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Finite Time Stability and Quasihomogeneous Control Synthesis of Uncertain Switched Systems with Application to Underactuated Manipulators

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Abstract— **Switched control synthesis is developed for underactuated mechanical systems. In order to locally stabilize an underactuated system around an unstable equilibrium, its output is specified in such a way that the corresponding zero dynamics is locally asymptotically stable. Once such an output has been chosen, the desired stability property of the closed-loop system is provided by applying a quasihomogeneous switched controller, driving the system to the zero dynamics manifold in finite time. Although the present synthesis exhibits an infinite number of switches on a finite time interval, it does not rely on the generation of sliding modes, while providing robustness features similar to those possessed by their sliding mode counterparts. Theoretical results are supported by an application to drive systems with backlash.**

I. INTRODUCTION

Stabilization of underactuated systems, forced by fewer actuators than degrees of freedom, presents a challenging problem [5], [22]. As well known (see, e.g., [2], [28]), these systems possess nonholonomic properties, caused by nonintegrable differential constraints, and therefore, they cannot be stabilized by means of smooth feedback. With this in mind, the present investigation on stabilization of underactuated mechanical systems is focused on switched control methods and it is based on the results recently published in [17].

The switched control synthesis to be developed is as follows. In order to locally stabilize an underactuated system around an unstable equilibrium, an output of the system is specified to ensure that the corresponding zero dynamics is locally asymptotically stable. Once such an output is chosen, the desired stability property of the closed-loop system is provided by applying a switched controller that drives the system to the zero dynamics manifold in finite time. Synthesis of such a controller that maintains an underactuated system on the zero dynamics manifold, which is typically of co-dimension greater than the control space dimension (the singular case in sliding mode control), is a main contribution of the paper.

The structure of the switched controller constructed is inspired from the quasihomogeneous controller of [16]-[17], stabilizing a one-link manipulator in finite time. Although that controller exhibited a so-called *Zeno* behavior with an infinite number of switches on a finite time interval (see [12], [14] for Zeno modes), it did not rely on the generation of sliding modes, while providing robustness features similar to those possessed by their sliding mode counterparts. A *second order sliding mode* (see the original work [11] for second order sliding modes and [1], [7], [8] for advanced results in the area) appeared in the equilibrium point only. In contrast to standard sliding mode control algorithms which are capable of providing the closed-loop manipulator with the ultimate boundedness property only [13], the afore-mentioned quasihomogeneous controller stabilized the manipulator in finite time, thus constituting an interesting alternative to standard sliding mode controllers.

The quasihomogeneous synthesis procedure is subsequently generalized for underactuated systems and capabilities of the procedure are illustrated by a regulation problem for a drive system with backlash, presenting a simple underactuated system.

For the drive system, which consists of a motor part, driven by DC motor, and a reducer part (load), operating in the presence of backlash, we ensure that all the trajectories of the closed-loop system are bounded and the angular position of the load asymptotically decays to a desired position. Due to practical requirements [10], the motor angular position and velocity are assumed to be the only information available for feedback.

The rest of the paper is outlined as follows. Section 2 is focused on the quasihomogeneous stabilization of a simple one degree-of-freedom manipulator, operating under uncertainty conditions. In Section 3, the proposed quasihomogeneous synthesis is extended to underactuated mechanical systems and its effectiveness is then illustrated in Section 4 by an application to a drive system with backlash. Finally, Section 5 presents some conclusions.

II. ROBUST FINITE TIME STABILIZATION OF ONE-LINK MANIPULATOR

The quasihomogeneous synthesis is first illustrated with a simple one degree-of-freedom mechanical manipulator, operating under uncertainty conditions. The dynamics of the manipulator is governed by

$$\ddot{y} = \omega(y, \dot{y}, t) + u \tag{1}$$

where y is the position of the manipulator, $\dot{y}$ is the velocity of the manipulator, u is the controlled input, $\omega(y, \dot{y}, t)$ is a piece-wise continuous nonlinearity that captures all forces (viscous and Coulomb frictions, gravitation, etc.), affecting the manipulator.

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Solutions of the differential equation (1) with piece-wise continuous right-hand side, are defined in the sense of Filippov [6] as that of a certain differential inclusion with a multi-valued right-hand side.

