = 2[(x_1 - \hat{x}_1)^T Q_{12} + x_2^T Q_{22}]$ and $\left.\frac{\partial W_{i,c}}{\partial x_2}\right|_{x+\hat{x}} = 2[(x_1 + \hat{x}_1)^T \tilde{P}_{12} + x_2^T \tilde{P}_{22}]$ must be equal with proper scaling σ^2. This is possible for all x and $\hat{x}$ if and only if $Q_{12} = \tilde{P}_{12} = 0$, $Q_{22} = \sigma^2 \tilde{P}_{22}$. Hence, Q and $\tilde{P}$ need to be block diagonal. This is obtained with linear coordinate transformations. If we change the coordinates as

$x = \phi(z) = Tz$, for some invertible matrix $T \in \mathbb{R}^{n\times n}$, see (5), then the quadratic Gramians that solve (11) and (12) transform as $W_{i,o}(z) = z^T(T^TQT)z$, $W_{i,c}(z) = z^T(T^T\tilde{P}T)z$. The eigenvalues of the product $\tilde{P}^{-1}Q$ are invariant under these coordinate transformations. The square roots of the eigenvalues are called the *Hankel singular values* of the system:

$$\sigma_i = \lambda_i^{1/2}(\tilde{P}^{-1}Q), \quad i = 1\ldots n.$$

Since $\tilde{P}$ and Q are not unique for a polynomial system G, the singular numbers are not unique. We can *balance the Gramians* if Q and $\tilde{P}$ are positive definite. That is, there is a linear coordinate transformation such that

$$\begin{aligned} Q &= \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}, \\ \tilde{P} &= \operatorname{diag}\{\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_n^{-1}\}, \end{aligned} \tag{19}$$

see [4]. The incremental Gramians then fulfill (7)–(8) in Section IV.

After balancing, it is easy to fulfill the condition (17) when x_2 is the last state in x, using $\sigma = \sigma_n$. More states can then be removed recursively by noticing that the reduced-order model $\hat{G}$ has the quadratic (diagonal) Gramians $\hat{x}^TQ_{11}\hat{x}$ and $\hat{x}^T\tilde{P}_{11}\hat{x}$. We can then derive the following proposition.

Proposition 6: Assume that the polynomial system G has balanced quadratic incremental Gramians (19). Then the difference between the outputs of (1) and a truncated $\hat{n}$-th order model $\hat{G}$ in (14) is bounded by $\|y - \hat{y}\| \leq 2\left(\sum_{i=\hat{n}+1}^{n}\sigma_i\right)\|u\|$, if the systems are initially at rest, and $u \in \boldsymbol{L}$ and $\|u\| \leq \gamma_R$.

Proof: Follows by the procedure described above together with the triangular inequality. Here γ_R is found through Proposition 3. ■

This is a generalization of the balanced truncation error bound in [2]. Similar results are available for linear parameter-varying and time-varying systems. See, for example, [15] and [9].

B. Approximate Quadratic Incremental Gramians

Let us for simplicity consider the case where the model (1) is given by $\dot{x}(t) = AZ(x(t)) + Bu(t)$, $y(t) = CZ(x(t))$, and see what (11) and (12) lead to. If we write out (11) and (12), we obtain

$$-2[x-\hat{x}]^TQ[A(Z(x)-Z(\hat{x}))] - |C(Z(x)-Z(\hat{x}))|^2 \geq 0, \tag{20}$$

$$-2[x+\hat{x}]^T\tilde{P}[A(Z(x)+Z(\hat{x}))+2Bu] + 4|u|^2 \geq 0, \tag{21}$$

for all $x, \hat{x} \in \boldsymbol{K}$ and $u \in \boldsymbol{L}$. It turns out that these inequalites are often hard to solve when $Z(x) \neq x$. However, one can argue that it is conservative to require the inequalities to hold when $\hat{x}$ and x are far apart. We are looking for good model approximations, and it is more important that (20) and (21) hold when $\hat{x} \approx x$.

Let $\Delta x = x - \hat{x}$ be small. Then a first approximation is $Z(x) - Z(\hat{x}) \approx \frac{\partial Z}{\partial x}(x)\Delta x$, and $Z(x) + Z(\hat{x}) \approx 2Z(x)$. Using these expressions in (20) and (21), we obtain

$$\Delta x^T\left[-2QA\frac{\partial Z}{\partial x}(x) - \left|C\frac{\partial Z}{\partial x}(x)\right|^2\right]\Delta x \geq 0, \tag{22}$$

$$-2x^T\tilde{P}[AZ(x)+Bu] + |u|^2 \geq 0, \tag{23}$$

for all $x \in \boldsymbol{K}$, Δx, and $u \in \boldsymbol{L}$. These inequalities are easier to solve than (20) and (21), but more studies are required to evaluate their usefulness as compared to the Gramians in Section III. Notice that solutions $W_o(x) = x^TQx$ to (3) should be solutions to (22) for small x. Furthermore, (23) is identical to (4) when $W_c(x) = x^T\tilde{P}x$. Hence, quadratic Gramians $W_o(x)$ and $W_c(x)$ from Section III can be used as approximate quadratic incremental Gramians $W_{i,o}(x)$ and $W_{i,c}(x)$ if $\boldsymbol{K}$ is small.

VI. Numerical Examples

We will now illustrate the methods described in the previous sections by considering some simple examples.

Example 7: Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2^3 + x_4^3, \\ \dot{x}_2 &= -x_1^3 - 0.25x_2^3 + x_3^3, \\ \dot{x}_3 &= 0.75x_1^3 + 0.5x_2^3 - x_3^3 + x_4^3 - u, \\ \dot{x}_4 &= -0.75x_3^3 - x_4^3 + u, \end{aligned}$$

with output $y = x_1^2x_2$. Note that the linearization of this system around the origin is zero, thus linear model reduction method cannot be applied. Using the methods described in Section III, homogeneous polynomial Gramians of degree 4 can be found. When evaluated for the basis vectors in the current coordinates, we have

$$\begin{aligned} W_o(e_1) &= 0.5071 & W_c(e_1) &= 0.9730 \\ W_o(e_2) &= 0.4506 & W_c(e_2) &= 1.2983 \\ W_o(e_3) &= 0.2812 & W_c(e_3) &= 5.9602 \\ W_o(e_4) &= 0.2962 & W_c(e_4) &= 3.6202. \end{aligned}$$

Thus, the Gramians satisfy the ordering property (9)–(10) for $i = 3$, suggesting that the third state is the least important from input-output perspective. When the third state is truncated, the response of the reduced order model for sinusoidal and step inputs are shown in Figure 1.

Example 8: Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1x_2 - 3x_2x_3 - x_1x_4, \\ \dot{x}_2 &= x_3 + 0.5x_1x_2 + 0.5x_2x_3 + x_1x_4, \\ \dot{x}_3 &= x_4 + 0.5x_1x_2 + 0.5x_2x_3 - 0.25x_1x_4, \\ \dot{x}_4 &= -x_1 - 3x_2 - 5x_3 - 7x_4 - 3x_1x_2 + 0.1x_2x_3 \\ &\quad + 0.3x_1x_4 + u, \end{aligned}$$

with output $y = x_1$. Since the linearization around the origin is stable, controllable, and observable, it is possible to compute quadratic Gramians, as well as a balancing transformation, based on this linearization. The Hankel singular values corresponding to these Gramians are 1.1028, 7.5260×10^{-1}, 1.5008×10^{-1}, and 2.2716×10^{-4}, which