Operating under uncertainty conditions implies imperfect knowledge of the nonlinearity $\omega(y, \dot{y}, t)$. This possibly destabilizing term

$$\omega(y, \dot{y}, t) = \omega^{nom}(y, \dot{y}, t) + \omega^{un}(y, \dot{y}, t) \tag{2}$$

typically contains an *apriori* known nominal part $\omega^{nom}(y, \dot{y}, t)$ to be handled through nonlinear damping and an uncertainty $\omega^{un}(y, \dot{y}, t)$ to be rejected. It is assumed that $\omega^{un}(y, \dot{y}, t)$ is locally bounded

$$|\omega^{un}(y, \dot{y}, t)| \leq N \ for \ all \ t \geq 0 \tag{3}$$

by an *apriori* known constant $N > 0$. Apart from this, both functions $\omega^{nom}(y, \dot{y}, t)$ and $\omega^{un}(y, \dot{y}, t)$ are assumed to be piece-wise continuous.

The following control law

$$u = -\omega^{nom}(y, \dot{y}, t) - asign(y) - bsign(\dot{y}) \\ -hy - p\dot{y} \tag{4}$$

subject to

$$N < b < a - N, \ h, p \geq 0 \tag{5}$$

appears to stabilize the uncertain system (1)-(3) in finite time. Apparently, the above controller (4), (5) consists of the nonlinear damping $-\omega^{nom}(z, t)$, the linear gain $-hy - p\dot{y}$, and the homogeneous switching part $\varphi(y, \dot{y}) = -asign(y)$ $-bsign(\dot{y})$ such that $\varphi(cy, c\dot{y}) = \varphi(y, \dot{y})$ for all $c > 0$.

Until recently, finite time stability of asymptotically stable homogeneous systems has been well-recognized for only continuous vector fields [3], [9]. Extending this result to switched systems has required proceeding differently [17] because a smooth homogeneous Lyapunov function, whose existence was proven in [23] for continuous asymptotically stable homogeneous vector fields, can no longer be brought into play.

The novel techniques that was developed in [17] has established that the finite time stability of a switched homogenous system is saved regardless of some inhomogeneous perturbations. In particular, it has been shown that the inhomogeneous system (1)-(5) is globally finite time stable whenever condition (3) holds globally. The local version of this result is as follows.

Theorem 1: Let a one-link manipulator (1) operate under uncertainty conditions (2), (3) and let it be driven by the quasihomogeneous switched controller (4), (5). Then the closed-loop system (1)-(5) is locally finite time stable, uniformly in the admissible uncertainties (2), (3).

Proof of Theorem 1 is nearly the same as that of Theorem 4.2 of [17] and it is therefore omitted.

The qualitative behavior of the one link manipulator (1)-(3), driven by the switched controller (4), (5), is depicted in Fig. 1 and it is as follows. While approaching the origin $y = \dot{y} = 0$, the system trajectories rotate around it. Since by Theorem 1, the closed-loop system is locally finite time stable, the switching times of the controller have a finite accumulation point.

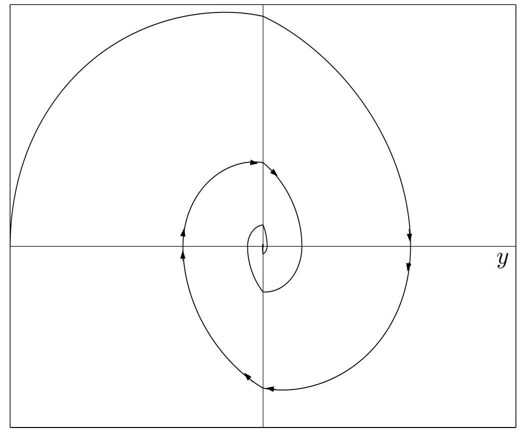

Fig. 1. Phase portrait of the controlled one-link manipulator.

Thus, system (1)-(5) does exhibit Zeno behavior with an infinite number of switches in a finite amount of time. This system does not generate sliding motions everywhere except the origin. If a trajectory starts there at any given finite time, the so-called sliding mode of the second order appears [11]. In a particular case, when the uncertainty $\omega(y, \dot{y}, t) = \omega^{un}(y, \dot{y}, t)$ has no nominal part and the control gains h, p are set to zero, the proposed control law (4) degenerates to the well-known homogeneous twisting algorithm [7], [8].

III. STABILIZATION OF UNDERACTUATED MECHANICAL SYSTEMS

In the present section, the quasihomogeneous synthesis is developed for stabilization of underactuated systems of the form

$$\ddot{q} = M^{-1}(q)[B\tau - C(q, \dot{q})\dot{q} - G(q)] \tag{6}$$

$$z = \Omega(q, \dot{q}). \tag{7}$$

In the above equation, $q \in \mathbb{R}^n$ is the joint position vector, $\tau \in \mathbb{R}^m$, $m < n$ is the input torque, $z \in \mathbb{R}^m$ is the output vector, $\dot{q}$ and $\ddot{q}$ are the velocity and acceleration vectors, respectively, $M(q) \in \mathbb{R}^{n \times n}$ is the inertia matrix, $C(q, \dot{q})\dot{q}$ represents centrifugal and Coriolis terms, $G(q)$ is the gravity vector, and B of rank m is the input matrix that maps the torque input τ of dimension m to the joint coordinates space of dimension n.

Under certain conditions system (6), (7) has m-vector relative degree $(2, \ldots, 2)^T$ at $x = 0$ and the distribution, spanned by the columns of the control matrix, is involutive. Just in case, it can be represented, by means of a nonlinear change of state coordinates and a feedback transformation (see [4] for details), in the form

$$\begin{aligned} \dot{x} &= g(x, \xi, \dot{\xi}) \\ \ddot{\xi} &= f(x, \xi, \dot{\xi}) + u. \end{aligned} \tag{8}$$

If in addition, this system is locally minimum phase and sufficiently smooth, it can locally be stabilized by a quasihomogeneous controller similar to (4). Throughout, the following assumptions are imposed on system (8).

1) The functions $g(x, \xi, \dot{\xi})$ and $f(x, \xi, \dot{\xi})$ are piece-wise continuous in all the arguments, and in addition, the function $g(x, \xi, \dot{\xi})$ is continuous in $(\xi, \dot{\xi})$ locally around $(\xi, \dot{\xi}) = 0$ for almost all x.
2) Given piece-wise differentiable functions $\xi(t), \dot{\xi}(t)$ of sufficiently small magnitudes, an arbitrary solution of the system
$$\dot{x} = g(x, \xi(t), \dot{\xi}(t)) \tag{9}$$
is bounded on any finite time interval.
3) The system
$$\dot{x} = g(x, 0, 0) \tag{10}$$
has 0 as a locally asymptotically stable equilibrium.

Solutions of the state and zero dynamics differential equations (8)-(10) with piece-wise continuous right-hand sides, are defined in the sense of Filippov [6] as that of a certain differential inclusion with a multi-valued right-hand side. Under Assumption 1 the existence of such a solution (possibly non-unique) of either equation with an arbitrary initial condition is guaranteed by Theorem 8 of [6, p. 85].

Other assumptions are made for technical reasons. Assumption 2 is introduced to avoid the destabilizing effect of the peaking phenomenon and particularly it holds whenever the function $g(x, \xi, \dot{\xi})$ satisfies a corresponding linear growth condition in x (a detailed treatment of the peaking phenomenon in the continuous setting can be found in [26]). Assumption 3 means that (8), specified with the output $z = \xi$, is a locally minimum phase system. The role of this notion is well-known from the theory of smooth fields [4] and it is now under study for switched nonautonomous systems.

As in the manipulator case (1), system (8) is operating under uncertainty conditions. The nonlinear gain g can not destabilize the closed-loop system because of the minimum phase hypothesis, which is why no more information is required for this gain. The destabilizing term

$$f(x, \xi, \dot{\xi}) = f^{nom}(x, \xi, \dot{\xi}) + f^{b}(x, \xi, \dot{\xi}) \tag{11}$$

is partitioned into a nominal part f^{nom}, known *a priori*, and an uncertain bounded gain f^b whose components $f_j^b,\ j = 1, \dots, m$ are locally upper estimated

$$|f_j^b(x, \xi, \dot{\xi})| \le N_j \tag{12}$$

by *a priori* known constants $N_j > 0$. Apart from this, both functions f^{nom} and f^b are assumed to be piece-wise continuous.