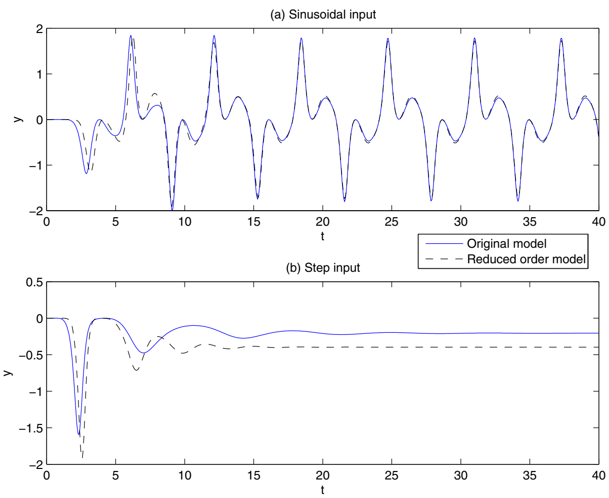

Fig. 1. Responses of the reduced order model obtained by truncating the third state of the system in Example 7. The inputs are $u(t) = \sin(t)$ in (a), and a unit step in (b).

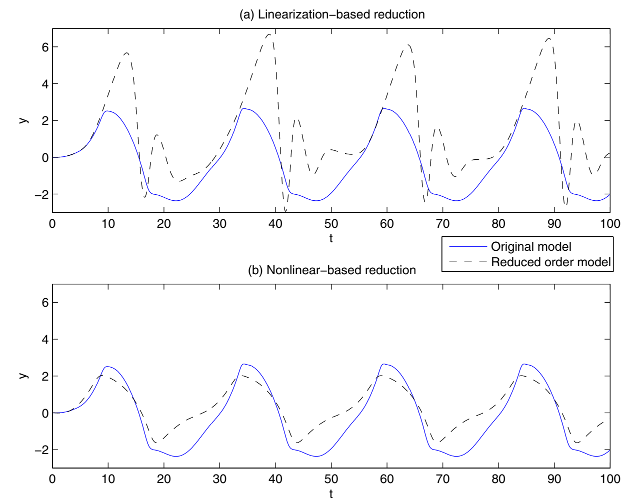

Fig. 2. Responses of the reduced order models in Example 8 for the sinusoidal input $u(t) = 2.5\sin(0.25t)$. Notice that the output of the reduced order model in (b) is qualitatively better than the one in (a).

hints that a very good approximation with a third order model is possible. We applied to the original *nonlinear* model the balancing transformation computed from the linearization, and truncated the least important state to obtain a nonlinear reduced order model. Unfortunately, although around the origin the reduced order model indeed approximates the original model very well, their behaviors in the nonlinear regime can be quite different. See Figure 2(a), where the outputs corresponding to a sinusoidal input are compared.

On the other hand, using the methods of Section III, it is also possible to compute quadratic Gramians using the original *nonlinear* model. We computed quadratic Gramians on the sets $\boldsymbol{K} = \{x \in \mathbb{R}^4 : x^T x \leq 9\}$ and $\boldsymbol{L} = \mathbb{R}$. In this case, the Hankel singular values are 1.8214, 1.8205, 1.6739, and 1.2065, which do not give an overly optimistic prediction like before, and hence one should not expect that a third order model will be able to approximate the original model very well. Applying the balancing transformation obtained from these Gramians to the nonlinear model and truncating the least important state, we obtained a third order nonlinear model which gives a response shown in Figure 2(b).

VII. Conclusions

In this paper, we have looked at generalizations of the balanced truncation procedure for polynomial systems. We started by obtaining generalized reachabiliy and observability Gramians that fulfill certain inequalities. The inequalities can be solved using sum of squares programming. Using the Gramians, we can identify coordinates that are hard to reach and to observe. Some heuristics were developed for the truncation procedure. Furthermore, we presented some tools for obtaining upper bounds on the truncation error. The corresponding inequalities are hard to solve, but with some approximations they become more tractable.

Future work should include how to compute more general balancing coordinate transformations that are applicable to non-quadratic Gramians.

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