Being inspired from the quasihomogeneous controller (4), (5), the following switched control law

$$\begin{aligned} u(x, \xi, \dot{\xi}) &= -f^{nom}(x, \xi, \dot{\xi}) - \alpha sign\ \xi \\ &\quad -\beta sign\ \dot{\xi} - H\xi - P\dot{\xi} \end{aligned} \tag{13}$$

with the parameter gains

$$H = diag\{h_j\},\ P = diag\{p_j\}, \tag{14}$$
$$\alpha = diag\{\alpha_j\},\ \beta = diag\{\beta_j\} \tag{15}$$

subject to

$$\begin{gathered} N_j < \beta_j < \alpha_j - N_j, \\ h_j, p_j \ge 0,\ j = 1, \dots, m \end{gathered} \tag{16}$$

is proposed to locally stabilize the uncertain system (8), (11), (12) whose state $(x, \xi, \dot{\xi})$ is available for measurements. Hereafter, the notation $diag$ is used to denote a diagonal matrix of an appropriate dimension; $sign\ \xi$ with a vector $\xi = (\xi_1, \dots, \xi_m)^T$ stands for the column vector $(sign\ \xi_1, \dots, sign\ \xi_m)^T$.

In what follows, the switched control law (13), (16) is shown to drive the uncertain system (8) to the zero dynamics manifold $\xi = \dot{\xi} = 0$ in finite time thereby yielding desired stability properties of the closed-loop system.

Theorem 2: Let Assumptions 1-3 be satisfied and let the uncertain system (8), (11), (12) be driven by the state feedback (13) such that condition (16) holds. Then the closed-loop system (8), (13)-(16) is locally asymptotically stable, uniformly in the admissible uncertainties (11), (12).

Proof: The closed-loop system (8) driven by (13) is represented as follows

$$\dot{x} = g(x, \xi, \zeta) \tag{17}$$
$$\begin{aligned} \dot{\xi}_j &= \zeta_j, \\ \dot{\zeta}_j &= f_j^b(x, \xi, \zeta) - \alpha_j sign\ \xi_j \\ &\quad -\beta_j sign\ \zeta_j - h_j \xi_j - p_j \zeta_j, \\ j &= 1, \dots, m. \end{aligned} \tag{18}$$

Due to Assumption 1, Theorem 8 of [6, p. 85] is applicable to system (17), (18), and by applying this theorem, the system has a local solution for all initial data and uncertainties (12). Let us demonstrate that each solution of (17), (18) is globally continuable on the right.

First, let us note that given $j \in (1, \dots, m)$, no motion appears on the axes $\xi_j = 0$ and $\zeta_j = 0$ except their intersection $\xi_j = \zeta_j = 0$. Indeed, if $\xi_j(t) = 0$ on a trajectory of (17), (18) then it follows from (17) that $\zeta_j(t) = 0$ along the trajectory. In turn, if $\zeta_j(t) = 0$ on a trajectory of (17), (18) then due to the parameter subordination (16), the second equation of (18) fails to hold for $\xi_j \neq 0$.

Next, let us compute the time derivative of the function $V_j(\xi_j, \zeta_j) = \alpha_j|\xi_j| + \frac{1}{2}(h_j\xi_j^2 + \zeta_j^2),\ j = 1, \dots, m$ along the trajectories of (18). Taking into account (12), one derives that

$$\dot{V}_j(\xi_j(t),\zeta_j(t)) = \alpha_j\zeta_j sign\ \xi_j + h_j\xi_j\zeta_j$$
$$+\zeta_j\{f_j^b(x,\xi,\zeta) - \alpha_j sign\ \xi_j - \beta_j sign\ \zeta_j$$
$$-h_j\xi_j - p_j\zeta_j\} \le -[\beta_j - f_j^b(x,\xi,\zeta) sign\ \zeta_j]$$
$$\times|\zeta_j| - p_j\zeta_j^2 \le -(\beta_j - N_j)|\zeta_j(t)| \quad (19)$$

everywhere but on the axis $\xi_j = 0$ where the function $V_j(\xi_j,\zeta_j)$ is not differentiable. Since no sliding motion appears on the axis $\xi_j = 0$ except the intersection $\xi_j = \zeta_j = 0$ where $\dot{V}_j(\xi_j(t),\zeta_j(t)) = 0$, inequality (19) remains in force for almost all t.

By virtue of (16), it follows that each solution of subsystem (18) subject to (12) is uniformly bounded in t. Coupled to Assumption 2, this ensures that all possible solutions of the over-all uncertain system (12), (17), (18) remain bounded on any finite time interval and by the property B of Theorem 9 of [6, p. 86], these solutions are globally continuable on the right.

Now let us observe that due to the local boundedness (12) of the uncertain terms $f_j^b(x,\xi,\zeta),\ j = 1,\dots,m$ and by virtue of the parameter subordination (16), each subsystem (18) satisfies all the conditions of Theorem 1 and by applying this theorem, (18) is locally finite time stable, uniformly in the admissible uncertainties (12). It follows that starting from a finite time moment, the over-all uncertain system (12), (17), (18) evolves on the zero dynamics manifold $\xi = \zeta = 0$ where its behavior is governed by the zero dynamics equation (10).

In order to complete the proof, it remains to note that by Assumption 3, the zero dynamics (10) is locally asymptotically stable. Coupled to the local uniform finite time stability of (18), this ensures that the closed-loop system (17), (18) is locally uniformly asymptotically stable. The proof of Theorem 2 is thus completed. ■

Summarizing, the following quasihomogeneity-based local stabilization procedure is proposed for underactuated systems. First, an output of the system is specified in such a way that the corresponding zero dynamics is locally asymptotically stable. Once such an output has been chosen, the underactuated system is transformed into the normal form (8), whose stabilization is achieved by applying the quasihomogeneous controller (13), (16).

To this end we note that the closed-loop system (8), (13)-(16) becomes globally asymptotically stable if Assumptions 1-3 as well as the upper estimate (12) hold globally. The proof of this fact is similar to that of Theorem 2 and it is omitted because of space limitations. Instead, the capability of the global stabilization is subsequently illustrated with a simple example.

IV. OUTPUT REGULATION OF DRIVE SYSTEM WITH BACKLASH

To demonstrate the effectiveness of the proposed synthesis procedure it is now applied to an electrical driver, consisting of a motor part and a reducer part (load). The objective is to drive the load to a desired position while providing the boundedness of the system motion and attenuating external disturbances, including nonlinear backlash effects. To meet practical requirements the motor angular position and velocity are assumed to be the only information available for feedback.

A. State equations and backlash model

The dynamic models of the angular position $q_i(t)$ of the DC motor and that $q_o(t)$ of the load are given according to [15]:

$$J_o N^{-1}\ddot{q}_o + f_o N^{-1}\dot{q}_o = T$$
$$J_i\ddot{q}_i + f_i\dot{q}_i + T = \tau_m \quad (20)$$
$$y = (q_i,\dot{q}_i)^T \quad (21)$$

Hereinafter, J_o, f_o, $\ddot{q}_o$ and $\dot{q}_o$ are, respectively, the inertia of the load and the reducer, the viscous output friction, the output acceleration and the output velocity. The inertia of the motor, the viscous motor friction, the motor acceleration and the motor velocity are denoted by J_i, f_i, $\ddot{q}_i$ and $\dot{q}_i$, respectively. The input torque τ_m serves as a control action, $y(t)$ is the measurement vector.

The transmitted torque T through a backlash with an amplitude j, located within $[-0.6, 0.6]$[1], is modelled as follows [15]:

$$T = K\Delta q + K\eta(\Delta q) \quad (22)$$

where[2]

$$\Delta q \quad = \quad q_i - Nq_o \in [-2j, 2j], \quad (23)$$
$$\eta(\Delta q) \quad = \quad -4j\frac{1-e^{-\gamma_s\Delta q}}{1+e^{-\gamma_s\Delta q}}, \quad (24)$$

K is the stiffness, N is the reducer ratio and $\gamma_s = \frac{1}{2j}$ is the positive graphic parameter.

Since the control input τ_m enforces the motor part only and it does not have a direct access to the load, the electrical driver presents a simple underactuated model.

B. Switched control synthesis

As mentioned above, our objective is to design a switched regulator, using the measurements of the motor angular position $q_i(t)$ and velocity $\dot{q}_i(t)$ only, so as to obtain the closed loop system in which all the trajectories are bounded and the output $q_o(t)$ asymptotically decays to a desired position q_d as $t \to \infty$ while also attenuating the influence of external disturbances.

To investigate the regulation problem, let us introduce the state deviation vector $x = [x_1\ x_2\ x_3\ x_4]^T$ with

$$\begin{aligned} x_1 &= q_o - q_d \\ x_2 &= \dot{q}_o \\ x_3 &= q_i - Nq_d \\ x_4 &= \dot{q}_i \end{aligned}$$

[1]This should be viewed as a restriction on the validity of the present backlash model.

[2]Implication (23) should be viewed as a physical constraint on the displacement Δq of the motor position q_i with respect to the reducer Nq_o.

where x_1 is the load position error, x_2 is the load velocity, x_3 is the motor position deviation from its nominal value, x_4 is the motor velocity. The nominal motor position value Nq_d has been pre-specified in such a way to guarantee that the deviation system

$$\dot{x} = Ax + \varphi(x) + B\tau_m \tag{25}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -J_o^{-1}KN^2 & -J_o^{-1}f_o & J_o^{-1}KN & 0 \\ 0 & 0 & 0 & 1 \\ J_i^{-1}KN & 0 & -J_i^{-1}K & -J_i^{-1}f_i \end{bmatrix},$$

$$\varphi(x) = \begin{bmatrix} 0 \\ J_o^{-1}KN\eta(x_3 - Nx_1) \\ 0 \\ -J_i^{-1}K\eta(x_3 - Nx_1) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ J_i^{-1} \end{bmatrix}$$

is minimum phase with respect to the output $x_3 = 0$. Indeed, according to [4] the zero dynamics of system (25) on the manifold $x_3 = 0$ is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= J_o^{-1}[-KN^2x_1 - f_ox_2 + NK\eta(-Nx_1)] \end{aligned} \tag{26}$$

and its global asymptotic stability is guaranteed by the following.

Theorem 3: Let the nonlinearity η be governed by (24). Then system (26) is globally asymptotically stable.

Proof: To begin with, let us consider a Lyapunov function of the form

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{2}KN^2J_o^{-1}x_1^2 + 8j^2KJ_o^{-1}[2\ln\frac{e^{\frac{Nx_1}{2j}}+1}{2} - \frac{Nx_1}{2j}]. \tag{27}$$

Since

$$2\ln\frac{e^{\frac{Nx_1}{2j}}+1}{2} \geq \frac{Nx_1}{2j} \ for\ all\ x_1 \in R$$

by inspection, the function V, governed by (27), appears to be positive definite.

Then, let us compute the time derivative of the Lyapunov function on the trajectories of (26):

$$\dot{V} = -J_o^{-1}(KN^2x_1 + f_ox_2 + 4jKN\frac{e^{\frac{Nx_1}{2j}}-1}{e^{\frac{Nx_1}{2j}}+1})x_2 + KN^2J_o^{-1}x_1x_2 + 8j^2KJ_o^{-1} \times[\frac{4Ne^{\frac{Nx_1}{2j}}}{4j(e^{\frac{Nx_1}{2j}}+1)}x_2 - \frac{Nx_2}{2j}] = -J_o^{-1}f_ox_2^2 \leq 0. \tag{28}$$

Now let us observe that the zero dynamics system (26) has no trivial solutions on the manifold $x_2 = 0$ where the time derivative of the Lyapunov function equals to zero. Indeed, if $x_2 = 0$ then due to (26)

$$\frac{Nx_1}{2j} + 2\frac{1-e^{\frac{Nx_1}{2j}}}{1+e^{\frac{Nx_1}{2j}}} = 0, \tag{29}$$

thus concluding that $x_1 = 0$. To reproduce the latter conclusion it suffices to represent equation (29) in terms of $z = \frac{Nx_1}{2j}$

$$z + 2\frac{1-e^z}{1+e^z} = 0 \tag{30}$$

and note that the left-hand side of (30) is a strictly increasing function of z because its derivative is positive definite by inspection:

$$1 - \frac{4e^z}{(1+e^z)^2} = \frac{(1-e^z)^2}{(1+e^z)^2} > 0\ for\ all\ z \neq 0. \tag{31}$$

In order to complete the proof it remains to apply the LaSalle-Krasovskii invariance principle to the system in question. ■

Our control strategy is to drive the system to the manifold $x_3 = 0$ in finite time, thereby imposing desired stability properties on the closed-loop system. The following switched control law

$$\tau_m = -k_1 sgn(x_3) - k_2 sgn(x_4) - hx_3 - px_4 \tag{32}$$

with $h, p \geq 0$ and $k_1 > k_2 > 0$ such that

$$6jK < k_2 < k_1 - 6jK \tag{33}$$

is deduced from the quasihomogeneous controller (13) to globally drive the system to the zero dynamics manifold $x_3 = 0$ in finite time thereby providing global asymptotic regulation of system (25) to the desired position q_d.

Theorem 4: The servomechanism (20)-(24) driven by the switched regulator (32) is globally asymptotically stable at the desired point $x = 0$.

Proof: First, let us note that the physical domain of the servomechanism (20)-(24) is governed by implication (23) and hence the transmitted torque (22) satisfies the linear growth condition on this domain.

Moreover, being restricted to the physical domain (23), the transmitted torque is estimated similarly to (12):

$$|T(\Delta q)| \leq 6jK, \tag{34}$$

and by virtue of the parameter subordination (33), the gain condition (16) holds everywhere on the physical domain.

Since in addition, the zero dynamics (26) on the manifold $y = 0$ is globally asymptotically stable by Theorem 3, the closed-loop system (25), (32) satisfies all the conditions of Theorem 5.1 from [17].

By applying [17, Theorem 5.1], the closed-loop system (25), (32), representing the state of the servomechanism (20)-

(24) in terms of the error vector x, is established to be globally asymptotically stable. Theorem 4 is thus proved. ■

It is worth of noting that applying Theorem 5.1 from [17] to the drive system (20)-(24) not only guarantees the stabilizing properties of the switched control law (32) but also ensures the desired robustness properties of the closed-loop system (25), (32) against matched external disturbances with norm bounds less than the maximum value of $k_1 - k_2$ and k_2.

Performance issues of the quasihomogeneous regulation of a drive system with backlash have been tested in a numerical study presented in [20] whereas an experimental support will appear elsewhere.

V. CONCLUSIONS

The quasihomogeneous control synthesis is developed to locally asymptotically stabilize underactuated mechanical systems. The stabilizing strategy is to drive the system to the zero dynamics manifold in finite time and maintain it there in spite of parameter uncertainties and external disturbances. Desired robustness properties and asymptotic stability of the closed-loop system are thus provided.

The proposed control synthesis presents an interesting alternative to the standard sliding mode control techniques. Although the resulting controllers do exhibit Zeno modes with an infinite number of switches on a finite time interval, however, they do not rely on the generation of sliding motions on the switching manifolds but on their intersections. Compared to sliding mode controllers, Zeno mode controllers exhibit additional attractive features recently illustrated in applications to fully actuated friction mechanical systems [16]-[17]. In those publications the controllers were demonstrated to be capable of providing the static position stabilization and the desired system performance in spite of significant uncertainties in the system description as is typically the case in control of electromechanical systems with complex hard-to-model nonlinear phenomena.

In the present work, the quasihomogeneity-based control synthesis is extended to underactuated mechanical systems. In order to locally stabilize an underactuated system around an unstable equilibrium, an output of the system is specified in such a way that the corresponding zero dynamics is locally asymptotically stable. Once such an output has been chosen, the desired stability property of the closed-loop system is provided by applying a quasihomogeneous controller, driving the system to the zero dynamics manifold in finite time.

As an illustration, the proposed quasihomogeneous synthesis is shown to be capable of providing the global asymptotic regulation of a mechanical drive system with backlash to a desired load position in a practical situation where the motor angular position and velocity are the only information available for feedback.

REFERENCES

[1] Bartolini G., Ferrara A. and Usai E. (2000). Chattering avoidance by second-order sliding mode control. *IEEE Trans. Autom. Contr.*, **43**, 241-246.

[2] Berkemeier M. D. and Fearing R. S. (1999). Tracking Fast Inverted Trajectories of the Underactuated Acrobot. *IEEE Trans. Robotics and Aut.*, **15**, 4, 740-750.

[3] Bhat S. P. and Bernstein D. S. (1997). Finite-Time Stability of Homogeneous Systems. *Proc. Amer. Contr. Conf.*, 2513-2514, Albuquerque, NM.

[4] Byrnes C.I. and Isidori A. (1991). Asymptotic stabilization of minimum phase nonlinear systems. *IEEE Trans. Automat. Contr.*, **36**, 1122–1137.

[5] Fantoni I., Lozano R. and Spong M. (2000). Energy Based Control of the Pendubot. *IEEE Trans. Aut. Contr.*, **45**, 4, 725-729.

[6] Filippov A. F. (1988). *Differential equations with discontinuous right-hand sides*. Dordrecht: Kluwer Academic Publisher.

[7] Fridman L. and Levant A. (1996). Higher order sliding modes as a natural phenomenon in control theory, in *Robust Control via variable structure and Lyapunov techniques*, Garafalo and Glielmo (eds.), Lecture notes in control and information science, Berlin, Springer, 217, 107-133.

[8] Fridman L. and Levant A. (2002). Higher order sliding modes, in *Sliding mode control in engineering*, W. Perruquetti and J.-P. Barbout (eds.), New York: Marcel Dekker, 53-102.

[9] Hong Y., Huang J. and Xu Y. (2001). On an output feedback finite-time stabilization problem. *IEEE Trans. Auto. Ctrl.*, **46**, 305-309.

[10] Lagerberg A., and Egardt B.S. (2003). Estimation of backlash with application to automotive powertrains. *Proceedings of the 42 Conference on Decision and Control*, Maui, Hawaii, USA, 4521-4526.

[11] Levant A. (1993). Sliding order and sliding accuracy in sliding mode control. *Int. J. Contr.*, **58**, 1247-1263.

[12] Liberzon D. (2003). *Switching in Systems and Control* Birkhauser: Boston.

[13] Lu X.Y. and Spurgeon S.K. (1997). Robust sliding mode control of uncertain nonlinear systems. *Syst. Contr. Lett.*, **32**, 75–90.

[14] Lygeros J., Johansson K.H., Simic S.N., Zhang J., and Sastry S.S. (2003). Dynamical properties of hybrid automata. *IEEE Trans. Automat. Contr.*, **48**, 2–17.

[15] Merzouki R. and Cadiou J. C. (2005) Estimation of backlash phenomenon in the electromechanical actuator. *Control Engineering Practice*, **8**, 973–983.

[16] Orlov Y. (2003). Extended invariance principle for nonautonomous switched systems. *IEEE Trans. Autom. Contr.*, **48**, 1448–1452.

[17] Orlov Y. (2005). Finite-time stability and robust control synthesis of uncertain switched systems. *SIAM Journal on Optimization and Control*, **43**, 1253–1271.

[18] Orlov Y., Aguilar L. and Cadiou J. (2003). Switched chattering control vs. backlash/friction phenomena in electrical servo-motors. *Int. J. Contr.*, **76**, 959-967.

[19] Orlov Y., Alvarez J., Acho L. and Aguilar L. (2003). Global position regulation of friction manipulators via switched chattering control. *Int. J. Contr.*, **76**, 1446-1452.

[20] Orlov Y., Acho L., Aguilar L., Cadiou J., and Merzouki R. (2004). Finite time output regulation of drive systems with backlash. *Proceedings of the 8 International Workshop on Variable Structure Systems*, Vilanova i la Geltri, Spain, CD-ROM.

[21] Orlov Y., Acho L., and Aguilar L. (2004). Quasihomogeneity approach to the pendubot stabilization around periodic orbits. *Proc. 2nd IFAC Symposium on Systems, Structure and Control*, Oaxaca, Mexico, CD-ROM.

[22] Ortega R., Gomez-Estern F., and Blankenstein G. (2002). Stabilization of a class of underactuated mechanical systems via interconnection and damping assignment. *IEEE Trans. Auto. Ctrl.*, **47**, 1218–1233.

[23] Rosier L. (1992). Homogeneous Lyapunov function for homogeneous conitnuous vector field. *Syst. Contr. Lett.*, **19**, 467-473.

[24] Spong M.W. (1995). The Swing Up Contol Problem for the Acrobot. *IEEE Control Systems Magazine*, 49-55.

[25] Spong M.W. and Praly L. (1997). Control of Underactuated Mechanical Systems Using Switching and Saturation. *Lecture Notes in Control and Information Sciences 222*, Springer Verlag, London, 163-172, 1997.

[26] Sussman H.J. and Kokotovic P.V. (1991). The peaking phenomenon and the global stabilization of nonlinear systems," *IEEE Trans. Automat.Contr.*, **36**, 424-440.

[27] Utkin V.I., Guldner J. and Shi J. (1999). *Sliding modes in Electromechanical Systems*. London: Taylor and Francis.

[28] Zhang M. and Tarn T.J. (2002). Hybrid control of the Pendubot. *IEEE Trans. Mechatronics*, **7**, 79-86.